Correction :

$G \times V \longrightarrow V$ represent. of a Lie group.

$G \times V^* \longrightarrow V^*$ $\quad (g \cdot \lambda)(v) = \lambda(g^{-1} \cdot v)$ $\quad \forall g \in G, \ \forall \lambda \in V^*$
$\forall v \in V$.

$\longrightarrow$ $\quad \mathfrak{g} \times V^* \longrightarrow V^*$ $\quad (X \cdot \lambda)(v) = -\lambda(X \cdot v)$.

II. BUNDES

We shall work in the smooth category, i.e. we will consider smooth fiber bundles, vector bundles and principal bundles.

2.1 Fiber bundles

Def. 2.1 Suppose F is a smooth mfd.

A (smooth) fiber bundle with standard fiber F is a smooth map $p : E \longrightarrow M$ between smooth mfds. E and M s.t. for any $x \in M$ $\exists$ an open neighbhd. U of x in M and a diffeom. $\phi : p^{-1}(U) \longrightarrow U \times F$ s.t

$$(*)\quad \begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

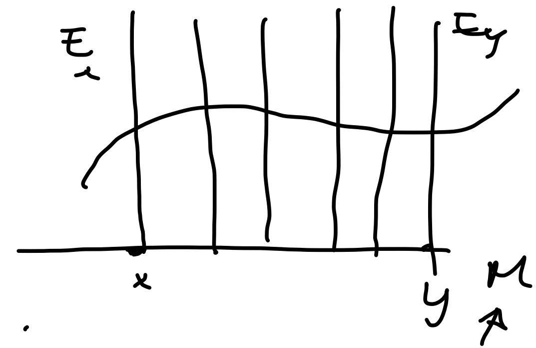
commutes.

- E is called the total space and M the base of $p: E \to M$.
- ϕ is called a fiber bundle chart or local trivialization of $p: E \to M$.
- for any $x \in M$, $E_x := p^{-1}(x)$ is called the fiber of x.

<u>Remark</u> If $U \subseteq M$ is open, then $p^{-1}(U) =: E|_U \xrightarrow{r} U$ is a fiber bundle with standard fiber F over U.

Def. 2.2 Suppose $p: E \to M$ is a fiber bundle.

- A (smooth) section of p is a smooth map $s: M \to E$ s.t. $\underline{p \circ s = \mathrm{id}_M}$, $s(x) \in E_x$.



- A local (smooth) section of p is a section of $E|_U$ for open subset $U \subseteq M$.

We write $\Gamma(E)$ resp. $\Gamma(E|_U)$ for the set of sections resp. local sections on U of $p: E \to M$.

Due to $(*)$, a fiber bundle has many local sections

(by $(*)$ they corresp. to maps $\underset{\subseteq M}{U} \to F$ $\quad x \mapsto (x, f(x))$ section $\begin{matrix} U \times F \\ \downarrow \\ U \end{matrix}$

$\underline{f : U \to F \text{ smooth}}$)

but global sections might not exist.

$\underset{\subseteq M}{U} \to F = \mathbb{R}$

$\begin{matrix} \cancel{M \times \mathbb{R}} \\ \downarrow \\ \cancel{M} \end{matrix}$ $\quad T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$

$\mathbb{R}^n \nearrow$

Existence of local sections (or $(*)$) implies that

$p : E \to M$ is a surjective submersion.

$\Rightarrow$ For $x \in M$, $E_x = p^{-1}(x) \subseteq E$ smooth submfd. of E

and it is diffeom. to F.

Def. 2.3

A morphism (resp. isomorphism) between two fiber bundle $p: E \to M$ and $\tilde{p}: \tilde{E} \to \tilde{M}$ is a smooth map (resp. a diffeom.) $f: E \to \tilde{E}$ that maps fibers to fiber ($f(E_{\underline{x}}) \subset \tilde{E}_{\hat{x}}$ for some $\hat{x} \in \tilde{M}$ dep. on x)

i.e. $\exists$ a map $\underline{f}: M \to \tilde{M}$ s.t. $\hat{x} = \underline{\underline{f}(x)}$

$$\begin{array}{ccc} E & \xrightarrow{f} & \tilde{E} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \tilde{p}} \\ M & \xrightarrow{\underline{f}} & \tilde{M} \end{array}$$

commutes. $\underline{\tilde{p} \circ f} = \underline{\underline{f} \circ p}$

By the universal property of surj. submersions, $\underline{f}$ is automatically smooth.

(If f is an isomorph. of fiber bundles, then f^{-1} is too).

Examples

① M and F mfds. $pr_1 : M \times F \longrightarrow M$ trivial fiber bundle over M with standard fiber F.

A fiber bundle is called trivializable (or trivial), if it is isomorphic to $M \times F \to M$.

Note that (*) says that any fiber bundle is locally trivializable.

2) Vector bundles (see Global Analysis)

Vector bundles are fiber bundles $p : V \longrightarrow M$ with standard fiber a vector space $\mathbb{V} \simeq \mathbb{R}^k$ s.t. for any $x \in M$ V_x is a vector space and there exists a local trivialization on a neighborhd. U of x, $\Phi : p^{-1}(U) \longrightarrow U \times \mathbb{V}$ s.t.

$$\longrightarrow \quad \Phi|_{V_y} : V_y \xrightarrow{\sim} \{y\} \times \mathbb{V} \simeq \mathbb{V}$$

is linear isomorphism. $\forall y \in U$.

Ex. (a) Tangent bundle $TM \to M$ of a mfd. M.

(b) Any tensor bundle $\underbrace{TM \otimes \dots \otimes TM}_{p} \otimes \underbrace{T^*M \otimes \dots \otimes T^*M}_{q} \to M$

$$\longrightarrow \Lambda^k T^* M \longrightarrow M$$

We have seen that (a) and (b) have always many global sections. Sections of (a) are vector fields and of (b) $\binom{p}{q}$-tensors resp. k-forms.

If G is a Lie group, we have seen that $TG \xrightarrow{p} G$ is trivializable (by left and right invariant vector fields on G).

$$\begin{tikzcd} TG \arrow[r] \arrow[dr, "p"'] & G \times \mathfrak{g} \arrow[d, "pr_1"] \\ & G \end{tikzcd} \qquad \begin{aligned} (g, \xi_g) &\longmapsto (g, T_g l_{g^{-1}} \xi_g) \\ L_X(g) &\longleftarrow (g, X) \end{aligned}$$

(3) Suppose G is a Lie group and $H \subseteq G$ a closed subgroup.

Then G/H is a smooth mfd and $p : \underline{G} \longrightarrow G/H$

a surjective submersion by Thm. 1.42.

In the proof we constructed a diffeom. $F : W \times H \longrightarrow U$, $(X, h) \longmapsto \exp(X)h$

$W \subseteq$ open neigh. of $0 \in \mathfrak{k}$; U open neigh. of H

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$

s.t.

$$\begin{tikzcd} U \arrow[r, "F^{-1}"] \arrow[dr, "p"'] & \underline{W} \times H \arrow[r, "p \circ \exp \times id"] & p(U) \times H \arrow[dl, "pr_1"] \\ & \underline{p(U)} & \end{tikzcd}$$

is a diffeom.

($p(U) \subseteq G/H$ is open)

$\Longrightarrow$ $p : G \to G/H$ is a fiber bundle with standard fiber H.

Suppose $p : E \to \underline{M}$ fiber bundle and let

$\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ and $\phi_\beta : p^{-1}(U_\beta) \to U_\beta \times F$ be two charts over open subsets $U_\alpha, U_\beta \subseteq M$ s.t. $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$.

Transition map / chart change is given of the form:

$$\phi_\beta \circ \phi_\alpha^{-1} : U_{\alpha\beta} \times F \longrightarrow U_{\alpha\beta} \times F$$
$$(y, f) \longmapsto (y, \phi_{\beta\alpha}(y, f))$$

$\phi_{\beta\alpha}(y, \_) : F \to F$ is a diffeom.

for smooth maps $\underline{\phi_{\beta\alpha}} : U_{\alpha\beta} \times F \to F$ s.t. for any $y \in U_{\alpha\beta}$,

$y \longmapsto \phi_{\beta\alpha}(y, -)$

$U_{\alpha\beta} \longrightarrow \mathrm{Diff}(F)$.

Local trivializations give rise to the notion of fiber bundle atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ for $p : E \to M$:

- $M = \bigcup_{\alpha \in I} U_\alpha$

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\phi_\alpha"] \arrow[dr] & & U_\alpha \times F \arrow[dl] \\ & U_\alpha & \end{tikzcd}$$

local trivialization.

- —

Prop. 2.4 Suppose E is a set, M and F are smooth mfds and $p : E \to M$ a map (between sets).

Assume there is

- $\{U_\alpha : \alpha \in I\}$ open cover of M
- for any $\alpha \in I$, a bijection $\phi_\alpha : p^{-1}(U_\alpha) \longrightarrow U_\alpha \times F$ s.t. $p|_{p^{-1}(U_\alpha)} = pr_1 \circ \phi_\alpha$

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\phi_\alpha"] \arrow[dr, "p"'] & & U_\alpha \times F \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

- for $\alpha, \beta \in I$ with $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ the transition map is of the following form

$$\phi_\alpha \circ \phi_\beta^{-1} : U_{\alpha\beta} \times F \longrightarrow U_{\alpha\beta} \times F$$
$$(y, f) \longmapsto (y, \phi_{\alpha\beta}(y, f))$$

for smooth maps $\phi_{\alpha\beta} : \underset{\subseteq M}{U_{\alpha\beta}} \times F \to F$.

Then E can be uniquely made into a smooth mfd. s.t.
$p : E \to M$ is a fiber bundle with fiber bundle atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$.

<u>Proof</u> (cf. Global Analysis, where we used this for vector bundles).

- Without loss of generality we may assume $\{(V_j, v_j) : j \in J\}$ is a countable atlas for M and that there are bijections $\phi_j : p^{-1}(V_j) \to V_j \times F$ $\forall j \in J$ which are restrictions of the given ϕ_α's, $\alpha \in I$.

- Equipp E with the following topology:

  the subsets $U \subseteq E$ s.t. $\phi_j(U \cap p^{-1}(V_j))$ is open in $V_j \times F$ $\forall j \in J$

  define a topology on E.

If $V \subseteq M$ is open, then $V \cap V_j$ is open in V_j $\forall j \in J$ and so $p^{-1}(V)$ is open in E, i.e. $p: E \to M$ is continuous.

For any $W \subseteq F$ open, $\phi_j^{-1}(V_j \times W) \subseteq E$ is open $\forall j \in J$. (check this).

- The topology on E is Hausdorff, since points in different fibers can be separated by open subsets of M and points in the same fiber can be separated by open subset of F.

Since M and F are second countable, so is E.

- Fix $\{(W_k, w_k) : k \in K\}$ an atlas for F.

Then the sets $\phi_j^{-1}(V_j \times W_k)$ for $j \in J$, $k \in K$ form an open cover of E and

$$v_j \times w_k \circ \phi_j : p^{-1}(V_j) \longrightarrow v_j(V_j) \times w_k(W_k)$$

is a homeomorphism onto the open subset $v_j(V_j) \times w_k(W_k)$
$\subseteq \mathbb{R}^{\dim(M) + \dim(F)}$

One verifies directly that transition maps of these maps are smooth. So, they define a smooth atlas on E with values in $\mathbb{R}^{\dim(M)+\dim(F)}$.

In these charts, p corresponds to π_1 and hence $p : E \to M$ is smooth.

Moreover, $\mathcal{A} = \{(V_j, \phi_j) : j \in J\}$ is a fiber bundle atlas for $p : E \to M$.

$\square$.

2.2 Bundles with structure group

Def. 2.5 Suppose $p: E \to M$ is a (smooth) fiber bundle with standard fiber F. Let G be a Lie group acting smoothly (from the left) on F.

(1) A G-atlas for $p: E \to M$ (or a reduction of structure group to G of $p: E \to M$ corresp. $G \times F \to F$) is a fiber bundle atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ for $p: E \to M$ s.t. for $(U_\alpha, \phi_\alpha), (U_\beta, \phi_\beta) \in \mathcal{A}$ the transition map is of the form:

G-action on F

$$\phi_\beta \circ \phi_\alpha^{-1} : U_{\alpha\beta} \times F \longrightarrow U_{\alpha\beta} \times F$$
$$(y, f) \longmapsto (y, \underline{\phi_{\beta\alpha}(y,f)}) = (y, \underline{\psi_{\beta\alpha}(y) \cdot f})$$

for smooth fcts $\varphi_{B\alpha} : U_{B\alpha} \longrightarrow G$.

(2) A fiber bundle with structure group G is a (or a fiber bundle with G-structure) fiber bundle $p: E \to M$ together with a maximal (or equivalence) G-atlas for some G-action on its standard fiber.

Without loss of generality, we may restrict to the case where $G \times F \longrightarrow F$ is effective, i.e., $G \longrightarrow \mathrm{Diff}(F)$ has trivial kernel.

If the action is not effective, then the kernel K of $G \to \mathrm{Diff}(F)$

is a closed normal subgroup of G and the fiber bundle with structure group G can be also seen as one with structure group G/K.

③ Two G-bundles are isomorphic if they are isomorphic as fiber bundles by an isomorphism identifying the G-structures.

Examples

① A (smooth) vectorbundle of rank k is the same as a fiber bundle with standard fiber a k-dim. vector space $F = \mathbb{V} \simeq \mathbb{R}^k$ with structure $GL(\mathbb{V}) \simeq GL(k, \mathbb{R})$ (where $GL(\mathbb{V})$ is acting on $\mathbb{V}$ by standard repres.).

(2) $p : G \longrightarrow G/H$ $\quad$ G Lie group, $H \subseteq G$ closed subgr.

is a fiber bundle with structure group H acting on H by left multiplication.

For any $x \in G/H$, $\exists$ an open neigh. U of x and a local section $s : U \longrightarrow G$ of p $\quad$ ($\underbrace{s(y)H}_{p(s(y))} = y$, $\forall y \in U \subseteq G/H$)

s gives rise to local trivialization of p :

$$
\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "p"'] & & U \times H \arrow[dl, "pr_1"] \\
& U &
\end{tikzcd}
$$

$$\phi(g) = \left(p(g), s(p(g))^{-1} \cdot g\right)$$

$$\phi^{-1}(y,h) = s(y)h$$

$\rightsquigarrow$ H-atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ in this way.

$$\phi_\beta \circ \phi_\alpha^{-1} : U_{\alpha\beta} \times H \longrightarrow U_{\alpha\beta} \times H$$

$$(y, h) \xrightarrow{\phi_\alpha^{-1}} (s_\alpha(y)h) \xrightarrow{\phi_\beta} \Big(p(s_\alpha(y)h), \; s_\beta\big(p(s_\alpha(y)h)\big)^{-1} \cdot s_\alpha(y)h \Big)$$

$$p(s_\alpha(y)h) = p(s_\alpha(y)) = y$$

$$= \Big(y, \; \underbrace{s_\beta(y)^{-1} s_\alpha(y)}_{\psi_{\beta\alpha}(y) \in H} \cdot h \Big)$$

$s_\alpha(y) = s_\beta(y) \cdot h'$ for some $h' \in H$.

since $p(s_\alpha(y)) = p(s_\beta(y)) = y$

Suppose $p : E \to M$ an effective fiber bundle with structure group G. Then we have given:

(i) effective smooth G-action of G on the standard fiber F

(ii) open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of M

(iii) smooth maps $\psi_{\beta\alpha} : U_\alpha \cap U_\beta \to G$ $\forall \alpha, \beta \in I$ related by the property that, if $\underbrace{U_\alpha \cap U_\beta \cap U_\gamma}_{=: U_{\alpha\beta\gamma}} \neq \emptyset$, then

$$(*) \quad \underline{\psi_{\gamma\beta}(y) \cdot \psi_{\beta\alpha}(y) = \psi_{\gamma\alpha}(y)} \qquad \forall y \in U_{\alpha\beta\gamma}.$$

$(*)$ is called the cocycle identity / equation

$$\Longrightarrow \psi_{\alpha\alpha}(y)\cdot\psi_{\alpha\alpha}(y) = \psi_{\alpha\alpha}(y) \Longrightarrow \psi_{\alpha\alpha}(y) = e \quad \forall y \in U_\alpha$$

$$\psi_{\gamma\beta}(y)\,\psi_{\beta\gamma}(y) = \psi_{\gamma\gamma}(y) = e \Longrightarrow \underbrace{\psi_{\gamma\beta}(y)}_{\in G} = \psi_{\beta\gamma}(y)^{-1} \quad \forall y \in U_{\beta\gamma} \quad \forall \beta,\gamma \in I$$

$$\psi_{\gamma\beta} : U_{\gamma\beta} \to G, \quad \psi_{\beta\gamma} : U_{\gamma\beta} \to G$$

Prop. 2.6 Suppose M and F are smooth mfds., G a Lie gr. and we are given the data (i)–(iii).

Let $E := \bigsqcup_{\alpha\in I}(U_\alpha \times F)/\sim \;= \{(\alpha, x, f) \mid \alpha \in I,\ x \in U_\alpha,\ f \in F\}/\sim$

where $(\alpha, x, f) \sim (\beta, x', f')$ if $x = x'$ and $f' = \psi_{\beta\alpha}(x)\cdot f$.

Then the natural projection $p : \tilde{E} \to M$ $\left(p([\alpha, x, f]) = x \right)$

can be made into a smooth fiber bundle with standard fiber F

and structure G.

<u>Proof:</u> $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$

$\phi_\alpha([\alpha, y, f]) = (y, f)$

$\phi_\alpha^{-1}(y, f) = [\alpha, y, f]$

bijections s.t. $p|_{p^{-1}(U_\alpha)} = pr_1 \circ \phi_\alpha$.

$p^{-1}(U_\alpha) = \{ [\alpha, y, f] : y \in U_\alpha, f \in F \}$

(any equiv. class in $p^{-1}(U_\alpha)$ has a unique representative with α in the first comp.

Transition maps :

$$\phi_\beta \circ \phi_\alpha^{-1} : U_{\alpha\beta} \times F \longrightarrow U_{\alpha\beta} \times F$$

$$\underline{(y, f)} \longmapsto \underline{[\alpha, y, f]} = [\beta, y, \psi_{\beta\alpha}(y) \cdot f]$$

$$\xmapsto{\phi_\beta} \underline{(y, \psi_{\beta\alpha}(y) \cdot f)}$$

By Prop. 2.4, $p : E \to M$ can be made into a smooth fiber bundle with fiber bundle atlas $\{U_\alpha, \phi_\alpha\}_{\alpha \in I}$, hence a fiber bundle with stand. fiber F and structure group G.

$\square$.

If we replace $\{\psi_{\beta\alpha}\}_{\beta\times\alpha\in I\times I}$ by a <u>cohomologous</u> cocycle $\{\psi'_{\beta\alpha}\}_{\beta\times\alpha\in I\times I}$ (assume (i) and (ii) are true):

$$\boxed{\psi_{\alpha\beta}(x)\, f_\beta(x) = f_\alpha(x)\, \psi'_{\alpha\beta}(x)} \quad \forall \alpha,\beta\in I \quad \forall x\in U_{\alpha\beta}$$

for smooth fcts $f_\alpha : \underline{U_\alpha} \to G \quad \forall \alpha\in I$.

Then the construction in the proof of Prop. 6.2 leads to an isomorphic fiber bundle with structure group G and M.

Conversely, any such fiber bundle with structure group G defines a cohomology class of cocycles.

Let us now look at two special cases of such bundles: vector bundles and principal bundles.

2.3 Vector bundles (recall from Global Analysis)

$p: V \longrightarrow M$ vector bundle

It is a fiber bundle with standard fiber a vector space $\mathbb{V}$ and structure group $GL(\mathbb{V})$.

- For any $x \in M$, V_x is a vector space: $y, y' \in V_x$, $t \in \mathbb{R}$

$$\underbrace{y + t y'}_{} := \phi^{-1}(v + t v') \qquad \phi : p^{-1}(U) \longrightarrow U \times \mathbb{V}$$

where $\phi(y) = (x, v)$ $\phi(y') = (x, v')$ local trivializ., $x \in U$.

Note that this is well-def., i.e. independent of the chart.

- Hence, $\Gamma(V)$ = space of sections is a vector space and a module over the ring $C^\infty(M,\mathbb{R})$ ($s, s' \in \Gamma(V)$

$$(s+s')(x) = \underbrace{s(x)}_{\in V_x} \underset{\uparrow}{+} \underbrace{s'(x)}_{\in V_x}$$

$$fs(x) = f(x)\,s(x) \qquad \forall x \in M.$$

- As we know from tensor fields, vector bundles have many global sections, since local sections can be extended by zero via bump functions.

<u>Def. 2.7</u> $p : V \to M$ and $q : W \to N$ two vector bundles with standard fibers $\mathbb{V}$ and $\mathbb{W}$

① Then a vector bundle morphism (resp isomorphism) between p and q is a fiber bundle morphism (resp. isomorph.) $f : V \to W$ with underlying map $\underline{f} : M \to N$ s.t. for any $x \in M$ $f|_{V_x} : V_x \to W_{\underline{f}(x)}$ is linear.

② In the special case $M = N$ and $\underline{f} = id_M$, a vector bundle morphism $f : V \to W$ induces a linear map

$$f_* : \Gamma(V) \to \Gamma(W)$$

$$f_*(s) = f \circ s \qquad (M \to W$$

Prop. 2.8 $p: V \to M$, $q: W \to M$ vector bundles and

$\Phi : \Gamma(V) \to \Gamma(W)$ a linear map.

Then $\Phi = f_*$ for a vector bundle morphism $f: V \to W$

$\iff$ Φ is linear $C^\infty(M, \mathbb{R})$.

Proof cf. analoguous statement for tensor fields in GA-(05)

Exercise.

$$\begin{tikzcd} TM \arrow[r, "Tf"] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

Example $f: M \to N$ C^∞-map between mflds., then

$Tf : TM \to TN$ is a vector bundle morphism covering f

Using Prop. 2.4. (cf. Global Analysis), we have

- $V \to M$, $V^* := \coprod_{x \in M} V_x^* \longrightarrow M$

  is a vector bundle and it is called the dual of V.

- $V \to M$, $W \to M$ vector bundle, then their tensor product

$$V \otimes W := \coprod_{x \in M} V_x \otimes W_x$$
$$\downarrow$$
$$M$$

is also a vector bundle over M.

- $\Lambda^k V$, $S^k V$ are also naturally vector bundles.

Remark K-Theory

Continuous vector bundles (usually complex vector bundles) over some nice topolog. X.

$V \to X$, $W \to X$ $V \oplus W \longrightarrow X$

Set of isomorphism classes of vector bundles over X is a commutative semi-group w.r. to $\oplus$ (unit 0, given by $id_{\cancel{X}} : \cancel{X} \longrightarrow \cancel{X}$ viewed as vector bd. over X with 0-dim. fibres).

$\underline{\mathbb{N}} \longrightarrow \mathbb{Z}$

One can construct an abelian group $K(X)$ out of this semi-group (Grothendieck group).

2.4. Principal fiber bundles

Def. 2.9 Let G be a Lie group. A principal fiber bundle with structure group G (or principal G-bundle) is a fiber bundle $p: P \longrightarrow M$ with standard fiber G and structure G acting on itself by left multiplication.

By definition, this means one has an atlas $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ for p s.t.

$$\phi_\beta \circ \phi_\alpha^{-1}(x, g) = (x, \psi_{\beta\alpha}(x) \cdot g)$$

$\psi_{\beta\alpha} : U_{\beta\alpha} \longrightarrow G$ smooth maps.

The fibers of a principal G-bundle are diffeom. to G, but they don't have a natural group structure, since left multiplications in a group are not group homomorphisms!

But each fiber P_x has a natural transitive right action of G, which is free, i.e. all isotropy group

$$G_u = \{e\} \quad \forall u \in P_x$$

$$(u \cdot g = u \implies g = e \quad \forall y \in P_x).$$

In particular, for any fixed $u \in P_x$, induces a diffeom.

$$\boxed{\begin{aligned} G &\xrightarrow{\sim} P_x \\ g &\longmapsto u \cdot g \in P_x \end{aligned}}$$

$$G/\underbrace{G_u}_{=\{e\}} \cong u \cdot G = P_x$$

How is this right-action of G on the fibers of p defined?

<u>Lemma 2.10</u> Suppose $p: P \to M$ is a principal G-bundle. For $u \in P$, $g \in G$ set

$$r(u,g) := \underline{u \cdot g} := \phi_\alpha^{-1}(x, \underline{h \cdot g}) \in P$$

where $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times G$ is a principal fiber bundle chart with $x \in U_\alpha$ and $\phi_\alpha(u) = \underline{(x, h)} \in U_\alpha \times G$.

Then $r: P \times G \to P$ defines a smooth right action of G on P, called the principal right action of G on P.

Moreover, this action preserves the fibers of p and restricts to a free and transitive right action of G on each fiber P_x.

<u>Proof.</u>

$r : P \times G \to P$ is well-defined, i.e. independent of the choice of chart, since left and right multiplication commute.

$u \in P_x$

Suppose ϕ_β is another chart around x and $\phi_\beta(u) = (x, h')$.

Then $h' = \psi_{\beta\alpha}(x) h$ and $\phi_\beta^{-1}(x, h'g) = \phi_\beta^{-1}(x, \psi_{\beta\alpha}(x) h g)$

$$= \phi_\beta^{-1} \phi_\beta \circ \phi_\alpha^{-1}(x, h \cdot g)$$

$$= \phi_\alpha^{-1}(x, h \cdot g).$$

Clearly, r is smooth and since right-mult. of G on G is right-action, r is a smooth right-action.

By construction, $r : P_x \times G \longrightarrow P_x$ $\forall x \in M$.

and it is transitive. It is also free.

$$\left(u \cdot g = u \iff h \cdot g = h \iff e = g \right)$$

$$\phi_\alpha(u) = (x, h)$$

$\square$