- $g(x)$ periodické s periodou T
  $g(x): \mathbb{R} \to \mathbb{R}$
- $F(x)$ Fourierova řada funkce $g(x)$

$\omega = \frac{2\pi}{T}$

$$F(x) = \sum_{n=0}^{\infty} \left(a_n \cos(n\omega x) + b_n \sin(n\omega x) \right)$$

$$\Rightarrow a_n = \frac{2}{T} \int_{x_0}^{x_0+T} g(x) \cos(n\omega x)$$

$$b_n = \frac{2}{T} \int_{x_0}^{x_0+T} g(x) \sin(n\omega x)$$

$$\boxed{F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (\quad \ldots \quad)}$$

Př 11.1 (i) $f(x) = \sin(2x)\cos(3x)$

lichá $\rightsquigarrow a_n = 0$

$$f(x) = \sin(2x)\left(\cos(2x + x)\right) =$$

$$= \sin(2x)\left(\cos(2x)\cos x - \sin(2x)\sin x\right)$$

$$= \frac{1}{2}\sin(4x)\cos x - \sin^2(2x)\sin x$$

$$= \frac{1}{2}\cos x\,(\sin 4x) - \frac{1-\cos(4x)}{2}\sin x$$

$$= -\frac{1}{2}\sin x + \frac{1}{2}\cos x\,(\sin 4x) + \frac{1}{2}\sin x\,(\cos 4x)$$

$$= -\frac{1}{2}\sin x + \frac{1}{2}\sin(5x)$$

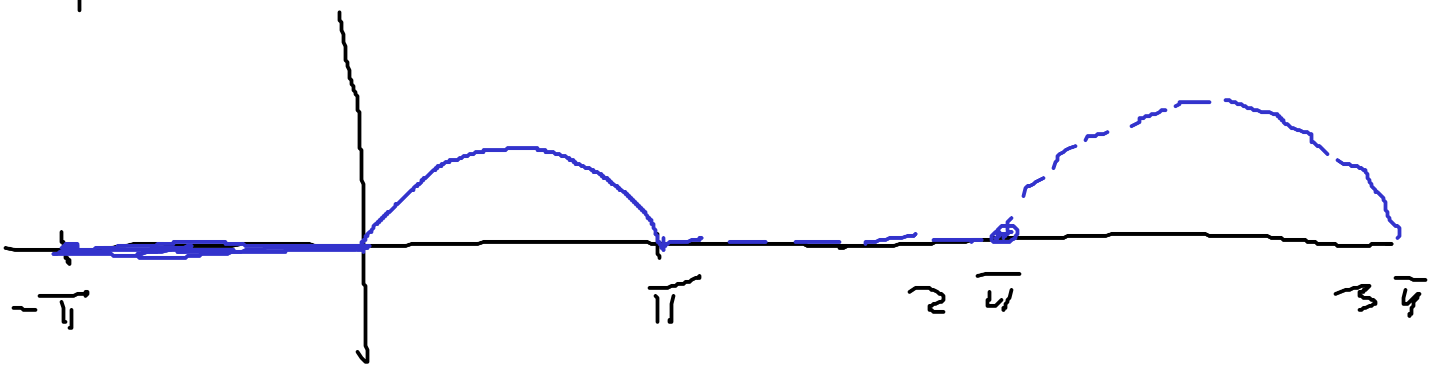
$a_n = 0 \quad \forall n \geq 0$

$b_1 = -\frac{1}{2} \qquad b_i = 0,\ i = 2, 3, 4, 6, \ldots$

$b_5 = \frac{1}{2}$

(ii) $g(x) = \begin{cases} 0 & x \in [-\pi, 0) \\ \sin x & x \in [0, \pi) \end{cases}$

periodické prodloužení



$T = 2\pi$

$\omega = \frac{T}{2\pi} = 1$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(nx)\, dx =$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin x \cos(nx)\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \frac{1}{2} \left(\sin(1+n)x + \sin(1-n)x \right) dx$$

$$\sin x \cos y = \frac{1}{2} \left(\underbrace{\sin(x+y)}_{\sin x \cos y + \cos x \sin y} + \underbrace{\sin(x-y)}_{\sin x \cos x - \cos x \sin y} \right)$$

$$= \frac{1}{2\pi} \int_0^{\pi} \sin(1+n)x \, dx + \frac{1}{2\pi} \int_0^{\pi} \sin(1-n)x \, dx$$

$$= \frac{1}{2\pi} \left[-\frac{1}{1+n} \cos(1+n)x \right]_0^{\pi}$$

$n \neq 1$

$$+ \frac{1}{2\pi} \left[-\frac{1}{1-n} \cos(1-n)x \right]_0^{\pi}$$

$$= \frac{1}{2\pi} \left(-\frac{1}{1+n} \right) \left[\cos(1+n)\pi - 1 \right]$$

$$+ \frac{1}{2\pi} \left(-\frac{1}{1-n} \right) \left[\cos(1-n)\pi - 1 \right]$$

n sudé $\Rightarrow$ $$a_n = -\frac{1}{2\pi} \left[\left(\frac{-1}{1+n} - 1 \right) + \left(\frac{-1}{1-n} - 1 \right) \right]$$

$$= \frac{1}{2\pi} \left[\frac{1+1+n}{1+n} + \frac{1+(1-n)}{1-n} \right]$$

$$= \frac{1}{2\pi} \left(\frac{n+2}{n+1} - \frac{n-2}{n-1} \right)$$

$$= \frac{1}{2\pi} \frac{(n+2)(n-1) - (n-2)(n+1)}{n^2-1}$$

$$= \frac{1}{2\pi} \frac{(n^2+n-2)-(n^2-n-2)}{n^2-1}$$

n sudé $\Rightarrow$

$$a_n = -\frac{1}{2\pi}\frac{1}{n+1}(-1-1)$$

$$-\frac{1}{2\pi}\frac{1}{1-n}(-1-1) =$$

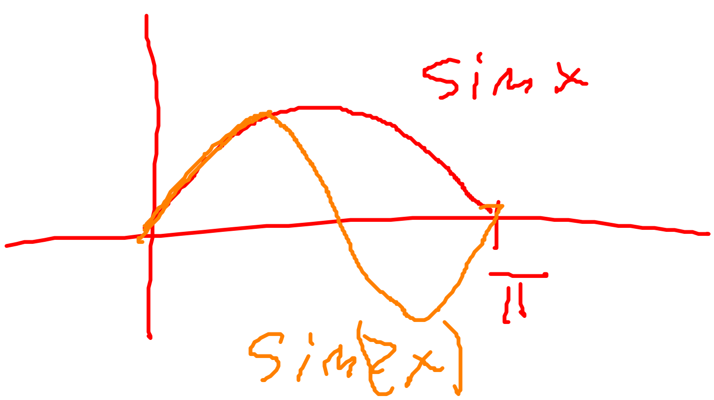
$$= \frac{1}{\pi}\left(\frac{1}{1+n}+\frac{1}{1-n}\right) = \frac{1}{\pi}\frac{(1-n)+(1+n)}{(1+n)(1-n)}$$

$$= \frac{2}{\pi(1-n^2)}$$

$$\boxed{a_n = \frac{2}{\pi(1-n^2)} \quad n \text{ sudé}}$$

$$a_0 = \frac{2}{\pi}$$

$$\boxed{n \text{ liché} \quad a_n = 0}$$



$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin(nx)\,dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \sin x \sin(nx)\,dx$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \frac{1}{2}\left(\cos(1-n)x - \cos(1+n)x\right)dx$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

$$\cos(x-y) - \cos(x+y) = 2 \sin x \sin y$$

$$= \frac{1}{2\pi} \int_{0}^{\pi} \cos(1-n)x\,dx - \frac{1}{2\pi} \int_{0}^{\pi} \cos(1+n)x\,dx$$

$$= \frac{1}{2\pi}\left(\left[\frac{1}{1-n} \sin(1-n)x\right]_0^{\pi} - \left[\frac{1}{1+n} \sin(1+n)x\right]_0^{\pi} \right)$$

$$= \frac{1}{2\pi}\left(\frac{\sin(1-n)\pi}{1-n} - \frac{\sin(1+n)\pi}{1+n} \right) = 0$$

$\forall n \geq 2$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2}\,(1-\cos 2x)\,dx$$

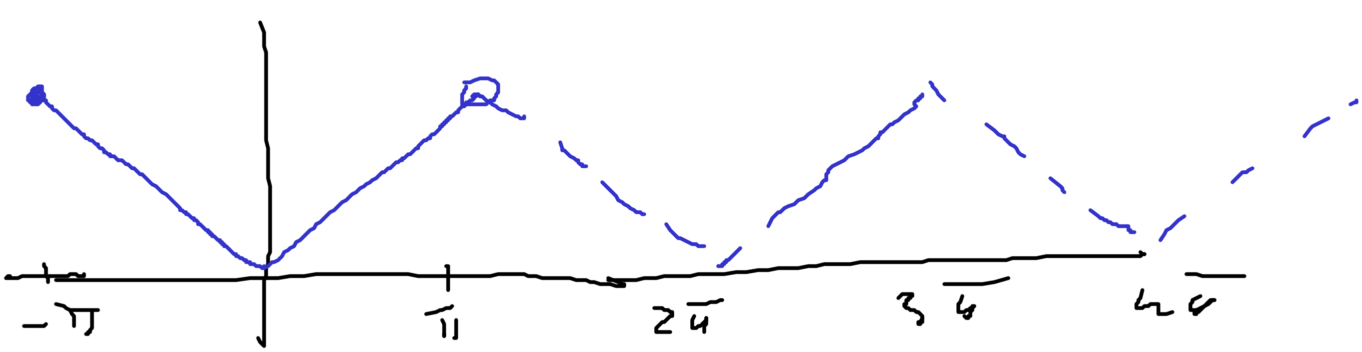
$$= \frac{1}{2\pi}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{1}{2}$$

Závěr : $F(x) = \dfrac{a_0}{2} + b_1 \sin x + \displaystyle\sum_{n=1}^{\infty} a_{2n} \cos(2nx)$

$$\boxed{F(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=1}^{\infty} \frac{2}{\pi(1-4n^2)} \cos(2nx)}$$

(iii) $g(x) = |x| \qquad x \in [-\pi, \pi)$

+ periodické prodloužení



$g(x)$ sudá $\Rightarrow b_n = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\underbrace{\cos(nx)}_{\text{sudá funkce}}dx$$

$$= 2\cdot\frac{1}{\pi}\int_0^{\pi} g(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx =$$

$$= \frac{2}{\pi}\left[\frac{1}{n}x\sin(nx)\right]_0^{\pi}$$

$u = x \qquad v' = \cos(nx)$

$u' = 1 \qquad v = \frac{1}{n}\sin(nx)$

$n \neq 0$

$$- \frac{2}{\pi}\int_0^{\pi}\frac{1}{n}\sin(nx)\,dx =$$

$$= -\frac{2}{\pi n}\int_0^{\pi}\sin(nx)\,dx =$$

$$= -\frac{2}{\pi n}\left[-\frac{1}{n}\cos(nx)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left(\cos(n\pi) - 1\right) = \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{1}{2}x^2\right]_0^{\pi} = \pi$$

Závěr

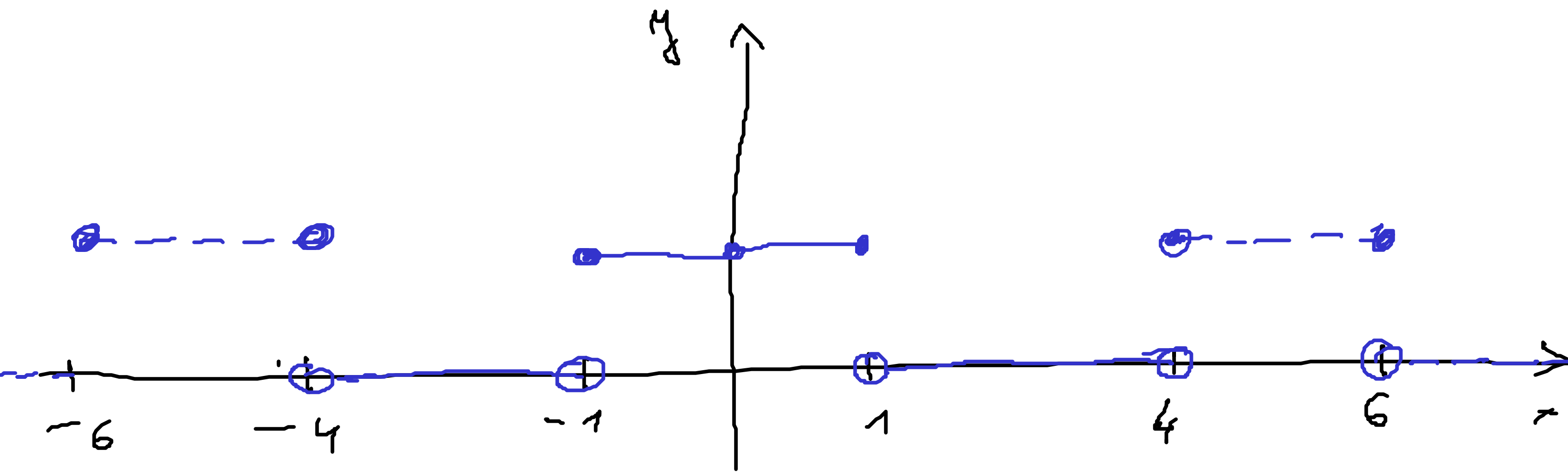
$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$F(x) = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\left((-1)^n - 1\right)\cos(nx)$$

11.2 Určete kosinovou F.ř. pro periodické prodloužení funkce

$$g(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \in (1,4) \end{cases}$$

$\Rightarrow$ chceme sudé periodické prodloužení



periodo $T = 8$

$$\omega = \frac{2\pi}{T} = \frac{\pi}{4}$$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega x) + b_n \sin(n\omega x)$$

$$a_n = \frac{2}{T}\int_{-4}^{4} \underbrace{g(x)} \cos(n\omega x)\, dx =$$

po periodickém $\overbrace{\text{prodloužení}}^{\text{sudá}}$

$$= \frac{4}{T}\int_{0}^{4} g(x)\cos(n\omega x)\, dx$$

$$= \frac{4}{T}\int_{0}^{1} \cos\left(n\cdot\frac{\pi}{4}x\right) dx$$

$\underline{\underline{n \neq 0}}$ $\frac{4}{8}\left[\frac{4}{\pi n} \cdot \sin\left(\frac{n\pi}{4}x\right)\right]_0^1$

$$= \frac{2}{\pi n}\left(\sin\frac{n\pi}{4} - 0\right)$$

$$a_0 = \frac{4}{1}\int_0^1 \cos(0x)\,dx$$

$$= \frac{4}{8}\int_0^1 dx = \frac{4}{8}\cdot 1 = \frac{1}{2}$$

$\underline{\text{Zdvěr}}$

$$F(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \underbrace{\frac{2}{\pi n}\cdot \sin\left(\frac{n\pi}{4}\right)}_{a_n} \cos\frac{n\pi x}{4}$$