$T: V \to V, \quad T(x)=0 \ \forall x \in V$

$u_1, \dots, u_k$ zvolíme bázi V

$a_1 = 1, \dots, a_k = 1$ Indukcí $n=1$

$V = [u]$

$T^n(u) = \underline{\underline{0}}$ nutně: $T(u)=0$,

$T(u) \neq 0$ $\quad T^n(u) = k^n u$

$= ku, \; k \neq 0 \quad = 0 \quad \neq 0$ Spor.

$\Rightarrow T$ je konstantní nulové zobrazení.

$n > 1 \quad \dim V = n$

$T: V \to V$

$\dim(\mathrm{Im}(T)) = 0 \iff \underline{\underline{T = 0}}$ ✓.

$\dim(\mathrm{Im}(T)) = n$

$\underbrace{\dim \mathrm{Ker}\, T}_{= 0} + \dim \mathrm{Im}\, T = n$

$\Rightarrow \mathrm{Ker}(T) = \{0\}$,

T je mocný! $u \neq 0$, $T^{n}(u) = 0 \Rightarrow T^{n-1}(u) = 0$

$\Rightarrow \ldots \Rightarrow T(u) = 0 \Rightarrow T \Rightarrow u = 0$

$0 < \dim \mathrm{Im}(T) < n$

$T' = T|_{\dim(T)}$

$T' : \mathrm{Im}(T) \to \mathrm{Im}(T)$

lze použít indukční předpoklad

$\exists\ v_1, \dots, v_l$ a přir. čísla $k_1, \dots, k_l$

tak, že $T^{k_1}(v_1) = 0, \dots\dots, T^{k_l}(v_l) = 0$

(#) $v_1, T(v_1), \dots T^{k_1-1}(v_1), \dots\dots, v_l,$

$T(v_l), \dots, T^{k_l-1}(v_l)$ je báze $\mathrm{Im}(T)$.

$v_1, \dots, v_\ell \in \mathrm{Im}\, T$

$\exists\, w_1, \dots, w_\ell$ tak, že

$T(w_1) = v_1, \dots, T(w_\ell) = v_\ell$.

$T^{k_1-1}(v_1), \dots, T^{k_\ell-1}(v_\ell)$ jsou LN

a zároveň leží v $\mathrm{Ker}\, T$

jádra

Dle Steinitzova lemmatu máme bázi V

$T^{k_1-1}(v_1), \dots, T^{k_\ell-1}(v_\ell), \mathbb{R}_1, \dots, \mathbb{R}_m$. $\square$

$w_1, \dots, w_\ell$

$T^j(w_i) = T^{j-1}(\cancel{x}\,\bar{v}_i)$, $\underline{i, j}$

$w_1, T(w_1), \dots, T^{k_1}(w_1), \dots\dots,$
$w_\ell, T(w_\ell), \dots, T^{k_\ell}(w_\ell), \bar{u}_1,$
$\dots, \bar{u}_m$?? ZDA je báze V.

(*)

Ukážeme, že (*) je LN.

$$a_{1,0} w_1 + a_{1,1} T(w_1) + \dots +$$
$$a_{1,b_1} T^{b_1}(w_1) + \dots +$$
$$a_{l,0} w_l + a_{l,1} T(w_l) + \dots +$$
$$a_{l,b_l} T^{b_l}(w_l) + \beta_1 R_1 + \dots + \beta_m R_m = \underline{\underline{0}}$$

Aplikujeme operátor T na výše uvedenou rovn.

Víme, že $T^{b_i+1}(w_i) = 0$, $T(R_i) = 0$

$$a_{1,0} v_1 + a_{1,1} T(v_1) + \dots + a_{1,k_1-1} T^{k_1-1}(v_1) +$$

$$\dots + a_{l,0} v_l + a_{l,1} T(v_l) + \dots$$

$$\dots + a_{l,k_l-1} T^{k_l-1}(v_l) = 0$$

z nez. množiny

$$\Rightarrow a_{1,0} = 0 = a_{1,1} = \dots = a_{1,k_1-1} =$$

$$\dots = a_{l,0} = \dots = a_{l,k_l-1}$$

$$a_{1,k_1} T^{k_1}(w_1) + \dots + a_{l,k_l} T^{k_l}(w_l) +$$

$$+ \beta_1 r_1 + \dots + \beta_m r_m = 0$$

z LN Relených máme

$a_{1,b_1} = 0 = \dots = a_{l,b_l} =$

$= \beta_1 = \dots = \beta_m.$

Podle (#) je báze $\operatorname{Im} T$

a (B) je báze $\operatorname{Ker} T$, máme

$\dim \operatorname{Im} T = b_1 + b_2 + \dots + b_l$

$\dim \operatorname{Ker} T = l + m$

$\dim V = \dim \operatorname{Im} T + \dim \operatorname{Ker} T \Rightarrow$

$$\dim V = k_1 + \dots + k_\ell + \ell + m$$
$$= (k_1+1) + \dots + (k_\ell+1) + m,$$

(což je počet vektorů naší LN posloupnosti (*))

(2) $W_i = \mathrm{Ker}(\varphi - \lambda\, id_V)^i$

$W_1 \subseteq W_2 \quad \overset{?}{\Longrightarrow} x \in W_2$

$x \in \mathrm{Ker}(\varphi - \lambda\, id_V)$

$\varphi(x) = \lambda x$

$(\varphi - \lambda\, id_V) \circ \underbrace{(\varphi - id_V)(x)} =$

$= (\varphi - \lambda id_V)(0) = 0 \Rightarrow x \in \mathrm{Ker}(\varphi - \lambda\, id_V)^2$

kvě 7-17:30

$W_1 \subseteq W_2 \subseteq \dots \subseteq W_N = W_{N+1} = W_{N+2} = \dots = W_k, \quad k \geq l$

(3) Ověříme, že $\mathrm{Ker}(\varphi - \lambda\, id_V)^l \cap \mathrm{Im}(\varphi - \lambda\, id_V)^l = \{0\}$

v leží v obou stranách $\Rightarrow (\varphi - \lambda\, id_V)^l(v) = 0$

$\exists\, w \in V \quad v = (\varphi - \lambda\, id_V)^l(w)$

$(\varphi - \lambda\, id_V)^{2l}(w) = 0 \Rightarrow w \in W_{2l} = W_l$

$w \in \mathrm{Ker}(\varphi - \lambda\, id_V)^{\nu}$

$v = (\varphi - \lambda\, id_V)(w) = \underline{\underline{o}}$

(4) $0 = \dim[\mathrm{Ker}(\varphi - \lambda\, id_V)^{\nu} \cap \mathrm{Im}(\varphi - \lambda\, id_V)^{\nu}]$

$\dim V = \dim \mathrm{Ker}(\varphi - \lambda\, id_V)^{\nu} +$
$+ \dim \mathrm{Im}(\varphi - \lambda\, id_V)^{\nu}$

kvě 7-17:39

(5) $\varphi(\underbrace{\mathrm{Ker}(\varphi - \lambda\, \mathrm{id}_V)^\ell}_{v \in}) \subseteq W_\lambda$

$(\varphi - \lambda\, \mathrm{id}_V)^\ell(v) = 0$

$\varphi \circ (\varphi - \lambda\, \mathrm{id}_V) = \varphi \circ \varphi - \varphi(\lambda\, \mathrm{id}_V)$

$= \varphi \circ \varphi - \lambda\, \mathrm{id}_V \circ \varphi = (\varphi - \lambda\, \mathrm{id}_V) \circ \varphi$

$(\varphi - \lambda\, \mathrm{id}_V)^\ell(\varphi(v)) = \ldots =$

$= \varphi \circ \underbrace{(\varphi - \lambda\, \mathrm{id}_V)^\ell(v)}_{= 0} = \varphi(0) = 0$

Ověříme, že $\varphi(\mathrm{Im}(\varphi-\lambda\, id_V)^k) \subseteq \mathrm{Im}(\varphi-\lambda\, id_V)^k$

$u \in \mathrm{Im}(\varphi-\lambda\, id_V)^k \Rightarrow \exists\, w \in V$

$u = (\varphi-\lambda\, id_V)^k(w)$

$\varphi(u) = (\varphi\circ(\varphi-\lambda\, id_V)^k)(w) =$

$= (\varphi-\lambda\, id_V)^k(\varphi(w)) \in \mathrm{Im}(\varphi-\lambda\, id)^k.$

$\varphi : V \to V$

báze $\mathrm{Ker}(\varphi - \lambda\, id_V)^k$ $\quad \alpha$

báze $\mathrm{Im}(\varphi - \lambda\, id_V)^k$ $\quad \beta$

$\gamma = (\alpha, \beta)$ báze V

$$\underbrace{(\varphi)_{\gamma,\gamma}}_{A} = \begin{pmatrix} \boxed{(\varphi)_{\alpha,\alpha}} & \underbrace{O}_{\overset{\shortparallel}{B}} \\ \underbrace{O}_{} & \underbrace{\boxed{(\varphi)_{\beta,\beta}}}_{\overset{\shortparallel}{C}} \end{pmatrix}$$

$$|A-\lambda I_n| = |B - \lambda I_2| \cdot |C - \lambda I_{n-2}|$$

$\lambda_2, \ldots, \lambda_n$

$\lambda_n \neq \lambda_1$

alg. násobnost λ_1 je 2