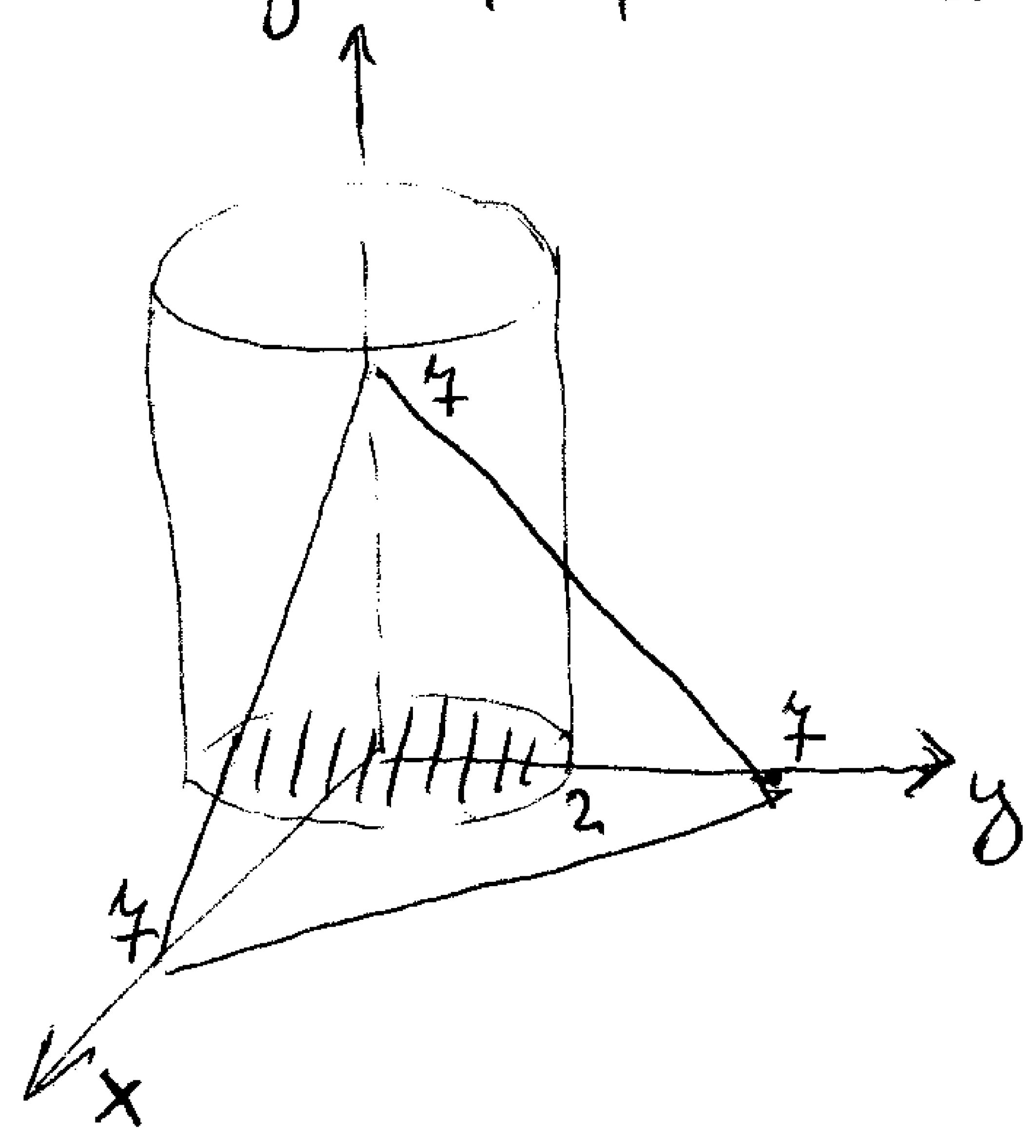
1) Ve válcových souřadnicích popište tělesa V

a) $x^2+y^2 \le 4$, $0 \le z \le 7-x-y$

Válec seříznutý rovinou



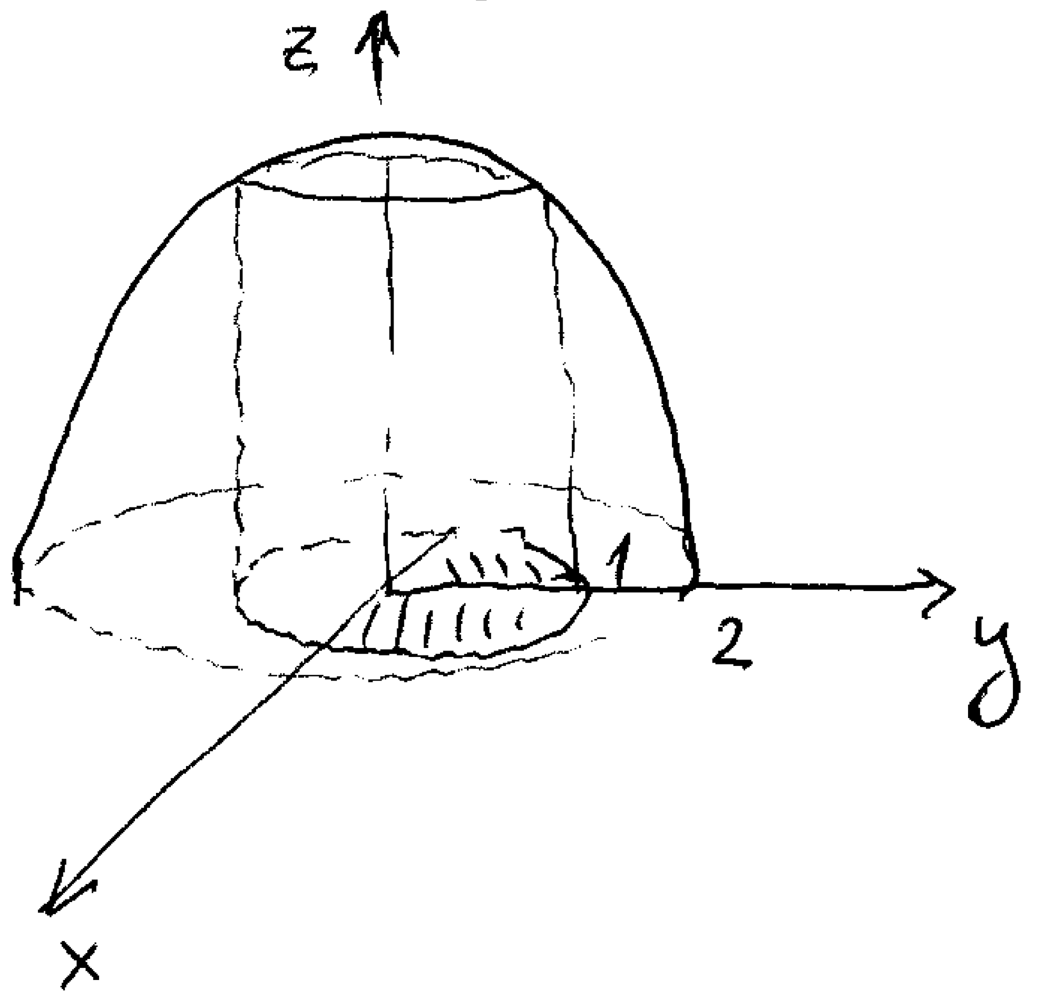
$0 \le \varrho \le 2$

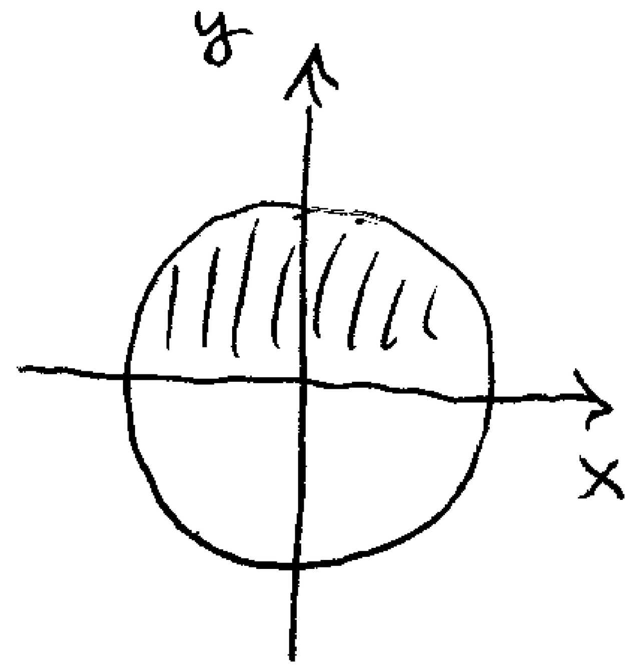
$0 \le \varphi \le 2\pi$

$0 \le z \le 7-\varrho\cos\varphi-\varrho\sin\varphi$

b) $x^2+y^2 \le 1$, $y \ge 0$, $z \ge 0$, $z \le 4-x^2-y^2$

polovina válce seříznutá paraboloidem



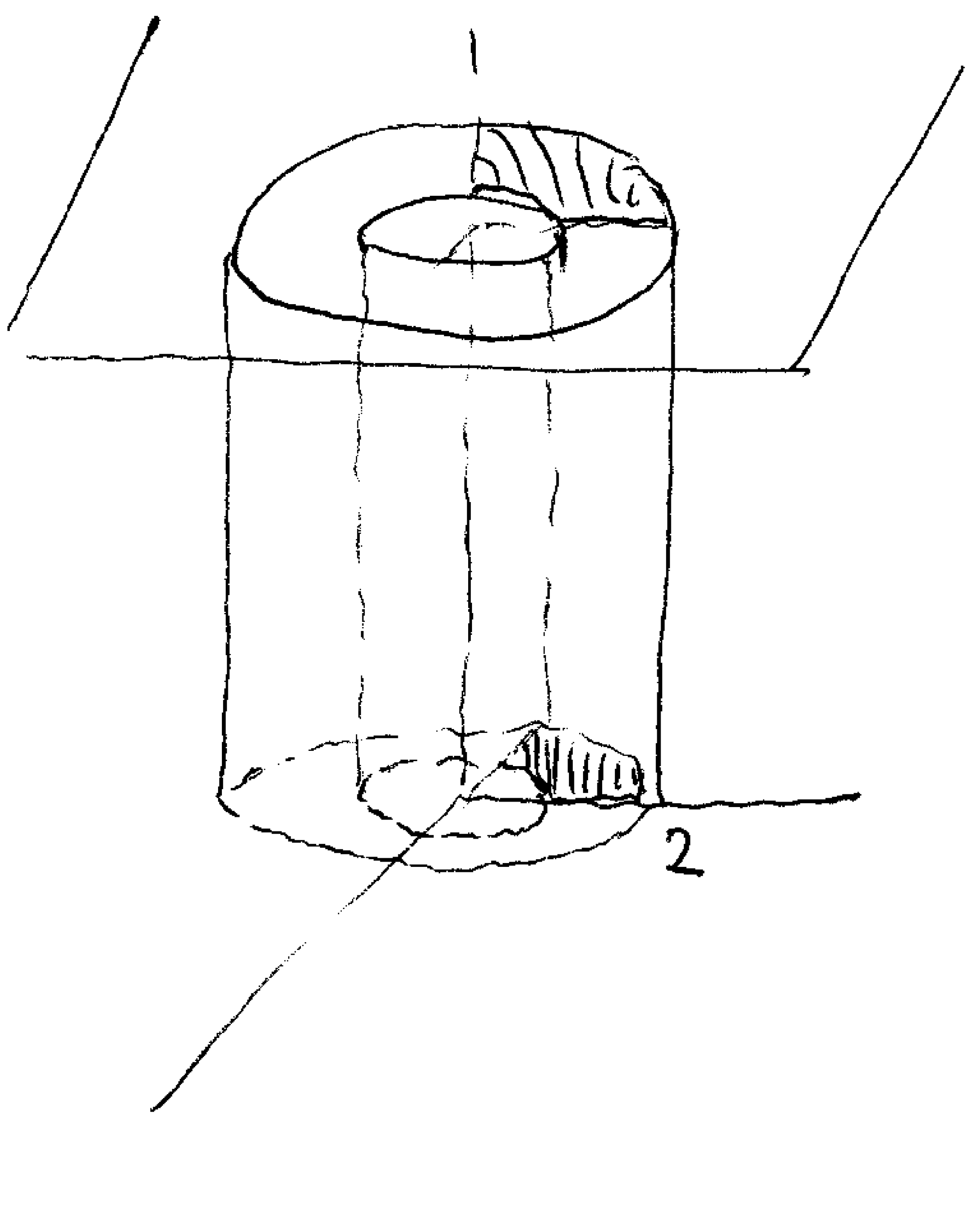


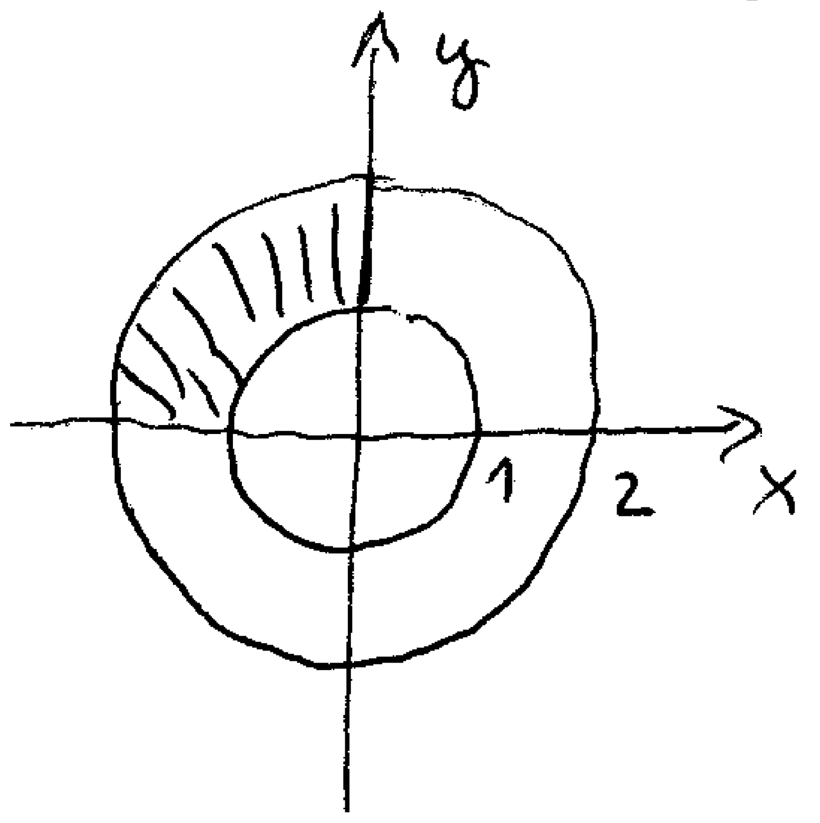
$0 \le \varrho \le 1$

$0 \le \varphi \le \pi$

$0 \le z \le 4-\varrho^2$

c) $1 \le x^2+y^2 \le 4$, $0 \le z \le 3$, $x \le 0$, $y \ge 0$



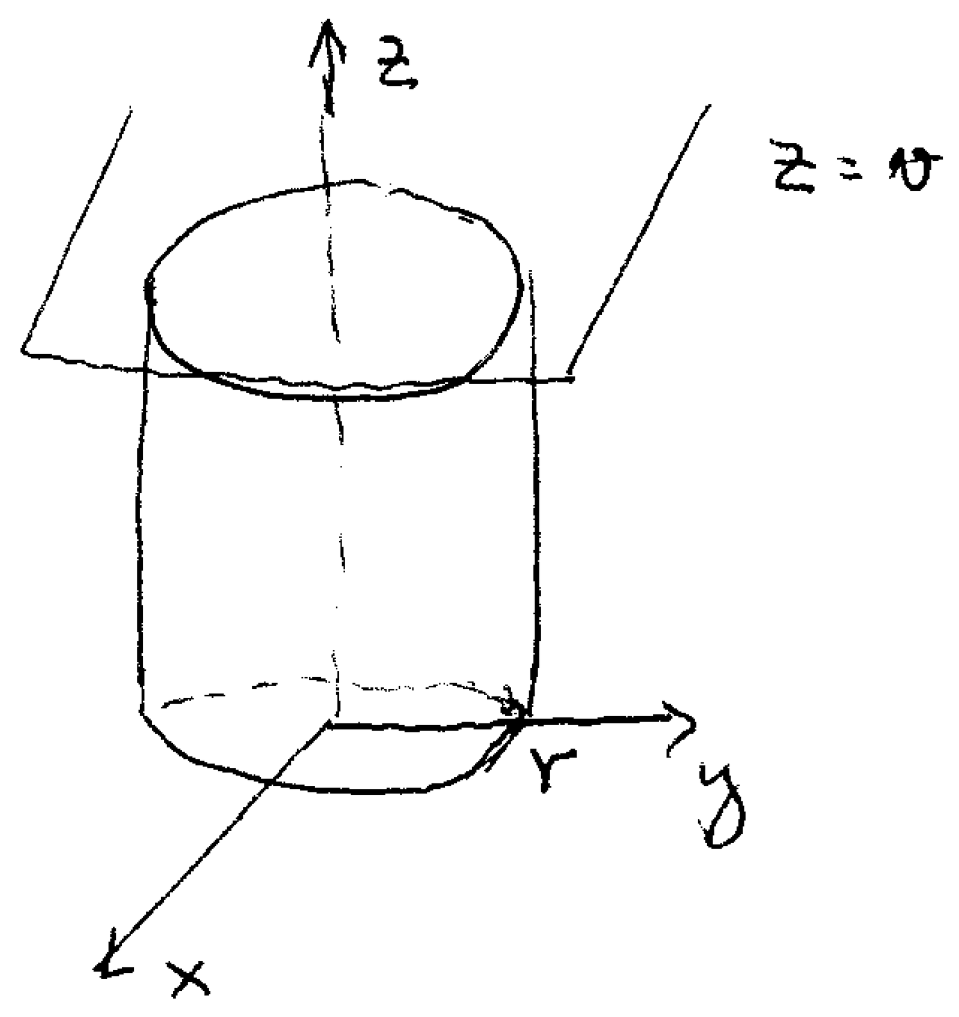


$1 \le \varrho \le 2$

$\frac{\pi}{2} \le \varphi \le \pi$

$0 \le z \le 3$

2) Odvoďte vzorec pro objem válce o poloměru r a výšce v.



$0 \le \varrho \le r$

$0 \le \varphi \le 2\pi$

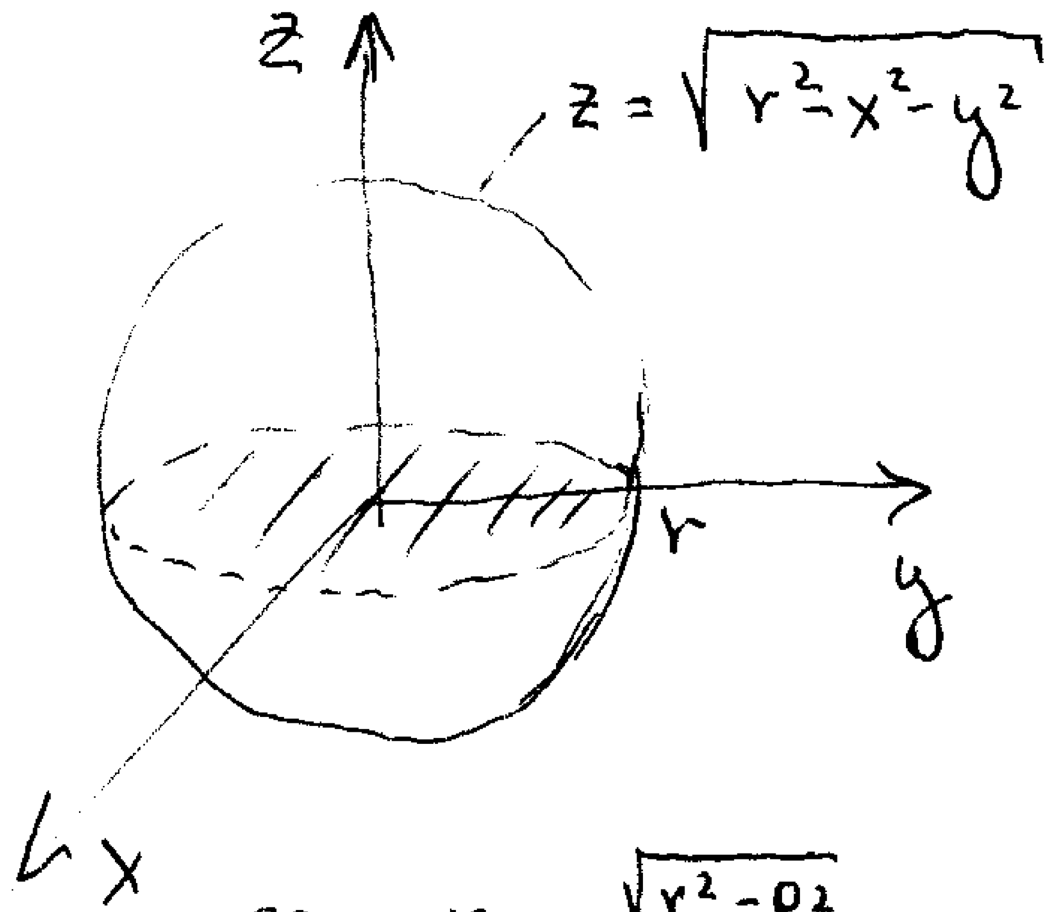
$0 \le z \le v$

konstantní meze !

$$V = \iiint_V dx\,dy\,dz = \int_0^{2\pi}\left(\int_0^{r}\left(\int_0^{v} \varrho\, dz\right) d\varrho\right) d\varphi = \int_0^{2\pi} d\varphi \cdot \int_0^{r} \varrho\, d\varrho \cdot \int_0^{v} dz$$

$$= [\varphi]_0^{2\pi} \cdot \left[\frac{\varrho^2}{2}\right]_0^{r} \cdot [z]_0^{v} = 2\pi \cdot \frac{r^2}{2} \cdot v = \pi r^2 v$$

3) Odvoďte vzorec pro objem koule o poloměru r.



Vypočteme horní polokouli a vynásobíme dvěma.

$x^2 + y^2 + z^2 \le r^2$

$0 \le z \le \sqrt{r^2 - x^2 - y^2}$

$0 \le \varrho \le r$

$0 \le \varphi \le 2\pi$

$0 \le z \le \sqrt{r^2 - \varrho^2}$

$$V = \int_0^{2\pi}\left(\int_0^{r}\left(\int_0^{\sqrt{r^2-\varrho^2}} \varrho\, dz\right) d\varrho\right) d\varphi = \int_0^{2\pi}\left(\int_0^{r} [\varrho z]_0^{\sqrt{r^2-\varrho^2}} d\varrho\right) d\varphi =$$

$$= \int_0^{2\pi}\left(\int_0^{r} \varrho\sqrt{r^2-\varrho^2}\, d\varrho\right) d\varphi = \int_0^{2\pi} d\varphi \cdot \int_0^{r} \varrho\sqrt{r^2-\varrho^2}\, d\varrho = 2\pi \cdot \int_0^{r} \varrho\sqrt{r^2-\varrho^2}\, d\varrho$$

$$= \left| \begin{array}{ll} r^2 - \varrho^2 = t & 0 \to r^2 \\ -2\varrho\, d\varrho = dt & r \to 0 \\ \varrho\, d\varrho = -\frac{1}{2} dt & \end{array} \right| = 2\pi \cdot \int_{r^2}^{0} -\frac{1}{2}\sqrt{t}\, dt = -\pi \left[\frac{\sqrt{t^3}}{\frac{3}{2}}\right]_{r^2}^{0} =$$

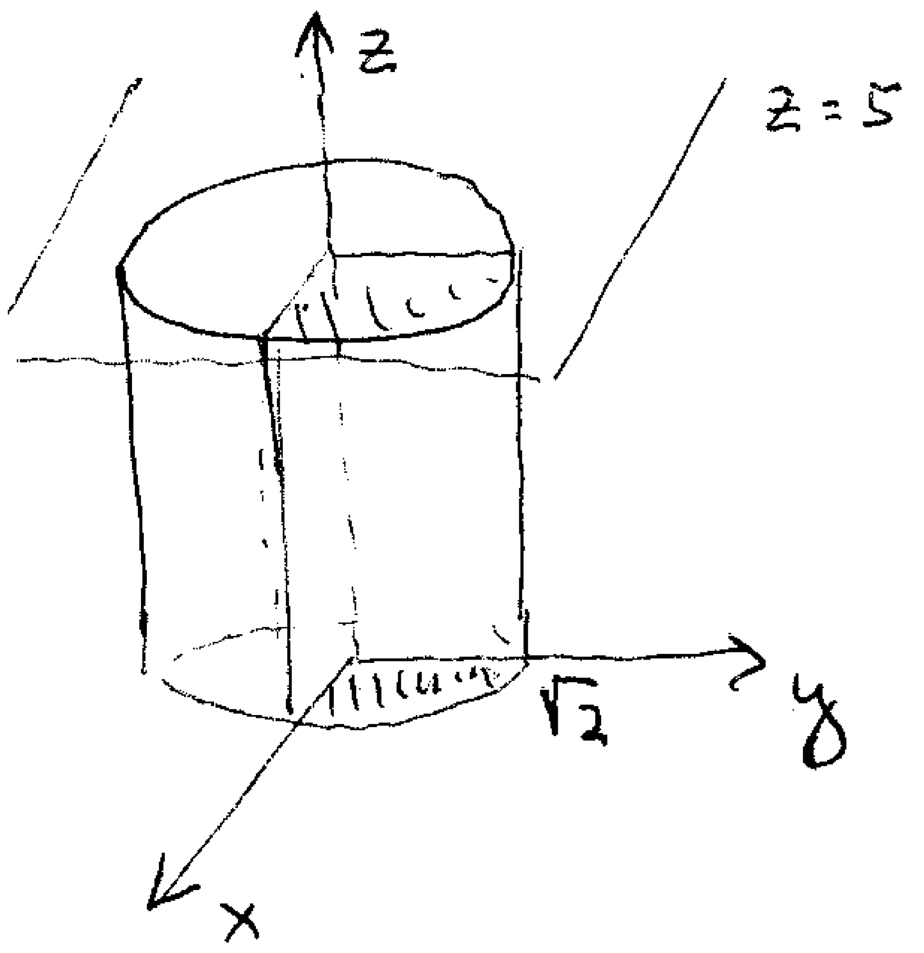
$$= -\frac{2\pi}{3} \cdot \left[0 - r^3\right] = \frac{2\pi}{3} r^3$$

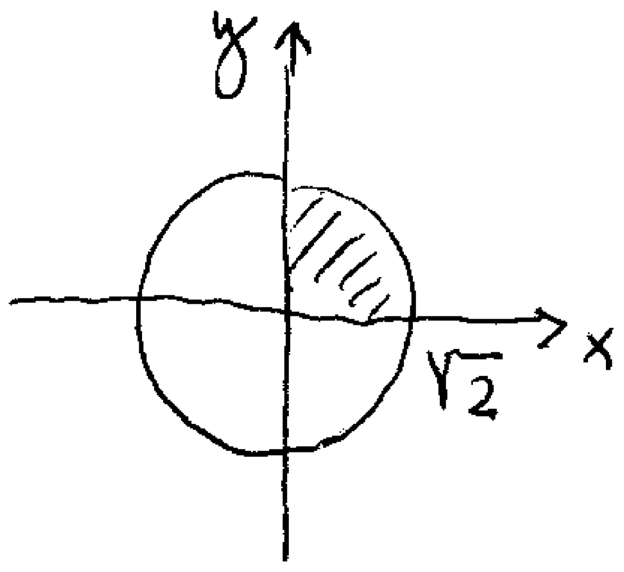
Tedy celá koule

$$2 \cdot \frac{2}{3}\pi r^3 = \frac{4}{3}\pi r^3$$

4) Vypočtěte trojné integrály

a) $\iiint\limits_V x \, dx\, dy\, dz$, V: $x^2+y^2 \le 2$, $0 \le z \le 5$, $x \ge 0$, $y \ge 0$





$0 \le \varrho \le \sqrt{2}$

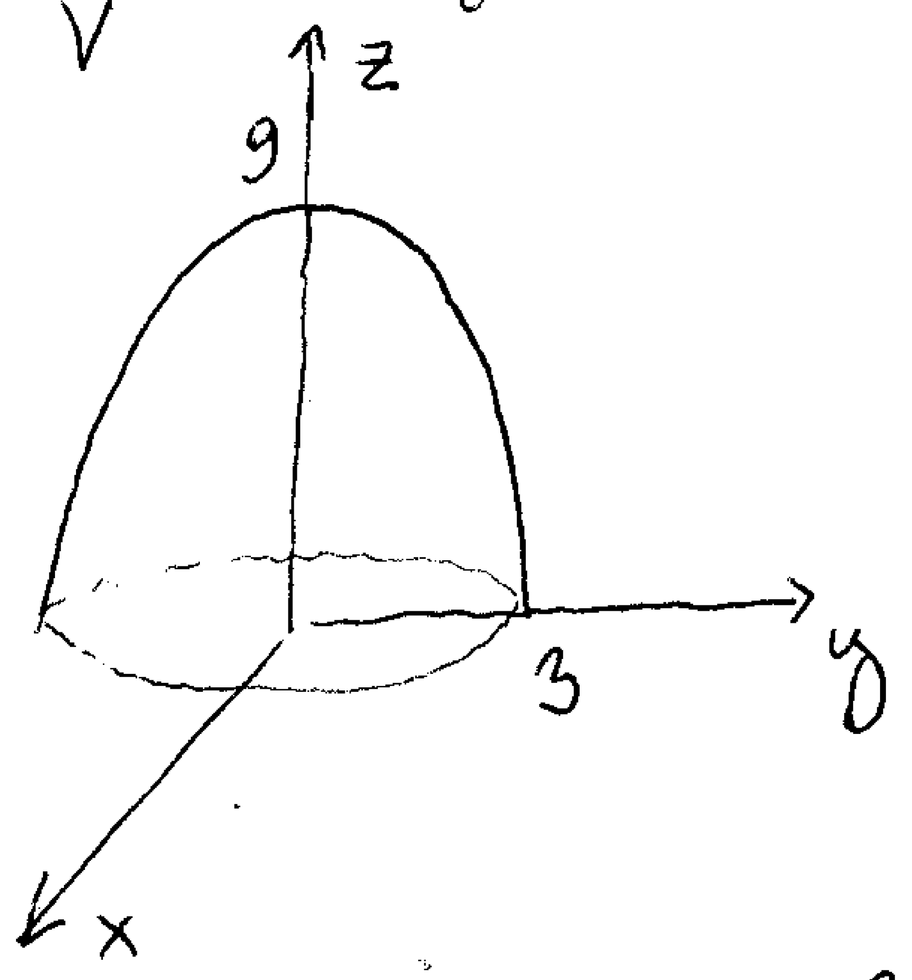
$0 \le \varphi \le \frac{\pi}{2}$

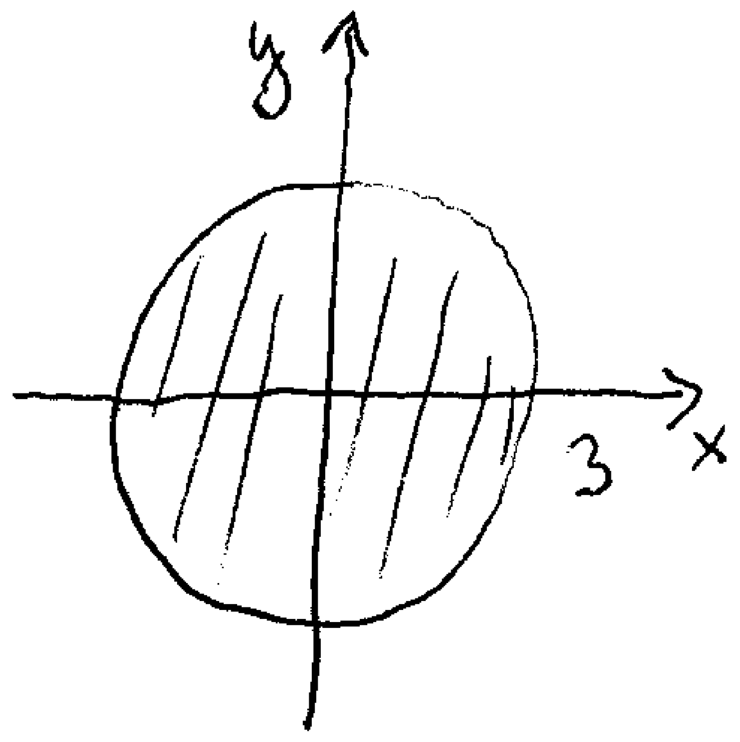
$0 \le z \le 5$

$$V = \int_0^{\pi/2} \left(\int_0^{\sqrt{2}} \left(\int_0^5 \varrho \cos\varphi \cdot \varrho \, dz \right) d\varrho \right) d\varphi = \int_0^{\frac{\pi}{2}} \cos\varphi \, d\varphi \cdot \int_0^{\sqrt{2}} \varrho^2 \, d\varrho \cdot \int_0^5 dz =$$

$$= \left[\sin\varphi \right]_0^{\pi/2} \cdot \left[\frac{\varrho^3}{3} \right]_0^{\sqrt{2}} \cdot \left[z \right]_0^5 = (1-0) \cdot \frac{1}{3} 2\sqrt{2} \cdot 5 = \frac{10}{3}\sqrt{2}$$

4b) $\iiint\limits_V z\,dx\,dy\,dz \quad , V: \quad z \leq 9 - x^2 - y^2 \;, \; z \geq 0$





Jak určíme poloměr ?

Řešíme

$z = 9 - x^2 - y^2$

a

$z = 0$

Tedy $\quad 0 = 9 - x^2 - y^2$

$x^2 + y^2 = 9 \nearrow r = 3$

$0 \leq \varrho \leq 3$

$0 \leq \varphi \leq 2\pi$

$0 \leq z \leq 9 - \varrho^2$

$$\iiint\limits_V z\,dx\,dy\,dz = \int_0^{2\pi} \left(\int_0^{3} \left(\int_0^{9-\varrho^2} z \varrho \, dz \right) d\varrho \right) d\varphi =$$

$$= \int_0^{2\pi} \left(\int_0^{3} \varrho \left[\frac{z^2}{2} \right]_0^{9-\varrho^2} d\varrho \right) d\varphi = \int_0^{2\pi} \left(\int_0^{3} \varrho \, \frac{1}{2} (9-\varrho^2)^2 \, d\varrho \right) d\varphi =$$

$$= \int_0^{2\pi} \left(\int_0^{3} \frac{1}{2} (81\varrho - 18\varrho^3 + \varrho^5) \, d\varrho \right) d\varphi = \int_0^{2\pi} d\varphi \cdot \int_0^{3} \frac{1}{2} (81\varrho - 18\varrho^3 + \varrho^5) \, d\varrho$$

$$= \frac{1}{2} \cdot \left[\varphi \right]_0^{2\pi} \cdot \left[81 \frac{\varrho^2}{2} - 18 \frac{\varrho^4}{4} + \frac{\varrho^6}{6} \right]_0^3 = \frac{1}{2} \cdot 2\pi \cdot \frac{243}{2} = \frac{243\pi}{2}$$