Recall: G is a Lie group

- representation of G: $\rho : G \to GL(V)$ Lie group homomorphism.
- Adjoint representation of G: $Ad : G \to GL(\mathfrak{g})$.

$$Ad(g) := T_e \, conj_g$$

Def. 1.25 Suppose $\mathfrak{g}$ is a Lie algebra over $\mathbb{K} = \mathbb{R}, \mathbb{C}$.
A representation of $\mathfrak{g}$ on a $\mathbb{K}$-vector space V is a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V) = \{\text{linear maps } V \to V\}$.

(i.e. ψ is linear and $\psi([X,Y]) = [\psi(X), \psi(Y)]$

$$= \psi(X)\circ\psi(Y) - \psi(Y)\circ\psi(X)$$

$\forall X,Y \in \mathfrak{g}$),

Equivalently, a bilinear map $\psi : \mathfrak{g}\times V \longrightarrow V$ s.t.

$$\psi([X,Y], v) = \psi(X, \psi(Y,v)) - \psi(Y, \psi(X,v))$$

$\forall X,Y \in \mathfrak{g}$ and $\forall v \in V$.

By Prop. 1.12, any representation $\varphi : G \longrightarrow GL(V)$ of a Lie group G induces a representation $\varphi' = T_e\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$

of its Lie algebra $\mathfrak{g}$.

For $G = GL(n, \mathbb{R})$, the standard representation φ of $GL(n, \mathbb{R})$ gives the standard representation of $\mathfrak{gl}(n, \mathbb{R})$ on $\mathbb{R}^n$:

$$\varphi' : \mathfrak{gl}(n, \mathbb{R}) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$(X, v) \longmapsto Xv$$

Similarly, for any matrix group and its standard representation.

For the adjoint repres. of a Lie group G, $Ad: G \to GL(\mathfrak{g})$, the induced representation of $\mathfrak{g}$, the so called adjoint representation of $\mathfrak{g}$, is given by

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}).$$

$$\mathrm{Ad}' = \mathrm{ad}(X)(Y) = [X,Y] \qquad \forall X,Y \in \mathfrak{g}.$$
$$= T_e \mathrm{Ad}$$

as the following proposition shows:

<u>Prop. 1.26</u> G Lie group with Lie algebra $(\mathfrak{g}, [\,,\,])$.

① For $X \in \mathfrak{g}$ and $g \in G$: $L_X(g) = R_{\mathrm{Ad}(g)X}(g)$.

② For $X, Y \in \mathfrak{g}$, $\mathrm{ad}(X)(Y) = [X,Y] \quad \forall X,Y \in \mathfrak{g}$.

③ For $X \in \mathfrak{g}$, $g \in G$ we have

$$\exp(t\,\mathrm{Ad}(g)(X)) = g \exp(tX) g^{-1} .$$

④ For $X, Y \in \mathfrak{g}$ we have :

$$\mathrm{Ad}(\exp(X))(Y) = e^{\mathrm{ad}(X)}(Y) = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\mathrm{ad}(X)^k Y}_{\underbrace{[X,[X,[X \dots,[X,Y]]\dots]}_{k}}$$

$$= Y + [X,Y] + \frac{1}{2}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \dots .$$

Proof.

(1) $L_X(g) = R_{\mathrm{Ad}(g)(X)}(g)$

$\lambda_g = \rho^g \circ \mathrm{conj}_g$

$$\Longrightarrow \underset{\overset{\|}{L_X(g)}}{T_e\lambda_g X} = T_e\rho^g \circ \underbrace{T_e \mathrm{conj}_g X}_{\mathrm{Ad}(g)X} = R_{\mathrm{Ad}(g)(X)}(g).$$

(2) Choose a basis $X_1, \dots, X_n$ of the vector space $\mathfrak{g}$.
Then $\mathrm{Ad}(g): \mathfrak{g} \to \mathfrak{g}$ corresponds to a $n \times n$ matrix $(a_{ij}(g))$ for any $g \in G$.

Note that $a_{ij} : G \longrightarrow \mathbb{R}$ are smooth, since Ad is smooth.

Matrix presentation of $\mathrm{ad}(X) : \mathfrak{g} \longrightarrow \mathfrak{g}$ equals

$$(X \cdot a_{ij}) = \underline{T_e a_{ij} X} = \underline{(L_X \cdot a_{ij})}(e) \ .$$

Any $Y \in \mathfrak{g}$ can be written as $Y = \sum_{i=1}^{n} y_i X_i$ and

$$\underline{L_Y(g)} \overset{(1)}{=} R_{\underbrace{\mathrm{Ad}(g)(Y)}}(g) = \underline{\sum_{i,j} y_i a_{ij}(g) R_{X_i}(g)}$$

$$\sum_{i,j} \underline{a_{ij}(g)} X_i y_j$$

$$\Longrightarrow \quad [L_X, L_Y] = \sum_{i,j} y_j \underbrace{[L_X, a_{ij} R_{X_i}]}_{= a_{ij} \underbrace{[L_X, R_{X_i}]}_{\substack{=0 \\ \text{by Prop. 1.14}}} + \underline{(L_X \cdot a_{ij}) R_{X_i}}} =$$

$$= \underline{\sum_{i,j} y_j (L_X \cdot a_{ij}) R_{X_i}} \,.$$

Evaluate at $e \in G$: $\quad \underline{[X, Y]} = [L_X, L_Y](e)$

$$= \sum_{i,j} y_j (X \cdot a_{ij}) X_i$$

$$= \underline{ad(X)(Y)} \,.$$

(3) Since $Ad(g) = T_e \operatorname{conj}_g$, the result follows directly from (1) of Thm. 1.23. $\big(\varphi : G \to H$

$\varphi \circ \exp^G = \exp^H \circ \varphi'$

$$\operatorname{conj}_g(\exp(tX)) = \exp(Ad(g)(tX)) = \exp(t\,Ad(g)(X)).$$

(4) Apply (1) of Thm. 1.23 to $Ad : G \to GL(\mathfrak{g})$.

$$Ad(\exp(X))(Y) = \exp(ad(X))(Y) = e^{ad(X)}(Y)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\, ad(X)^k(Y).$$

$\square$.

Prop. 1.27 Suppose G is a Lie group with Lie algebra $(\mathfrak{g}, [\,,\,])$. Let $\varphi : G \longrightarrow GL(V)$ be a Lie group representation of G with induced representation $\varphi' : \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$.

① $\varphi(\exp(tX))(v) = \exp(t\varphi'(X))(v)$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \varphi'(X)^k v = v + t\varphi'(X)v + \frac{t^2}{2!}\varphi'(X)\varphi'(X)v + \ldots$$

$\forall X \in \mathfrak{g}, v \in V, t \in \mathbb{R}$.

② $X \cdot v := \varphi'(X)(v) = \frac{d}{dt}\Big|_{t=0} \varphi(\exp(tX))v = \frac{d}{dt}\Big|_{t=0} \exp(tX) \cdot v$

Proof.

(1) follows from (1) of Thm. 1.23 and (2) follows immediately from (1).

We will later develop the basic representation theory of Lie groups and Lie algebras.

1.3 Lie subgroup and virtual Lie subgroups

We already defined what a Lie subgroup of a Lie group is.

<u>Prop. 1.28</u> Suppose H is a Lie subgroup of a Lie group G. Then H is closed as a subset of the topological space G.

<u>Proof</u>.

Any submfd. N of a mfd. M is locally closed, i.e. N is open in its closure $\overline{N}$ ($\Longleftrightarrow$ every $x \in N$ has neighbhd U in M s.t. $U \cap N$ is closed in U).

For any subgroup H of a topological group G, $\overline{H}$ is also a subgroup of G $\left(\underset{\in H}{h_n} \xrightarrow{n\to\infty} h \in \overline{H} \quad \underset{\in H}{g_n} \longrightarrow g \in \overline{H} \Longrightarrow \underset{\in H}{h_n g_n} \xrightarrow[n\to\infty]{} h \cdot g \in \overline{H} \right)$

If H is a Lie subgroup of a Lie group G, H is open and dense $\overline{H}$.

Hence, for $\underline{g \in \overline{H}}$, $\lambda_g(H) \subseteq \overline{H}$ is open in $\overline{H}$ ($\lambda_g : \overline{H} \to \overline{H}$ homeomorphism)

Since H is dense in $\overline{H}$, $\overline{\lambda_g(H) \cap H \neq \emptyset}$, ~~with~~ which implies $g \in H$. ($\exists\, h, h' \in H$ s.t. $g \cdot h' = h \Longrightarrow g \in H$).

$\square$

Conversely, one has:

<u>Thm. 1.29</u> Suppose H is a subgroup of a Lie group G that is closed as a subset of the topolog. space G.
Then H is a Lie group.

<u>Proof.</u> We write $\mathfrak{g}$ for the Lie alg. of G and

$$\mathfrak{h} := \left\{ c'(0) \;:\; \begin{array}{l} c : \mathbb{R} \to G \text{ is smooth, } c(0) = e \\ \text{and } c \text{ has values in } H \end{array} \right\}.$$

$$\subseteq \mathfrak{g}.$$

<u>Claim 1:</u> $\mathfrak{h}$ is a linear subspace.

If $c_1, c_2 : \mathbb{R} \longrightarrow H \subseteq G$ are C^∞- curves and $c_1(0) = c_2(0) = e$,

then $c(t) := c_1(t)\, c_2(at)$ is a C^∞- curve with values in H

and $c(0) = e$. $(a \in \mathbb{R})$.

Then, $c'(0) \in \underline{\mathfrak{g}}$

$$\begin{aligned} c'(0) = \frac{d}{dt}\Big|_{t=0} \mu(c_1(t), c_2(at)) &= T_{(e,e)}\mu\left(c_1'(0),\, a c_2'(0)\right) \\ &= \underbrace{c_1'(0)}_{\text{Lemma } 1.5} + \underline{a}\, \underline{c_2'(0)} \end{aligned}$$

<u>Claim 2</u>: Suppose $(X_n)_{n \in \mathbb{N}}$ is a sequence in $\mathfrak{g}$ with $\lim_{n \to \infty} X_n = X \in \mathfrak{g}$ and let $(t_n)_{n \in \mathbb{N}}$ is a sequence in $\mathbb{R}_{>0}$ s.t.,

$\lim_{n\to\infty} t_n = 0$.

Then if $\underline{\exp(t_n X_n) \in H}$ $\forall n \in \mathbb{N}$, then $\underline{\exp(tX)} \in H$ $\forall t \in \mathbb{R}$.

Fix $t \in \mathbb{R}$. For $n \in \mathbb{N}$ let a_n be the largest integer $\leq \frac{t}{t_n}$.

Then, $a_n t_n \leq t$ and $\underline{t - a_n t_n} < \underline{t_n}$, so

$$\lim_{n\to\infty} a_n t_n = t$$

$H \subseteq G$ is closed

$$\Longrightarrow \lim_{n\to\infty} \Big(\underbrace{\exp(t_n X_n)^{a_n}}_{H} \Big) = \lim_{n\to\infty} \exp(\underline{a_n t_n} \underline{X_n}) = \exp(tX) \in H$$

by continuity of exp in G

by assumption + H subgroup

~~since~~

<u>Claim 3</u> $\boxed{\mathfrak{h} = \{X \in \mathfrak{g} : \underline{\exp(tX) \in H \quad \forall t \in \mathbb{R}}\}}$.

RHS $\subseteq \mathfrak{h}$ by definition of $\mathfrak{h}$.

To show $\mathfrak{h} \subseteq$ RHS, let $c: \mathbb{R} \to H \subseteq G$ be a C^∞-curve with $c(0) = e$ $(c'(0) \in \mathfrak{h})$.

Then $\exists \varepsilon > 0$ and a C^∞-curve $v: (-\varepsilon, \varepsilon) \to \mathfrak{g}$ s.t. $c(t) = \exp(v(t))$ $\forall t \in (-\varepsilon, \varepsilon)$. $(v(0) = 0 \in \mathfrak{g})$.

$\Longrightarrow$

$$v'(0) = \lim_{t \to 0} \frac{v(t)}{t}$$

$$c'(0) = \frac{d}{dt}\Big|_{t=0} \exp(v(t)) = \underbrace{T_0 \exp}_{= \mathrm{Id}_{\mathfrak{g}}} v'(0) = v'(0) = \underline{\lim_{n \to \infty} n\, v\left(\tfrac{1}{n}\right)}$$

Set $t_n := \frac{1}{n}$ and $X_n := n\, v(\frac{1}{n})$, then

$$\exp(t_n X_n) = \exp\left(v\left(\tfrac{1}{n}\right)\right) \underset{\substack{\uparrow \\ \text{for big } n}}{=} c\left(\tfrac{1}{n}\right) \in H$$

By Claim 2, $\exp(t\,c'(0)) \in H \quad \forall t \in \mathbb{R}$.

<u>Claim 4.</u> Write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ as a vector space ($\mathfrak{k}$ is linear complement of $\mathfrak{h}$ in $\mathfrak{g}$.)

Then $\exists$ an open neighbhd. W of $0 \in \mathfrak{k}$ in $\mathfrak{k}$ s.t.

$$\exp(W) \cap H = \{e\}.$$

TO BE CONTINUED...