Klein geometry $\rightleftharpoons$ $M$ $C^\infty$-mfd. + transitive smooth left-action of a Lie group $G$.

$M \simeq G/H$ and left actions becomes left multipl. by $G$ on $G/H$.

- Euclidean geometry.

$M = \mathbb{R}^n$ equipped with standard inner product $\langle \, , \rangle$

$\mathrm{Euc}(n) = \{ f : \mathbb{R}^n \to \mathbb{R}^n : f \text{ preserves the distance} \}$

$\mathrm{dist}(x,y) := \| x - y \|$

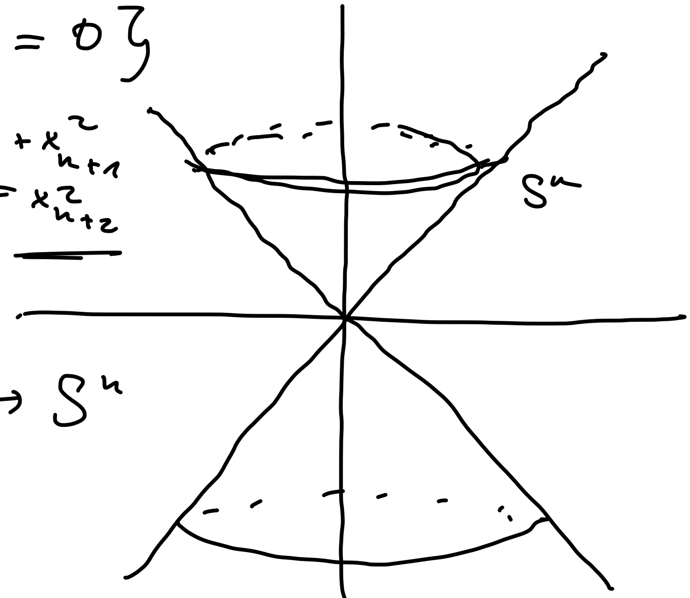
- Affine geometry
- $(S^n, g_{rd})$ , $(H^n, g_{hyp})$
- $\mathbb{R}P^n$

(6) Consider $\mathbb{R}^{n+1,1} = \mathbb{R}^{n+2}$ equipped with Lorentzian inner product $\langle x,y\rangle := x^t \begin{pmatrix} I_{n+1} & \\ & -1 \end{pmatrix} y$.

$$C := \{ x \in \mathbb{R}^{n+1,1} : x \neq , \ \langle x,x\rangle = 0 \}$$

$$\langle x,x\rangle = x_1^2 + \dots + x_{n+1}^2 \quad \underline{= x_{n+2}^2}$$

= null cone or light-cone




Consider the space of null lines

$$\mathbb{P}C = C/\sim \;\simeq\; S^n \qquad \pi : C \longrightarrow S^n \text{ smooth}$$

$(x \sim y : x = \lambda y, \ \lambda \in \mathbb{R})$

$$[(x_1, \dots, x_{n+2})] \longmapsto \frac{1}{x_{n+2}}(x_1, \dots, x_{n+1}) \qquad \left(\text{inverse } (x_1, \dots, x_n) \longmapsto [(x,1)]\right)$$

$T_x C = \{ y \in \mathbb{R}^{n+2} : \langle y, x \rangle = 0 \}$

$\cup$

$\mathbb{R}x$

$T_x \pi : T_x C \to T_{\pi(x)} S^n$ induces an isomorphism

$$T_x C / \mathbb{R}x \simeq T_{\pi(x)} S^n \quad (*)$$

Since $x$ is null, $\langle , \rangle$ induces a positive definite inner product on $T_x C / \mathbb{R}x$:

$\langle y + \mathbb{R}x, y' + \mathbb{R}x \rangle = \langle y, y' \rangle$ is well-def. $\forall y, y' \in T_x C$.

$\rightsquigarrow$ gives rise to inner product via $(*)$ ~~of~~ on $T_{\pi(x)} S^n$.

Replacing $x$ by $\lambda x$ ($\lambda \in \mathbb{R}\setminus\{0\}$) changes the isomorphism (*)

$$T_{\lambda x} C \big/ \mathbb{R}\lambda x \xrightarrow{\sim} T_{\pi(\lambda x)} S^n = T_{\pi(x)} S^n$$ by a non-zero multiple.

Hence, the induced inner product on $T_{\pi(x)} S^n$ changes by $\lambda^2$.

$\Longrightarrow$ $\langle \, , \rangle$ induces on $S^n \simeq \mathbb{P}C$ only a Riemannian metric up to multiplication by a smooth fct.

$\Longrightarrow$ $(S^n, [g_{rd}])$

conformal equivalence class of $g_{rd}$

Two metrics $g$ and $\tilde{g}$ are conformally equiv., if $\exists$ $C^\infty$-fct. $f$ s.t. $g = f \tilde{g}$.

$\ell : O(n+1,1) \times S^n \longrightarrow S^n$ transitive left-action

$$\Rightarrow \quad S^n \simeq \underline{O(n+1,1)} \Big/ P$$

↗ as a conformal mfld.

↑ stabilizer of a null-line

$$O(n+1,1) \simeq \mathrm{Conf}(S^n, [g_{rd}]) = \{ f : S^n \to S^n \text{ diffeom.} : f^* g_{rd} \in [g_{rd}] \}$$

$$f^*[g_{rd}] = [g_{rd}].$$

Geometry in the sense of Klein does however not comprise the other significant generalization of Euclidean geometry in the 19th. century, namely Riemannian geometry!

Only homogeneous Riemannian mf. can be described as Klein geometries.

Common generalization of ~~the~~ both of these notions of geometry was given by Cartan at the beginning of the 20th century ⇝ Cartan geometry

## 1.5 Further existence results and the classification of Lie groups

**Lemma 1.44** Suppose $\varphi : G \to H$ a Lie group homomorphism, and let $K := \ker(\varphi) \subseteq G$ be the normal Lie subgr. given by the kernel of $\varphi$.

① The Lie algebra of $K$ is given by the following ideal of $\mathfrak{g}$:

$$\mathfrak{k} = \ker(\varphi') \subseteq \mathfrak{g} .$$

② Multiplication on $G$ descends to a smooth multiplication map $G/K \times G/K \longrightarrow G/K$, i.e. $G/K$ is a Lie group.

Proof.

① $K = \{ g \in G : \varphi(g) = e \}$

$$T_e K = \ker(T_e \varphi) = \ker(\varphi')$$
$$\| $$
$$\mathfrak{k}$$

$$\mathfrak{k} = \{ X \in \mathfrak{g} : \exp(tX) \in K \ \forall t \in \mathbb{R} \}$$
$$= \{ X \in \mathfrak{g} : \exp(t\varphi'(X)) = e \ \forall t \}$$
$$= \{ X \in \mathfrak{g} : \varphi'(X) = 0 \}.$$

② By Thm. 1.42, $G/K$ is smooth mfd.

Since $K$ is a normal subgroup, $G/K$ is also a group.

$$\begin{tikzcd} G \times G \arrow[r, "\pi \times \pi"] \arrow[rr, bend right] & G/K \times G/K \arrow[r, "\mu^{G/K}"] & G/K \end{tikzcd}$$

$\underline{\pi : G \to G/K}$

$\underline{\mu}^{G/K} \circ \pi \times \pi = \underline{\pi \circ \mu G}$ is smooth as comp. of smooth maps.

$\Rightarrow$ $\mu^{G/K}$ is smooth by universal property of surj. submersions. $\square$

Recall the following definition from topology:

**Def. 1.45** Suppose $p : Y \to X$ is continuous map between topolog. spaces $X$ and $Y$. Then $p$ is called **a covering map**, if for each point $x \in X$ $\exists$ an open neighbhd $U$ of $x$ in $X$, a discrete space $D$ and a homeomorphism $\varphi : p^{-1}(U) \to U \times D$ s.t.

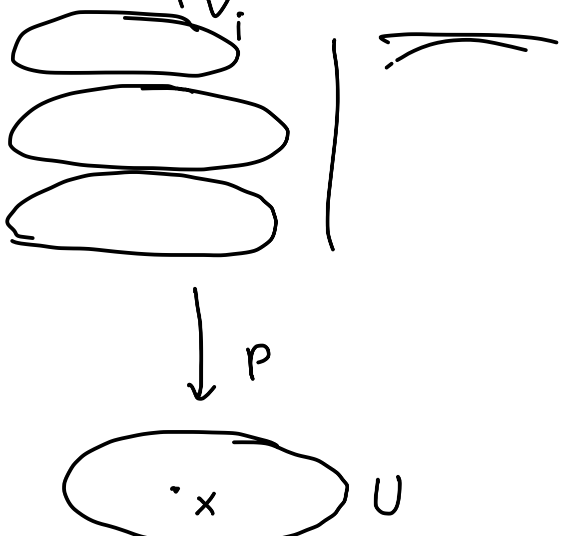
$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & U \times D \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

commutes.

Equivalently, for each $x \in X$ $\exists$ an open neigh. $U$ of $x$ s.t.

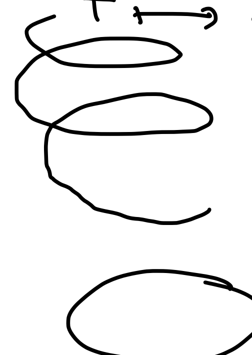
$p^{-1}(U) = \bigcup_{i \in I} V_i$ for pairwise disjoint open subsets $V_i$ of $Y$

and $p|_{V_i} : V_i \to U$ is a homeomorphism.




Ex. $\mathbb{R} \to S^1 = U(1)$

$t \mapsto e^{it}$



If $Y$ and $X$ are smooth mfds, then one can talk about smooth coverings / covers, $p: Y \to X$ requiring everything in Def. 1.45 to be smooth.

Thm. 1.46 Let $\varphi: G \to H$ be a Lie group homomorphism between connected Lie groups $G$ and $H$.

1. If $\varphi': \mathfrak{g} \to \mathfrak{h}$ is injective, then $\ker(\varphi)$ is a discret normal subgroup of $G$ contained in $Z(G)$.
2. If $\varphi': \mathfrak{g} \to \mathfrak{h}$ is surjective, then $\varphi$ surjective and descends to a Lie group isomorphism $G/\ker(\varphi) \xrightarrow{\sim} H$.

③ If $\varphi'$ is bijective, then $\varphi : G \to H$ is a smooth covering map and a local diffeomorphism.

Proof

① By Lemma 1.44, the Lie algebra of $\ker(\varphi)$ is zero.

$\Longrightarrow$ $\ker(\varphi) \subseteq G$ is a submfd. of dimension 0.

Submfd charts for $\ker(\varphi)$ give rise to an open subset $U \subseteq G$, $e \in U$ with $U \cap \ker(\varphi) = \{e\}$

For $g \in \ker(\varphi)$, $U_g := \lambda_g(U) \subseteq G$ is open neighbh. of $g$ with $U_g \cap \underline{\underline{\ker(\varphi)}} = \underline{\underline{\{g\}}}$ $\Longrightarrow$ subspace topology on $\ker(\varphi)$ is discret.

Since $\ker(\varphi)$ is normal,

the curve $c : t \mapsto \underline{\exp(tx)\, g\, \exp(tx)^{-1}}$ $\quad \forall t \in \mathbb{R}, \forall x \in \mathfrak{g}$
$\underline{\forall g \in \ker(\varphi)}$

is continuous with values in $\ker(\varphi)$.

By discreteness, $c(t) = g \quad \forall t \in \mathbb{R}$

$\Rightarrow \exp(tx) g = g \exp(tx) \quad \forall t \in \mathbb{R}, \forall x \in \mathfrak{g}$.

$\Rightarrow g \in \ker(\varphi)$ commutes with all elements of the subgroup generated by $\underline{\exp(\mathfrak{g})}$, which coincides with $G$, since $G$ is connected.

$\Rightarrow \ker(\varphi) \subseteq Z(G)$.

② $\varphi' : \mathfrak{g} \to \mathfrak{h}$ surjective.

$\varphi(\underline{\exp(\mathfrak{g})}) = \exp(\underline{\varphi'(\mathfrak{g})}) = \exp(\mathfrak{h})$ $\quad$ <u>Thm. 1.23</u>

$\Rightarrow \underbrace{\exp(\mathfrak{h})}_{\nearrow} \subseteq \underset{\substack{\uparrow \\ \text{subgroup}}}{\varphi(G)} \subseteq H \Rightarrow \varphi(G) = H$

$\Rightarrow \varphi$ induces a group isomorphism $\underline{\varphi} : G/_{\ker(\varphi)} \simeq H$.

It is an isomorphism of Lie groups: $\quad p : G \to G/_{\ker(\varphi)}$

$\underline{\varphi} = \varphi\_ \circ p$ is smooth + $p$ surj. submersion

$\Rightarrow \varphi\_$ is smooth.

$\varphi' : \mathfrak{g} \twoheadrightarrow \mathfrak{h}$ $\quad p' : \mathfrak{g} \to \mathfrak{g}/_{\ker(\varphi')}$)

$\ker(\varphi') = \ker(p')$

$\Rightarrow$ $\varphi'_-$ is an isomorphism $\quad (\varphi = \varphi_- \circ p)$.

$\Rightarrow$ $\varphi_-$ is a local diffeom., which together with bijectivity, implies $\varphi_-$ is a diffeomorphism.

3 By 2 we may assume $H = \ker(\varphi)$ and $\varphi = p : G \to G/_{\ker(p)}$ is the natural projection.

By proof of 1, $\exists$ an open neighbhd. $U$ of $e$ in $G$ s.t. $\ker(\varphi) \cap U = \{e\}$.

By continuity of $\mu$ and $\nu$, $\exists$ an open neighbh. $V$ of $e$ s.t.

$$\boxed{g, h \in V \Rightarrow h^{-1}g \in U} \Longleftarrow$$

In particular, $V \subseteq U$.

For $g \in G$, set $V_g := \rho^g(V)$.

- For $g \neq g' \in \ker(\varphi)$ one has $V_g \cap V_{g'} = \emptyset$.

$$h \in V_g \cap V_{g'} \Rightarrow h = v \cdot g = v'g' \quad v, v' \in V$$

$$\Rightarrow hg^{-1} = v = v'\underbrace{g'g^{-1}}_{\ker(\varphi)} \in V$$

$$\Rightarrow g'g^{-1} \in U \cap \ker(\varphi) = \{e\}$$

$$\Rightarrow g' = g.$$

- $p|_V : V \longrightarrow p(V)$ is bijective $\Big( \underset{\in V}{v = v'g}\quad g \in \ker(\varphi) \Rightarrow g \in \underbrace{(U \cap \ker(\varphi))}_{=e} \Big)$

$$\Rightarrow p^{-1}(p(V)) = \bigcup_{g \in \ker(\varphi)} V_g$$

Moreover, $\underline{p|_{V_g} : V_g \xrightarrow{\sim} p(V_g) = p(V)}$ is a bijective local diffeom. and hence a diffeomorphism.

For $g \in G$, $\underline{p(V_g)}$ is an open neighb. of $g\ker(\varphi) \in \underline{G/\ker(\varphi)}$ with $p^{-1}(p(V_g))$ again a union of pairwise disjoint open set.

$\square$

<u>QUESTION</u>: Suppose $G$ and $H$ are Lie groups and $\varphi : \mathfrak{g} \to \mathfrak{h}$ homomorphism between their Lie algebras.

Does there $\exists$ a Lie group homomorphism $\psi : G \to H$ s.t. $\psi' = \underline{\varphi}$ ?

<u>Def. 1.47</u> $G, H$ Lie groups. A local homomorphism from $G$ to $H$ is given by an open neighbhd $U$ of $e$ in $G$ and a $C^\infty$-map

$$\psi : \underset{\subseteq G}{U} \longrightarrow H$$

s.t. $\psi(e) = e$, $\psi(gh) = \psi(g)\psi(h)$ whenever $g, h$ and $gh$ lie in $U$.

Note that $T_e \psi =: \psi' : \mathfrak{g} \to \mathfrak{h}$ is again Lie algebra homomorphism.

<u>Thm. 1.48</u> Let $G$ and $H$ be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of Lie algebras.

Then $\exists$ a local homomorphism $\psi : \underset{\subseteq G}{U} \to H$ s.t. $\psi' = \varphi$.

If $G$ is simply connected, then there exists a homomorphism of Lie groups $\psi : G \to H$ s.t. $\psi' = \varphi$. $\square$