11. Hall effect and magnetoresistivity

The motion of free carriers in the external electric field can be influenced rather significantly by an applied magnetic field.

Hall effect in three-dimensional gas of free carriers

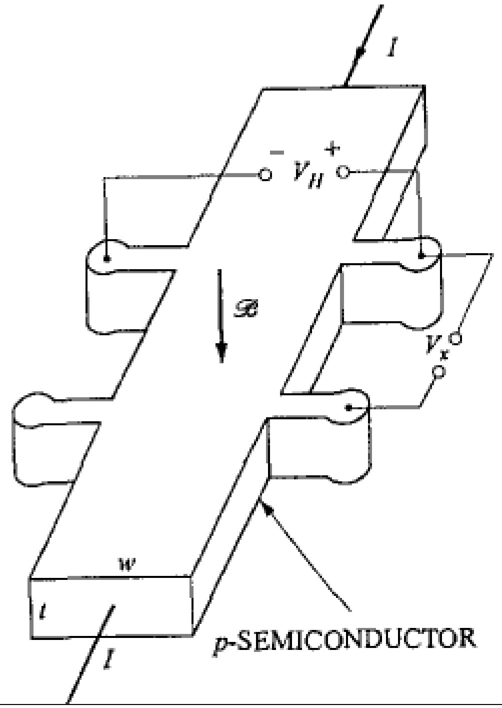
In the approximation of relaxation time, the quasiclassical equation motion with the electric filed intensity E and magnetic induction B for the particle carrying the charge e is

$$m^* \frac{d^2\vec{r}}{dt^2} + \frac{m^*}{\tau}\frac{d\vec{r}}{dt} = e\vec{E} + e\frac{d\vec{r}}{dt}\times\vec{B} \; , \tag{11.1}$$

where τ is the mean time interval between collisions restoring the zero value of the drift velocity. The last term in Eq. (11.1) is the Lorentz force. Let us assume the orientation of the magnetic field to be along the z axis; the equation of motion along x and y reads

$$\frac{m^*}{e}\left(\frac{dv_x}{dt} + \frac{v_x}{\tau}\right) = E_x + v_y B_z \; , \quad \frac{m^*}{e}\left(\frac{dv_y}{dt} + \frac{v_y}{\tau}\right) = E_y - v_x B_z \; . \tag{11.2}$$

We assume the external electric field aligned along x (this is "longitudinal direction"), the electric field along y is due to the Lorentz force (acting in the "transverse direction").



A suitable shape of samples for the measurements of conductivity and Hall voltage. The polarity of the latter is shown for the p-type semiconductor; it is reversed for n-type sample.

The stationary state means the zero acceleration of carriers; in Eqs. (11.2), the v_x component from the first one can be use in the second, leading to the following equation for v_y:

$$v_y\left(\frac{m^*}{\tau}+\frac{e^2B_z^2\tau}{m^*}\right)=eE_y-\frac{e^2B_z\tau}{m^*}E_x \ . \tag{11.3}$$

Recalling the convenient quantity of the cyclotron frequency,

$$\omega_c = \frac{eB_z}{m^*} \ , \tag{11.4}$$

the equation of motion (11.3) assumes the following form,

$$v_y \left(1 + \omega_c^2 \tau^2\right) = \frac{e\tau}{m^*} \left(E_y - \omega_c \tau E_x\right) \ . \tag{11.5}$$

Measurements of the Hall voltage are typically performed at weak magnetic fields; the square of the product of the cyclotron frequency and relaxation time is negligible, and Eq. (11.5) simplifies to

$$v_y = \frac{e\tau}{m^*} \left(E_y - \frac{eB_z \tau}{m^*} E_x \right) \ . \tag{11.6}$$

The current density *j* results from the multiplication of the carrier velocity by the charge and concentration. Let us assume the presence of electrons with the density n_e and effective mass m_e , and holes with the density n_h and effective mass m_h. The total current density along *y* is given by the sum of the two contributions, and is zero in the stationary state (after vanishing of transient phenomena):

$$j_y = j_{ye} + j_{yh} = \frac{n_e e^2 \tau_e}{m_e} \left(E_y + \frac{eB_z \tau_e}{m_e} E_x \right) + \frac{n_h e^2 \tau_h}{m_h} \left(E_y - \frac{eB_z \tau_h}{m_h} E_x \right) = 0 \ . \tag{11.7}$$

This condition can be written down in the following form, which uses the symbol μ for the mobility of electrons and holes,

$$E_y\left(n_e\mu_e+n_h\mu_h\right)=E_x\left(n_h\mu_h^2-n_e\mu_e^2\right)B_z\ ,$$
$$\mu_e=\frac{e\tau_e}{m_e}\ ,\quad \mu_h=\frac{e\tau_h}{m_h}\ . \tag{11.8}$$

The electric field intensity along x con be expressed in terms of the current density j_x and conductivity σ as

$$E_x=\frac{j_x}{\sigma}=\frac{j_x}{e\left(n_e\mu_e+n_h\mu_h\right)}\ . \tag{11.9}$$

Consequently, Eq. (11.8) leads to the proportionality of the transverse electric field E_y to the longitudinal current density j_x and magnetic induction B_z, expressed with the help of the *Hall coefficient*, R_H:

$$E_y=R_H\,j_xB_z=\frac{n_h\mu_h^2-n_e\mu_e^2}{e\left(n_e\mu_e+n_h\mu_h\right)^2}\,j_xB_z\ . \tag{11.10}$$

The (SI) units of the previous equations are:

$$\frac{\mathrm{V}}{\mathrm{m}}=\frac{\mathrm{m}^3}{\mathrm{C}}\,\frac{\mathrm{A}}{\mathrm{m}^2}\,\mathrm{T}=\frac{\mathrm{m}^3}{\mathrm{A\cdot s}}\,\frac{\mathrm{A}}{\mathrm{m}^2}\,\frac{\mathrm{Wb}}{\mathrm{m}^2}=\frac{\mathrm{m}^3}{\mathrm{A\cdot s}}\,\frac{\mathrm{A}}{\mathrm{m}^2}\,\frac{\mathrm{V\cdot s}}{\mathrm{m}^2}\ .$$

The Hall coefficient in the material containing a single type of the carriers of the concentration n is

$$R_H=\pm\frac{1}{en}\ . \tag{11.11}$$

Due to the thermal (chaotic) movement, the charge carriers have different energies, and the scattering processes are energy dependent. The previous development can be modified by finding the mean (expectation) values of relevant functions of the relaxation time. Equation (11.8) will be replaced by

$$\begin{aligned} &E_y\left(n_e\left\langle\mu_e\right\rangle+n_h\left\langle\mu_h\right\rangle\right)=E_x\left(n_h\left\langle\mu_h^2\right\rangle-n_e\left\langle\mu_e^2\right\rangle\right)B_z\ , \\ &\left\langle\mu_e\right\rangle=\frac{e\left\langle\tau_e\right\rangle}{m_e}=\frac{e}{m_e}\int_{-\infty}^{\infty}\tau_e(E)f(E)dE\ ,\ \left\langle\mu_h\right\rangle=\frac{e\left\langle\tau_h\right\rangle}{m_h}=\frac{e}{m_h}\int_{-\infty}^{\infty}\tau_h(E)f(E)dE\ , \\ &\left\langle\mu_e^2\right\rangle=\frac{e^2\left\langle\tau_e^2\right\rangle}{m_e^2}=\frac{e^2}{m_e^2}\int_{-\infty}^{\infty}\tau_e^2(E)f(E)dE\ ,\ \left\langle\mu_h^2\right\rangle=\frac{e^2\left\langle\tau_h^2\right\rangle}{m_h^2}=\frac{e^2}{m_h^2}\int_{-\infty}^{\infty}\tau_h^2(E)f(E)dE\ , \end{aligned} \tag{11.12}$$

where *f(E)* is the probability density of the occupation of the state having the energy *E*. Consequently, the mean values of the relaxation time and its square enter Eq. (11.10). In the material containing a single type of carriers with the concentration *n*, the Hall coefficient of Eq. (11.11) is replaced by

$$R_H=\pm\frac{\left\langle\tau^2\right\rangle}{en\left\langle\tau\right\rangle^2}=\pm\frac{r_H}{en}\ . \tag{11.13}$$

The symbol r_H denotes the *Hall factor*:

$$r_H=\frac{\left\langle\tau^2\right\rangle}{\left\langle\tau\right\rangle^2}\ . \tag{11.14}$$

Whenever its value is close to unity, the mobility becomes the product of the Hall coefficient and conductivity,

$$\mu=\left|R_H\right|\sigma\ , \tag{11.15}$$

and a simultaneous measurement of the Hall voltage and conductivity determines the carrier density and their relaxation time. As a rule, the energy dependence of the relaxation time cannot be neglected; the product of Eq. (11.15) differs from the mobility, and is called *Hall mobility*:

$$\mu_H = r_H \mu \ . \tag{11.16}$$

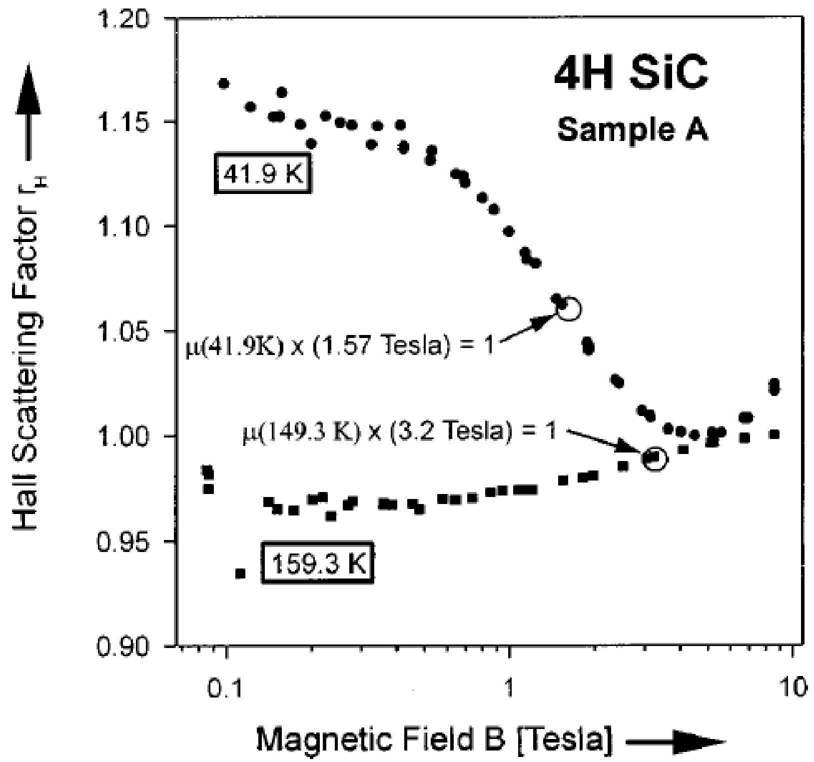


FIG. 1. Hall scattering factor r_H as a function of magnetic field in Tesla in 4H SiC epitaxial sample A. $n(300\text{ K}) = 5.9\times10^{15}\text{ cm}^{-3}$. At 41.9 K, the Hall mobility is 6400 cm^2/V s and at 159.3 K it is 3100 cm^2/V s.

Experimental results for the Hall factor,

from Rutsch et al., JAP 1998

Magnetoresistance

In the case of the same concentration of both electrons and holes, and the zero total transverse current, the nonzero contributions the electrons and holes to the transverse current leads to their movement caused by the magnetic field in the opposite directions along the longitudinal direction *x*. This leads to the magnetoresistance, i.e., the dependence of the longitudinal resistivity on the magnetic field. Within the model used above, the first of the equation of motion (11.2) results in the following relations for the velocity along *x* (in the stationary state, with vanishing acceleration),

$$v_x = \frac{\tau e}{m^*}\left[E_x + \frac{\tau e}{m^*}\left(E_y - v_x B_z\right)B_z\right] = \mu\left[E_x + \mu\left(E_y - v_x B_z\right)B_z\right] \ ,$$
$$v_x = \frac{\mu E_x + \mu^2 E_y B_z}{1+\mu^2 B_z^2} \ . \tag{11.17}$$

The longitudinal current density is composed of the electrons and holes,

$$j_x = j_{xe} + j_{xh} = n_e e v_{xe} + n_h e v_{xh} \ . \tag{11.18}$$

For a small transverse electric field (small Hall voltage), Eqs. (11.12) and (11.13) lead to the linear dependence of the longitudinal current on E_x:

$$j_x \approx e\left(\frac{\mu_e n_e}{1+\mu_e^2 B_z^2}+\frac{\mu_h n_h}{1+\mu_h^2 B_z^2}\right)E_x = \sigma_x(B_z)E_x \ , \tag{11.19}$$

with the conductivity dependent on the magnetic induction. In the two obvious limits of a weak and strong magnetic field, we arrive at

$$\begin{aligned} &\sigma_x(B_z) \approx \sigma_x(0) - e\left(n_e\mu_e^3 + n_h\mu_h^3\right)B_z^2 \ , \ \text{for } \mu_h^2 B_z^2 \ll 1 \ , \\ &\sigma_x(B_z) \approx \left(\frac{n_e}{\mu_e}+\frac{n_h}{\mu_h}\right)\frac{e}{B_z^2} \ , \ \text{for } \mu_h^2 B_z^2 \gg 1 \ . \end{aligned} \tag{11.20}$$

In the second case, the resistance is proportional to the square of B_z.

Quantum Hall effect in two-dimensional free carrier gas

A gas of free carriers can be localized to a very thin layer in the (*x*,*y*) plane, preferably within a heterostructure; this localization reduces the number of degrees of freedom by one. In an external magnetic field along *z*, the energies are quantized in the scheme of Landau levels,

$$E(l,B_z) = \left(l+\frac{1}{2}\right)\hbar\omega_c \pm \frac{1}{2}g^*\mu_B B_z = \left(l+\frac{1}{2}\right)\hbar\frac{eB_z}{m^*} \pm \frac{1}{2}g^*\mu_B B_z \ , \tag{11.21}$$

i.e., assuming discrete values separated by the quanta related to the cyclotron frequency. Each of the landau level is split into a spin doublet, with the energy given by the Bohr magneton μ_B and effective *g*-factor. This change of the electronic structure changes substantially both longitudinal and transverse currents.

The two-dimensional current densities along *x* and *y* in the limit of weak fields are

$$J_x = \sigma_{xx}E_x + \sigma_{xy}E_y \ , \ J_y = -\sigma_{xy}E_x + \sigma_{xx}E_y \ , \tag{11.22}$$

since

$$\sigma_{xx} = \sigma_{yy} \ , \ \sigma_{yx} = -\sigma_{xy} \tag{11.23}$$

hold for isotropic materials. Denoting the width of the 2D "channel" by *w*, the longitudinal current density and the transverse electric field intensity are

$$J_x = \frac{I_x}{w} \ , \ E_y = \frac{V_y}{w} \ , \tag{11.24}$$

where I_x and V_y are the longitudinal current and transverse voltage, respectively. The two-dimensional current density is the current per unit width.

In the stationary state, the transverse current along y vanishes; the resistivity (which is the inverse conductivity) results from Eq. (11.22) as

$$\rho_{xx} \equiv \frac{E_x}{J_x} = \frac{\sigma_{xx}}{\sigma_{xx}^2 + \sigma_{xy}^2} \ , \ \rho_{xy} \equiv \frac{E_y}{J_x} = \frac{\sigma_{xy}}{\sigma_{xx}^2 + \sigma_{xy}^2} \ . \tag{11.25}$$

The unit of diagonal and off-diagonal conductivities σ is 1/Ω, that of the resistivity ρ is Ω. They are alternatively called 1/ Ω and Ω per square, as the same values result for sample of the square form of arbitrary dimensions. With the channel length and width of l and w, respectively, the resistance along x is the product of the resistivity ρ_{xx} and the dimensionless ratio l/w.

Using the approximation of the constant relaxation time τ, Eq. (11.5) leads to the following condition of vanishing transverse velocity of the carriers:

$$v_y \propto E_y - \frac{eB_z\tau}{m^*}E_x = 0 \quad \rightarrow \quad \frac{E_y}{E_x} = \omega_c\tau \ . \tag{11.26}$$

The longitudinal current density is the product of charge, area density, area density n_s , and velocity v_x; the first of Eqs. (11.2) leads to the stationary velocity

$$v_x = \frac{e\tau}{m^*}E_x = \frac{e\tau}{m^*}\frac{E_y}{\omega_c\tau} = \frac{E_y}{B_z} \ . \tag{11.27}$$

Consequently,

$$J_x = -en_s v_x = -en_s\frac{E_y}{B_z} \ , \tag{11.28}$$

$$E_y = -\frac{1}{en_s} J_x B_z = R_H J_x B_z \ , \tag{11.29}$$

$$R_H = -\frac{1}{en_s} \ . \tag{11.30}$$

The Hall coefficient is determined by the area density of electrons. Equation (11.24) leads to

$$R_H = \frac{V_y}{I_x B_z} \ . \tag{11.31}$$

In a strong magnetic field, the product of cyclotron frequency and relaxation time is much larger than unity; consequently, Eq. (11.26) implies much larger transverse electric field intensity than its longitudinal counterpart. Following Eq. (11.25), the modulus of the off-diagonal element of the conductivity tensor is much larger than that of the diagonal element. The consequence for the resistivities of Eq. (11.25) is the following approximate relations,

$$\rho_{xx} \approx \frac{\sigma_{xx}}{\sigma_{xy}^2} \ , \ \rho_{xy} \approx \frac{1}{\sigma_{xy}} = \frac{V_y}{I_x} = R_H B_z \ . \tag{11.32}$$

Both conductivity and resistivity in the magnetic field change dramatically due to the occupation of the discrete Landau levels The area density of electrons in the state of all levels up to the ν-th occupied, and all higher unoccupied, is

$$n_s(\nu) = \nu \frac{eB_z}{h} \ , \tag{11.33}$$

and the Hall resistivity is quantized,

$$\rho_{xy} = \frac{V_y}{I_x} = \frac{1}{\nu}\frac{h}{e^2} = \frac{25813}{\nu}\ \Omega\ ,\ \nu = 1,2,... \tag{11.34}$$

This quantization lead to the use of the term “QHE” (Quantum Hall Effect“). In addition to the Hall voltage, remarkable changes occur also in the longitudinal resistivity; the explanation is based on the occurrence of “conductive” state of the electron gas with partly occupied Landau levels, and “insulating” state in the case of the Frmi level placed between a full and empty level.

The relative accuracy of the Hall resistance measurements in the QHE is ~10^{-8}, it has been adopted as the standard of electrical resistivity. At the same time, it provides very precise value of the fine structure constant,

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137.035963(15)}\ , \tag{11.35}$$

which contains the elementary charge and three fixed (defined) constants.

New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance

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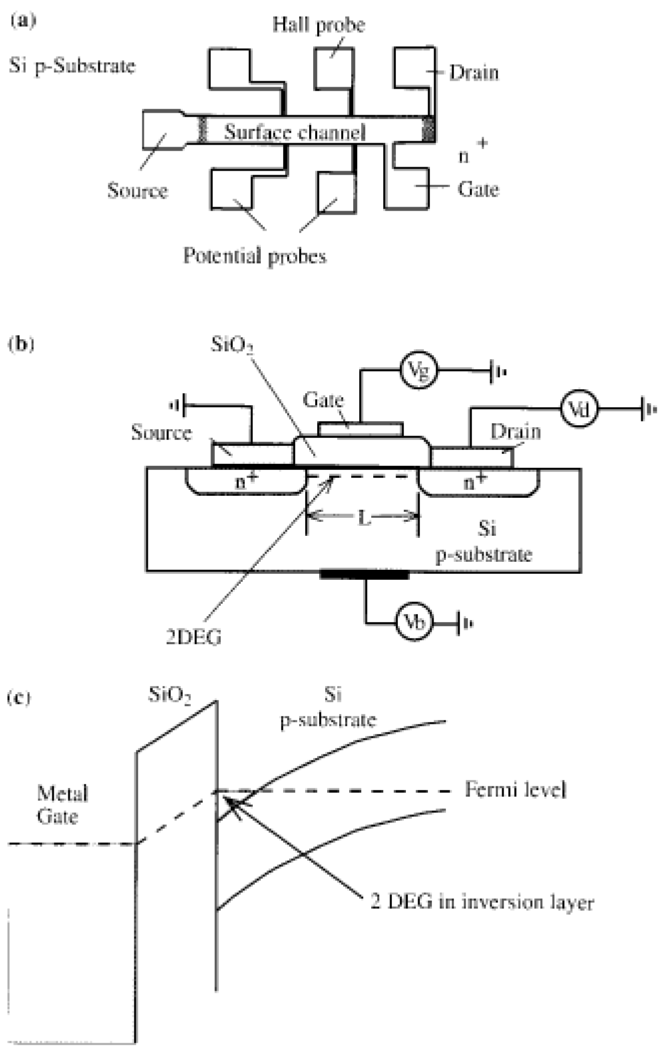
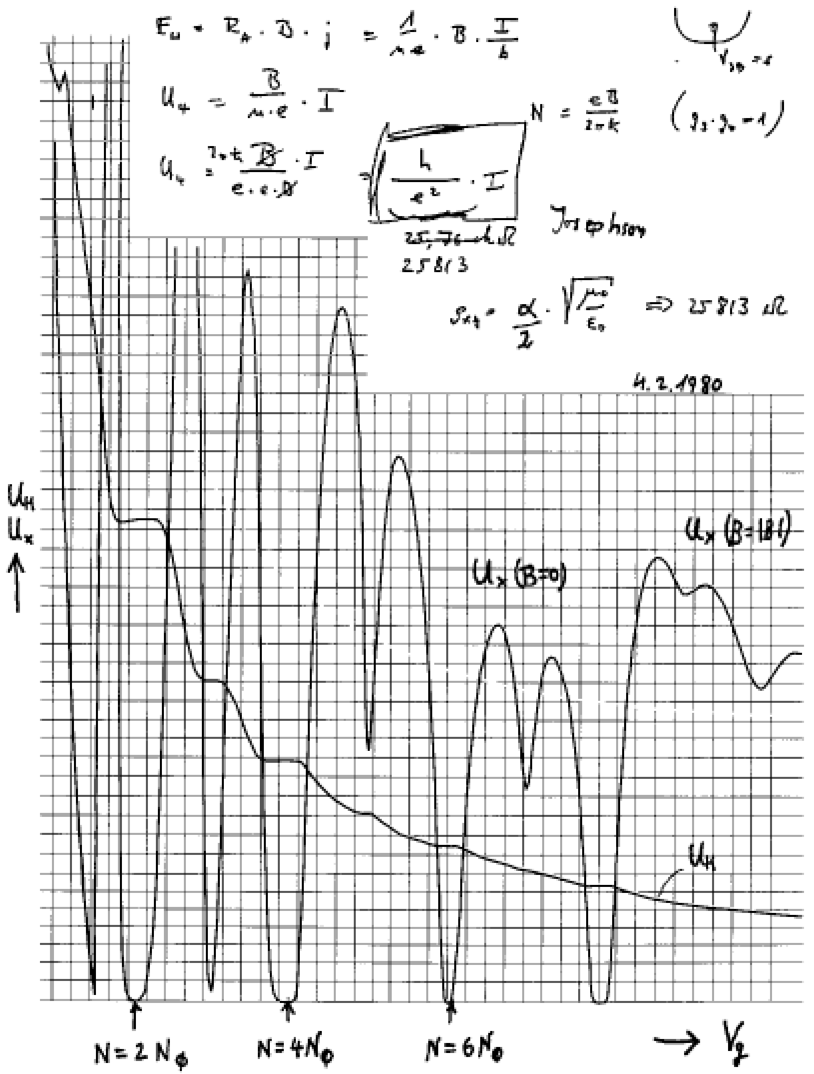
(Received 30 May 1980)

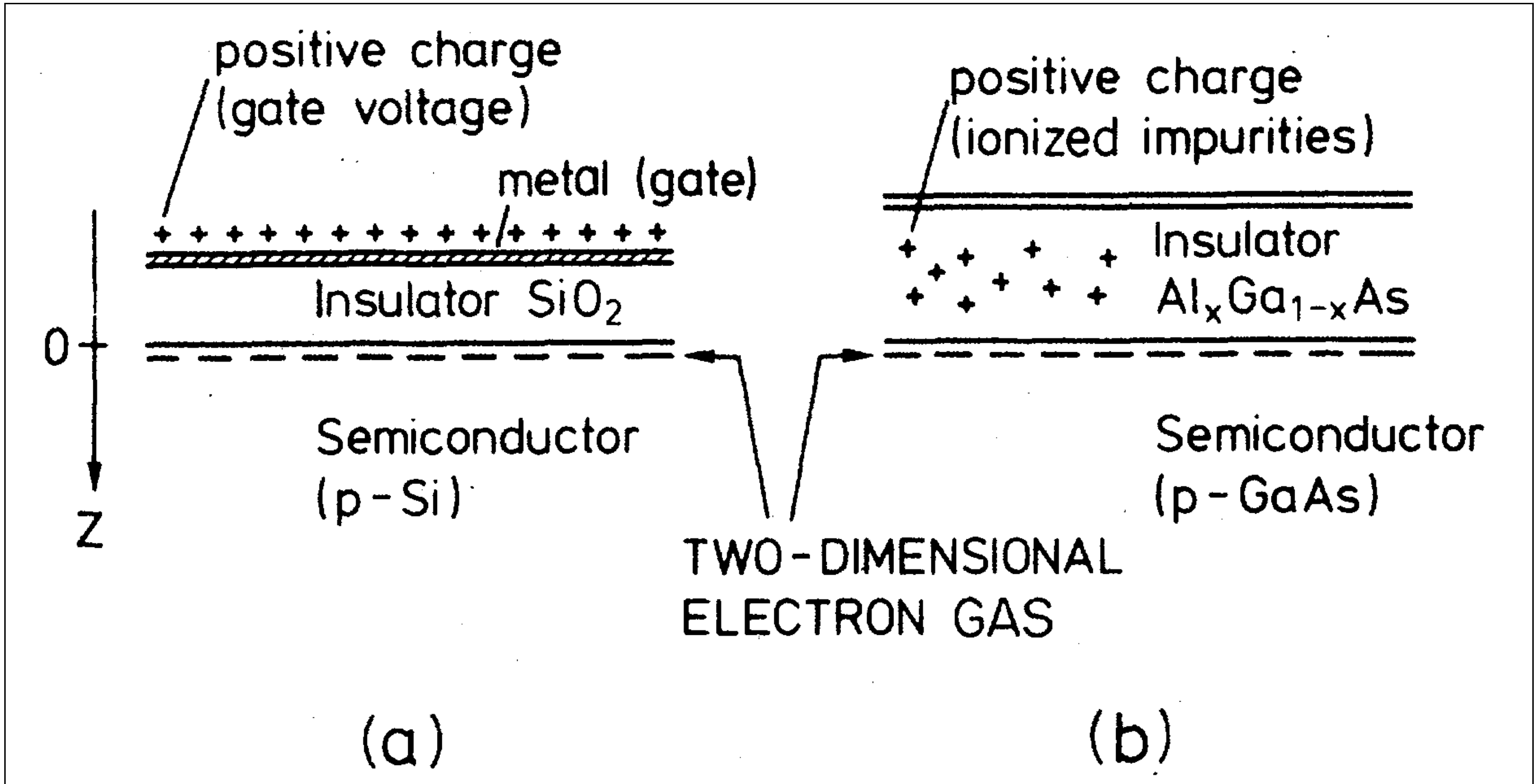
Measurements of the Hall voltage of a two-dimensional electron gas, realized with a silicon metal-oxide-semiconductor field-effect transistor, show that the Hall resistance at particular, experimentally well-defined surface carrier concentrations has fixed values which depend only on the fine-structure constant and speed of light, and is insensitive to the geometry of the device. Preliminary data are reported.



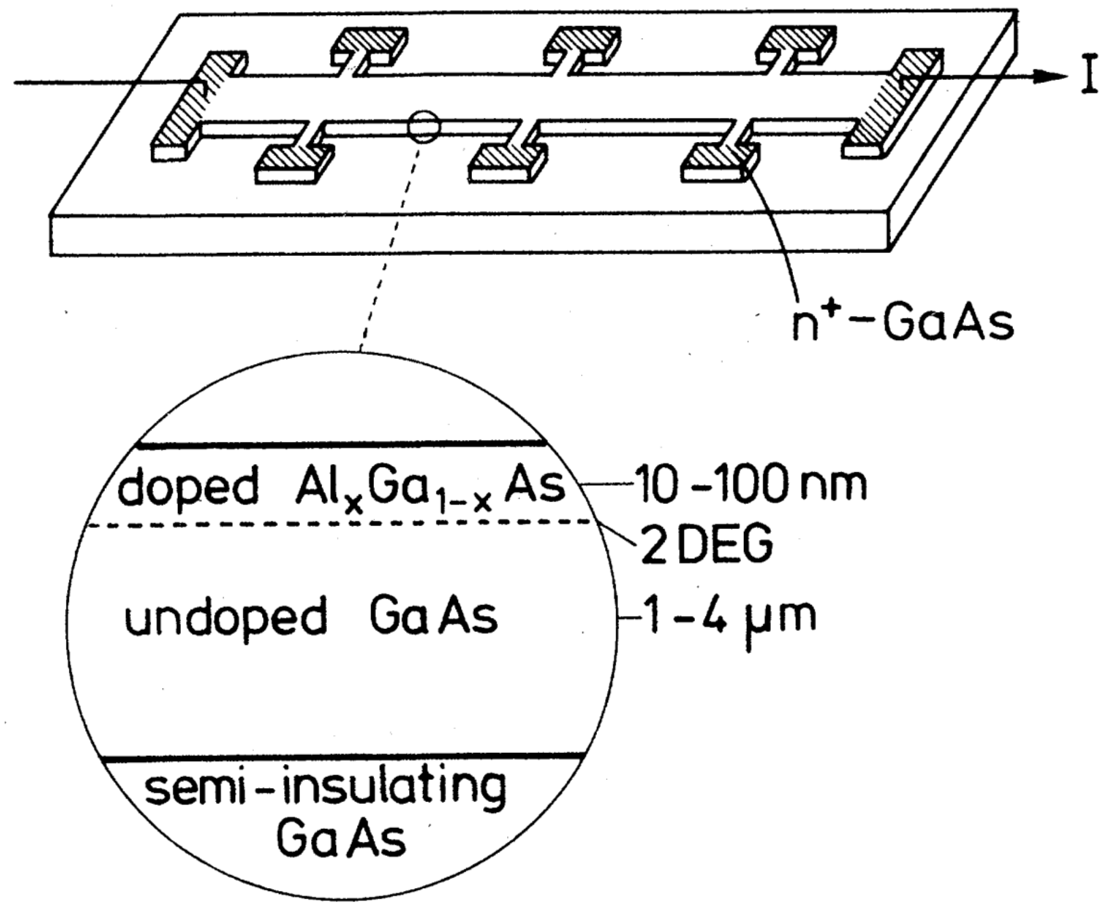
Setup of the experiment by Klitzing, Dorda and Pepper, leading to the discovery of quantum Hall effect in 1980 (Nobel prize – von Klitzing 1985).



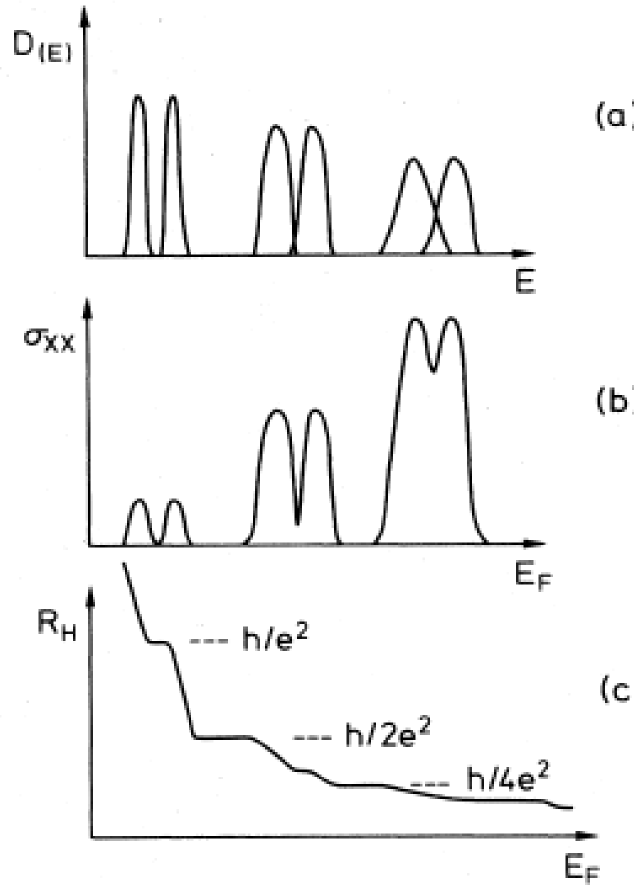
Original plot and notes, identifyinfg the origin of QHE (K. von Klitzing).



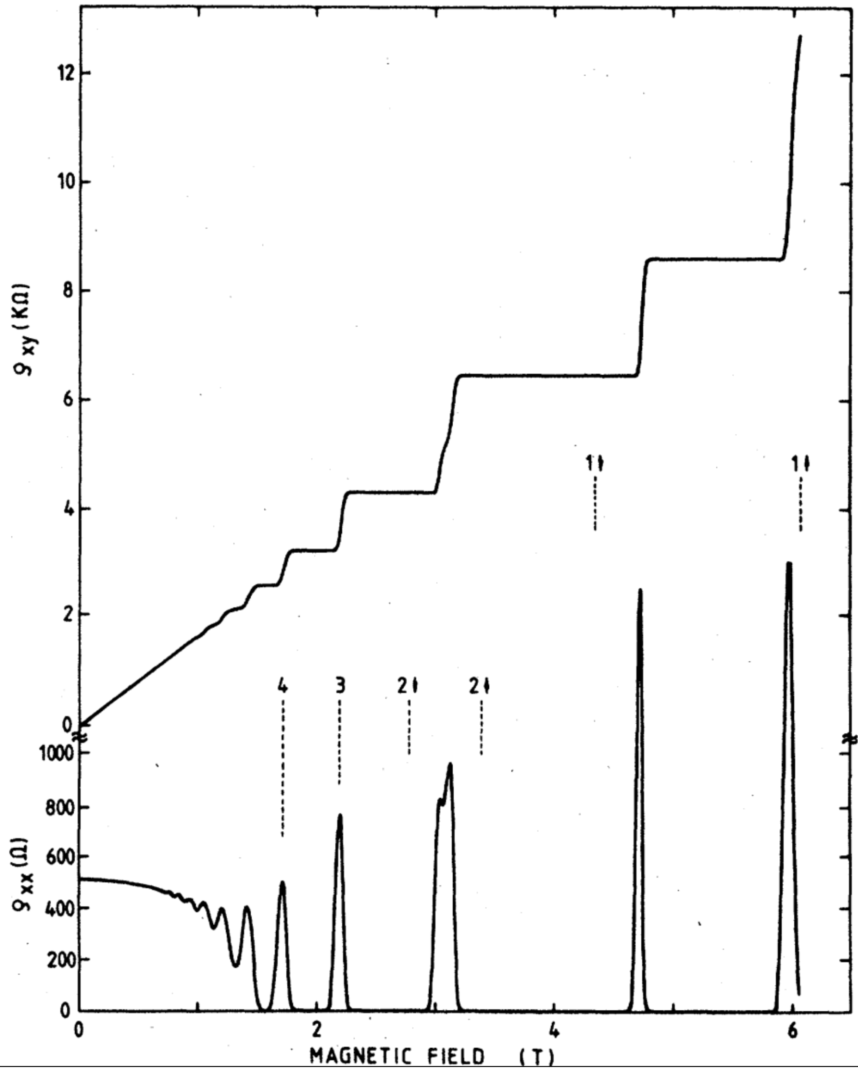
2D electron gas in a silicon MOS structure (a), and in a GaAs/AlGaAs heterostructure (b).



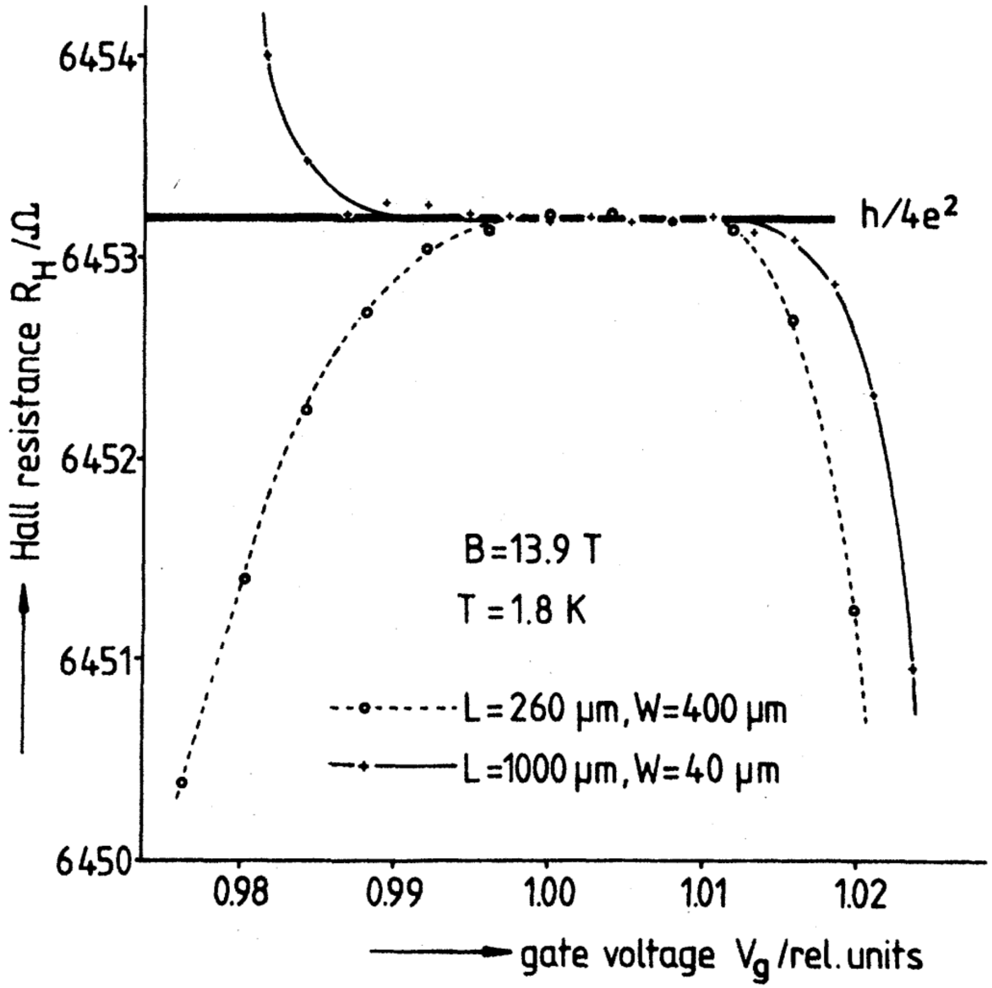
Heterostructure for QHE measurements (Klitzing, RMP 1986).



Density of states, lateral conductivity and Hall coefficient (Klitzing, RMP 1986).



Hall and lateral resistivity in a GaAs-AlGaAs heterostructure at the temperature of 8 mK (Klitzing, RMP 1986).



Hall resistivity on expanded scales, at the temperature of 1.8 K and magnetic field 13.9 T, for two different samples (Klitzing, RMP 1986).