Corollary 1.21 Let $\varphi : H \to G$ be a continuous group homomorph. between two Lie groups H and G. Then φ is smooth.

Proof

First we show the statement for $H = (\mathbb{R}, +)$ (i.e. φ is a continuous 1-parameter subgroup).

Claim If $\varphi = \alpha : (\mathbb{R}, +) \to G$ is a contin. 1-parameter subgr., then α is smooth.

By Thm. 1.19, $\exists\, r > 0$ s.t. $\exp : \underset{\subseteq \mathfrak{g}}{B_{2r}(0)} \longrightarrow \exp(B_{2r}(0))$

open ball around $0 \in \mathfrak{g}$ with radius $2r$

is a diffeom. onto an open neighbhd. of $e \in G$.

Since α is continuous (and $\alpha(0) = e$), $\exists\, \varepsilon > 0$ s.t. $\alpha([-\varepsilon, \varepsilon]) \subseteq \exp(B_r(0)) =: B_r(e)$.

Define

$$\beta : [-\varepsilon, \varepsilon] \longrightarrow B_r(0)$$

$$\beta = \exp\big|_{B_r(0)}^{-1} \circ \alpha$$

For $|t| < \frac{\varepsilon}{2}$ we have:

$$\underline{\exp(\beta(2t))} = \alpha(2t) = \overbrace{\alpha(t)}^{\exp(\beta(t))} \cdot \alpha(t) = \underline{\exp(2\beta(t))}$$

$\Longrightarrow \beta(2t) = 2\beta(t) \Longrightarrow \beta\left(\frac{s}{2}\right) = \frac{1}{2}\beta(s) \quad \forall s \in [-\varepsilon, \varepsilon].$

By induction, one gets : $\underline{\beta\left(\frac{s}{2^k}\right) = \frac{1}{2^k}\beta(s)}$ $\quad \forall s \in [-\varepsilon, \varepsilon]$ $\forall k \in \mathbb{N}$,

$\Longrightarrow$ For $\underline{k, n \in \mathbb{N}}$ one has :

$$\alpha\left(\frac{n\varepsilon}{2^k}\right) = \alpha\left(\frac{\varepsilon}{2^k}\right)^n = \exp\left(\beta\left(\frac{\varepsilon}{2^k}\right)\right)^n = \exp\left(\frac{n}{2^k}\beta(\varepsilon)\right)$$

$$\Longrightarrow \boxed{\underline{\alpha(t)} = \underline{\exp\left(t\frac{1}{\varepsilon}\beta(\varepsilon)\right)}} \quad \forall t \in \overbrace{\left\{\frac{n\varepsilon}{2^k} : k \in \mathbb{N}, \underline{\underline{n \in \mathbb{Z}}}\right\}}^{=: D}$$

since $\alpha(t)^{-1} = \alpha(-t)$ and $\exp(-x) = \exp(x)^{-1}$

Since $D \subseteq \mathbb{R}$ is dense and both sides of $\boxed{\cdot = \cdot\cdot}$ are continuous, we deduce that $\alpha(t) = \exp\left(t \frac{1}{\varepsilon} \beta(\varepsilon)\right) \; \forall t \in \mathbb{R}$.

So α is smooth, since the right-hand side is.

Now consider the general case: $\varphi : H \to G$.

Take a basis $X_1, \dots, X_n$ of $\mathfrak{h}$. Then

$$u^{-1}(t_1, \dots, t_n) = \exp(t_1 X_1) \dots \exp(t_n X_n)$$

defines a diffeom. from a neighbhd. of $0 \in \mathbb{R}^n$ to a neighbhd. of $e \in H$ and its inverse u is a chart.

Then

$$(\varphi \circ u^{-1})(t_1, \dots, t_n) = \varphi(\exp(t_1 X_1) \dots \exp(t_n X_n))$$
$$= \varphi(\exp(t_1 X_1)) \dots \varphi(\exp(t_n X_n))$$

continuous 1-parameter subgroups, hence smooth by what we have shown.

$\Longrightarrow$ $\underline{\varphi \circ u^{-1}}$ is smooth and hence φ is smooth locally around $e \in H$.

$\Longrightarrow$ $\lambda_{\varphi(h)} \circ \varphi = \varphi \circ \lambda_{h^{-1}}$ implies that φ is smooth locally around any $h \in H$. $\square$

Prop. 1.22 For a Lie group G we denote by $G_0 \subseteq G$ the connected component of G containing $e \in G$, which is called the connected component of the identity of G.

① G_0 is an open and closed subset of G. In particular, G_0 is a submfd. of G of the same dimension as G.

② G_0 is a normal subgroup of G.

Hence, G_0 is a Lie subgroup of G and G/G_0 is a discrete topological group, called the component group of G.

Proof

(1) ✓

(2) $g, h \in G_0 \implies \exists$ continuous curves $c_g, c_h : [0,1] \to G$

s.t. $c_g(0) = c_h(0) = e \in G$

and $c_g(1) = g$, $c_h(1) = h$.

$\implies t \mapsto \mu(c_g(t), c_h(t)) = c_g(t) \cdot c_h(t)$

is a continuous curve connecting $e \in G$ with gh.

$\implies gh \in G_0$.

Since, $t \mapsto \nu(c_g(t))$ is continuous, also $g^{-1} \in G_0$ for any $g \in G_0$.

It is a normal subgroup: for $g \in G_0$, $k \in G$,

$t \to k c_g(t) k^{-1}$ is continous curve connecting $e \in G$ with kgk^{-1}

$\Longrightarrow$ $kgk^{-1} \in G_0$.

① and ②

$\Longrightarrow$ $G_0 \subseteq G$ is a Lie subgroup and G/G_0 is group, which is a discret topolog. group.

($\pi : G \to G/G_0$, quotient topolog. on G/G_0.

$U \subseteq G/G_0$ is open $\Longleftrightarrow$ $\pi^{-1}(U)$ is open in G.

$g \cdot G_0 \in G/G_0$ $\qquad \pi^{-1}(g \cdot G_0) = \{ g \cdot h \; : \; h \in G_0 \}$

$$= \lambda_g(\underline{\underline{G_0}}) \subseteq G.$$

<u>Thm. 1.23</u> G and H are Lie groups with Lie algebras $\square$.
$\mathfrak{g}$ and $\mathfrak{h}$. Then one has:

① If $\varphi : G \to H$ is a Lie group homomorphism, then

$$\varphi \circ \underline{\exp^G} = \exp^H \circ \varphi' \longleftarrow$$

where $\varphi' := T_e \varphi : \mathfrak{g} \to \mathfrak{h}$.

② G_0 coincides with the subgroup generated by $\exp(\mathfrak{g}) \subseteq G$.

③ If $\varphi, \psi : G \to H$ are Lie group homomorphism s.t.

$\varphi' = \psi'$. then $\varphi|_{G_0} = \psi|_{G_0}$.

In particular, if G is connected, then $\varphi = \psi$.

Proof.

① Recall that $T_g\varphi\, L_x(g) = L_{\varphi'(x)}(\varphi(g))$

(see proof of ① of Prop. 1.12 : L_x and $L_{\varphi'(x)}$ are φ-related),

which implies $\varphi \circ Fl_t^{L_x} = Fl_t^{L_{\varphi'(x)}} \circ \varphi$.

Hence,

$$\underline{\varphi(\exp(X))} = \varphi\left(Fl_1^{L_X}(e)\right) = Fl_1^{L_{\varphi'(X)}}(\underbrace{\varphi(e)}_{=e})$$

$$= \underline{\exp(\varphi'(X))}$$

$\forall X \in \mathfrak{g}$.

(2) If $\tilde{G}$ is the subgroup generated by $\exp(\mathfrak{g}) \subseteq G$,

then $\tilde{G} \subseteq G_e$, since $t \mapsto \exp(tX)$ is a C^∞-curve connecting e to $\exp(X)$

To see that $G_0 \subseteq \tilde{G}$, note that, since $\exp$ restricts to a diffeom. ~~from~~ on a neighbor. of $0 \in \mathfrak{g}$,

$\exp(\mathfrak{g})$ (and thus also $\tilde{G}$) contains an open neighbhd.
$U \subseteq G$ of $e \in G$.
Then for $g \in \tilde{G}$, $\lambda_g(U)$ is an open neighbhd. of
g containend in $\tilde{G}$, since $\tilde{G}$ is a subgroup.
Thus, $\tilde{G}$ is an open subset of G.
But $\tilde{G} \subseteq G$ is also closed, equivalently $G \setminus \tilde{G}$
is open, since for $g \in G \setminus \tilde{G}$, $\lambda_g(\nu(U)) = \{g \cdot h^{-1} : h \in U\}$
is an open neighbh. of g contained in $G \setminus \tilde{G}$. $\quad g \cdot h^{-1} \in \tilde{G}$
Thus, $\tilde{G}$ is also closed. $\Longrightarrow$ $\tilde{G} = G_0$. $\quad \Rightarrow \underbrace{g \cdot h^{-1} h}_{g} \in \tilde{G}$

3 By 1, φ and ψ coincide on $\underline{\exp(\mathfrak{g})}$.

Since they are group homomorphism, $\varphi|_{\hat{G}} = \psi|_{\hat{G}}$.

and so the claim follows from 2.

$\square$.

An application of 1 of Thm. 1.23 is for example:

<u>Example</u>

$\det : GL(n,\mathbb{R}) \longrightarrow (\mathbb{R}\setminus\{0\}, \cdot)$ is a Lie group homomorphism,

$\det(A\cdot B) = \det(A)\det(B)$.

In Global analysis we saw: $\det' := T_{Id}\det = \mathrm{trace} : \mathfrak{gl}(n,\mathbb{R}) \to \mathbb{R}$.

By ① of Thm. 1.23:

$$\boxed{\det(e^X) = e^{\operatorname{tr}(X)} \quad \forall X \in \mathfrak{gl}(n, \mathbb{R})}$$

1.2 Representations of Lie groups and Lie algebras

Def. 1.24 Suppose G is a Lie group.
Then a representation of G on a finite-dimensional real vector space V is a Lie group homomorphism

$$\varphi : G \longrightarrow GL(V).$$

Equivalently, it is a smooth map $\varphi : G \times V \to V$ s.t.

- $\varphi(g, -) : V \to V$ is linear $\forall g \in G$.
- $\varphi(e, v) = v \quad \forall v \in G$

$\varphi(g, \varphi(h, v)) = \varphi(gh, v) \quad \forall g, h \in G, \ \forall v \in V.$

Remark Often we refer to V as ~~the~~ a representation of G when φ is understood.

Notation : $\varphi(g, v) =: g \cdot v =: gv$

Examples

① $G = GL(V)$

Defining representation / standard representation of $GL(V)$:

$$\varphi : GL(V) \times V \longrightarrow V$$
$$(A, v) \longmapsto Av$$

Via a choice of basis, we can identify $GL(V) \simeq GL(n, \mathbb{R})$ and $V \cong \mathbb{R}^n$ and $\varphi : GL(n, \mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$

becomes multiplication of $\underset{\in GL(n,\mathbb{R})}{A}$ with a vector $v \in \mathbb{R}^n$.

Similarly, any matrix group $H \subseteq GL(V)$ has a standard representation on V.

(2) Adjoint representation of a Lie group G

Denote by $\mathrm{conj}_g : G \to G$ conjugation by $g \in G$.

$$\mathrm{conj}_g(h) := g h g^{-1} \quad \forall h \in G.$$

It is a Lie group homomorphism and

$$Ad : G \longrightarrow GL(\mathfrak{g}) = \{\text{lin. isomorph. } \mathfrak{g} \to \mathfrak{g}\}$$
$$Ad(g) := T_e \mathrm{conj}_g : \mathfrak{g} \to \mathfrak{g} .$$

is a representation of G on $\mathfrak{g}$, called the adjoint representation of G.

Let us check this is a representation:

$$\mathrm{conj}_g = \lambda_g \circ \rho^{g^{-1}} = \rho^{g^{-1}} \circ \lambda_g$$

$$\Longrightarrow T_e \mathrm{conj}_g = T_{g^{-1}} \lambda_g \circ T_e \rho^{g^{-1}} = T_g \rho^{g^{-1}} \circ T_e \lambda_g . \quad (*)$$

$$\mathrm{conj}_{gh} = \mathrm{conj}_g \circ \mathrm{conj}_h \Longrightarrow \underline{Ad(gh) = Ad(g) \circ Ad(h)}$$

$$\mathrm{conj}_{g^{-1}} = (\mathrm{conj}_g)^{-1} \Longrightarrow Ad(g^{-1}) = Ad(g^{-1})$$

$\Longrightarrow \quad Ad : G \to GL(\mathfrak{g})$ is group homomorphism.

To see that Ad is smooth, it sufficient to show that $(g, X) \longmapsto Ad(g)(X)$ is smooth.

Set $F : G \times \mathfrak{g} \longrightarrow TG \times TG \times TG$

$$F(g, X) \longmapsto (0_g, X, 0_{g^{-1}}).$$

It is smooth and so is also

$$T\mu \circ (id_G \times T\mu)(F(g, X)) = T_{g^{-1}}\lambda_g \circ T_e \rho^{g^{-1}} X \overset{(*)}{=} Ad(g)(X).$$

If $G = GL(n,\mathbb{R})$, then

$$\mathrm{conj}_A(B) = A B A^{-1}$$

is a linear map $\mathrm{conj}_A : M_n(\mathbb{R}) \to M_n(\mathbb{R})$.

$$\Longrightarrow \mathrm{Ad}(A)(X) = T_{id}\,\mathrm{conj}_A X = \mathrm{conj}_A X = A X A^{-1}$$

$\forall X \in \mathfrak{gl}(n,\mathbb{R})$
$\forall A \in GL(n,\mathbb{R})$.