$$0 \neq v \in \operatorname{Ker}(\varphi - \lambda\, \mathrm{id}_V) \iff$$

$$(\varphi - \lambda\, \mathrm{id}_V)(v) = 0 \iff$$

$$\varphi(v) - \lambda \cdot v = 0 \iff$$

$$\varphi(v) = \lambda v, \; v \neq 0$$

dub 16-16:20

$S \subseteq V$ $\quad \varphi : V \to V$

inv. $\quad \varphi(S) \subseteq S$

$u_1, \ldots, u_k$ báze S

$\alpha = (u_1, \ldots, u_k, u_{k+1}, \ldots u_n)$ báze V

$$(\varphi)_{\alpha,\alpha} = \left(\begin{array}{c|c} A_1 & M \\ \hline O & A_2 \end{array}\right) \qquad (\varphi(u_1))_\alpha = \begin{pmatrix} X \\ \hline O \end{pmatrix}$$

$$\chi_\varphi(x) = \left|\begin{array}{c|c} A_1 - xI_k & M \\ \hline O & A_2 - xI_{n-k} \end{array}\right| = |A_1 - xI_k| \cdot |A_2 - xI_{n-k}|$$

$$ch_{\varphi_1}(x) = |A_1 - xI_k|$$

$$\Uparrow$$

$$(\varphi_1)_{\alpha',\alpha'} = A_1$$

$$\alpha' = (u_1, \dots, u_k)$$

$$\varphi_2 = \varphi|_T$$

$$ch_{\varphi_2}(x) = |A_2 - xI_{n-k}|$$

$V = S \oplus T$, S, T invariantní vůči φ

$$\varphi(S) \subseteq S, \ \varphi(T) \subseteq T$$

$$ch_\varphi(x) = ch_{\varphi_1}(x) \cdot ch_{\varphi_2}(x) = \left|\begin{pmatrix} A_1 & O \\ O & A_2 \end{pmatrix}\right|$$

$S = \mathrm{Ker}(\varphi - \lambda\, id_V)$ je invariantní podp. V

$\dim S \leq$ alg. násob. vl. č. λ

$x \in S \quad x = 0, \varphi(0) = 0 \in S$

$x \neq 0 \Rightarrow x$ je vl. číslo přísl. k λ

$\varphi(x) = \lambda x \in S$ je on vlastní vekt. přísl. k λ

$k = \dim S \quad B = (v_1, \dots, v_k)$ báze S

$\varphi_1 = \varphi|_S \quad$ ch$_{\varphi_1}(x) = \mathbb{Q}$

$(\varphi_1)_B =$

$(\varphi_1(u_1))_B = (\lambda u_1)_B = \lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$(\varphi_1)_B = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$ $\quad ch_{\varphi_1}(x) = (\lambda - x)^k$

$(\lambda - x)^k$ dělí $ch_\varphi(x) \Rightarrow k \le$ alg. nás. v.č. λ.

dub 16-17:08

$$J_n = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{pmatrix}$$

$$\lambda = 0$$

$$J_n = J_n(0)$$

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$$

alg. mn. $= n$

$$ch_{J_n(\lambda)}(x) = \begin{vmatrix} \lambda - x & 1 & & 0 \\ & \lambda - x & 1 & \\ & & \ddots & \\ 0 & & & \lambda - x \end{vmatrix} = (\lambda - x)^n$$

$$\begin{pmatrix} \lambda - x & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda - x \end{pmatrix}$$

$(x-\lambda)^n$

$x = \lambda$

$\Downarrow$

$$h\left(\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}\right) = n-1 \qquad \dim \mathrm{Ker}(J_n) = 1$$

geom. násob. $= 1$

$$A_{m\times m} = P^{-1} B P, \quad P \ \underline{\text{reg.}}$$

$$\exists\ \varphi : \mathbb{K}^m \to \mathbb{K}^m$$

+ báze α, β prostoru $\mathbb{K}^m$ $|\alpha| = |\beta|$ (?)

$$\underset{\substack{\| \\ A}}{(\varphi)_{\alpha,\alpha}} = \underset{P^{-1}}{P_{\alpha,\beta}}\ \underset{B}{(\varphi)_{\beta,\beta}}\ \underset{P}{P_{\beta,\alpha}}$$

(I) $\Rightarrow$ (II) φ diag., bază $\alpha = (v_1, \dots, v_n)$ v. p.

$$(\varphi)_{\alpha\alpha} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

λ_1

$$\begin{pmatrix} 0 & & & \\ & \underline{\lambda_2 - \lambda_1} & & \\ & & \ddots & \\ & & & \lambda_n - \lambda_1 \end{pmatrix}$$

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$(II) \Rightarrow (III)$

$$n = \sum_{\lambda \text{ vl. hodnota } \varphi} \text{geom. nás.} \le \sum_{\lambda \text{ vl. hodnota } \varphi} \text{alg. nás.} \le n$$

$(III) \Rightarrow (IV)$ zřejmé, protože alg. nás. = geom. nás.

alg. váha = n.

Alg. i geom. váha $= n$

$(IV) \Rightarrow (I)$

$$\text{geom. násob. } \overset{g_\lambda}{\le} \text{ alg. násob. } \overset{a_\lambda}{}$$

$$\lambda \qquad = \qquad \lambda$$

$$\dim \operatorname{Ker}(\varphi - \lambda \,\mathrm{id}_V) = g_\lambda = a_\lambda$$

$$\sum g_\lambda = \sum a_\lambda = n$$

$$\lambda \quad V_1^\lambda, \dots, V_{g(\lambda)}^\lambda$$

$$\begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$$

$$V = [\underbrace{V_1^\lambda, \dots, V_{g(\lambda)}^\lambda, \ \lambda \ \text{vl. hodnoc}}_{\alpha}]$$

$$(\varphi)_{\alpha,\alpha} = \begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_1 \end{matrix}} & 0 & & 0 \\ 0 & \boxed{\begin{matrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_2 \end{matrix}} & & 0 \\ & & \ddots & \\ 0 & & & \boxed{\begin{matrix} \lambda_r & & \\ & \ddots & \\ & & \lambda_r \end{matrix}} \end{pmatrix}$$

(1) $\|\varphi(u)\| = \|u\|$

$$\langle \varphi(u), \varphi(u)\rangle = \langle u, u\rangle$$

$$\|\varphi(u)\|^2 = \|u\|^2 \checkmark$$

$$\Downarrow \quad \|\varphi(u)\| = \|u\|$$

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(2) $u \perp v \Rightarrow \varphi(u) \perp \varphi(v)$

$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$

$$0 = \langle u, v \rangle \Rightarrow 0 = \langle \varphi(u), \varphi(v) \rangle$$

(3) φ mono? $\varphi(x) = 0 \overset{?}{\Rightarrow} x = 0$

$$0 = \langle \varphi(x), \varphi(x) \rangle = \langle x, x \rangle$$

(4) Plyne z (1) a (2)

$$\cos\alpha = \frac{\langle u, v\rangle}{\|u\|\,\|v\|} = \frac{\langle \varphi(u), \varphi(v)\rangle}{\|\varphi(u)\|\,\|\varphi(v)\|}$$

$$\alpha = \arccos\left(\;\right)$$

$$A^T A = I_n \qquad A^T = A^{-1}$$

$$A \cdot A^{-1} = I_n \Rightarrow A \cdot A^T = I_n$$

$$c_i(A^T) \cdot c_j(A) = \delta_{ij} \qquad r_i(A) \cdot r_j(A^T) = \delta_{ij}$$

dub 16-18:04

(1) φ je unitární nad $\mathbb{C}$

$\Downarrow$

(2) ? $\varphi(u_1), \ldots, \varphi(u_n)$ je ONB

zřejmé (z předch. Lemmatu).

(2) $\Rightarrow$ (1) ? φ zachovává součin

v, w $v = \sum c_j u_j$, $w = \sum d_j u_j$

$\langle v, w \rangle = \langle \sum_j c_j u_j, \sum_i d_i u_i \rangle = \ldots$

dub 16-18:09

$$\ldots = \sum_i \sum_j c_i \bar{d}_i \langle u_i, u_j \rangle$$

$$= \sum_j c_j \bar{d}_j$$

$$\langle \varphi(u), \varphi(v) \rangle = \left\langle \varphi\left(\sum_j c_j u_j\right), \varphi\left(\sum_i d_i u_i\right) \right\rangle$$

$$= \left\langle \sum_j c_j \varphi(u_j), \sum_i d_i \varphi(u_i) \right\rangle =$$

$$= \sum_{i,j} c_j \langle \varphi(u_j), \varphi(u_i) \rangle \bar{d}_j = \sum c_j \bar{d}_j$$

(1) $\langle u, v \rangle = \langle \varphi(u), \varphi(v) \rangle$

$\Downarrow$

(3) $A = (\varphi)_{\alpha\alpha}$ je unitární

$\alpha = (u_1, \ldots, u_n)$ ON

$(\varphi(u_i))_\alpha \perp (\varphi(u_j))_\alpha \quad i \neq j$

$(\varphi(u_i))_\alpha^T \cdot (\varphi(u_i))_\alpha = 1$

$i \neq j \Rightarrow \langle u_i, u_j \rangle = 0 \Rightarrow \langle \varphi(u_i), \varphi(u_j) \rangle = 0$

dub 16-18:19

$$(u_1, \dots, u_n)\, s_j(A) = \varphi(u_j)$$

$$(u_1, \dots, u_n)\, s_i(A) = \varphi(u_i)$$

$$\left\langle \sum_{k=1}^{m} a_{kj} u_k, \sum_{l=1}^{m} a_{li} u_l \right\rangle =$$

$$= \dots\dots = \sum_{k=1}^{m} a_{kj} \overline{a_{ki}} = s_j(A)^T\, \overline{s_i(A)}.$$

(3) ⟹ (1) Nechť A je unitární

matice $\quad \overline{A}^T A = I_n$

$u = \sum c_j u_j \qquad \langle u, v \rangle =$

$v = \sum_i d_i u_i \qquad \dots = \sum_j c_j \overline{d_j}$

$\langle \varphi(u), \varphi(v) \rangle =$

$= \sum_{j,i} c_j \overline{d_i} \langle \varphi(u_i), \varphi(u_j) \rangle$

$$\varphi(u_i) = (u_1, \dots, u_n)\, r_i(A)$$
$$\varphi(u_j) = (u_1, \dots, u_n) \cdot r_j(A)$$
$$\langle \varphi(u_j), \varphi(u_i) \rangle = \left\langle \sum_{k=1}^{m} a_{kj} u_k, \sum_{l=1}^{m} a_{li} u_l \right\rangle = \sum_{k=1}^{m} a_{kj} \overline{a_{ki}} = 0 \quad i \neq j$$
$$= 1 \quad \underline{i = j}$$

dub 16-18:36

$$|\det A| = 1 \qquad \overline{\det A} = \overline{\sum_{\sigma \in S_n} (-1)^{n(\sigma)} a_{\sigma(1)1} \cdots a_{\sigma(n)n}}$$

$$A^{-1} = \overline{A}^T$$

$$|\det A|^2 = \det A \cdot \overline{\det A} = \sum_{\sigma \in S_n} (-1)^{n(\sigma)}\, \overline{a_{\sigma(1)1}} \cdots \overline{a_{\sigma(n)n}}$$

$$= \det \overline{A}$$

$$\det A \det \overline{A}^T = \det A \det A^{-1} = 1$$

$\varphi(x) = \lambda x \qquad x \neq 0$

$\qquad\qquad\qquad\qquad x$ s.t.

$$0 \neq \langle x, x \rangle = \langle \varphi(x), \varphi(x) \rangle =$$

$$= \langle \lambda x, \lambda x \rangle = \lambda \bar{\lambda} \langle x, x \rangle \Big| : \langle x, x \rangle$$

$$1 = \lambda \bar{\lambda} = |\lambda|^2 \Rightarrow \underline{\underline{|\lambda| = 1}}$$

dub 16-18:47

(2) $\lambda \neq \mu$ vl. čísla

$$\varphi(u) = \lambda u, \quad u \neq 0$$

$$\varphi(v) = \mu v, \quad v \neq 0$$

$$\langle u, v\rangle = \langle \varphi(u), \varphi(v)\rangle =$$

$$\langle \lambda u, \mu v\rangle = \lambda \overline{\mu} \langle u, v\rangle$$

SPOR

a) $\langle u, v\rangle \neq 0 \Rightarrow 1 = \lambda \overline{\mu} \Rightarrow \mu = \lambda$

$$(\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta) = 1 \Rightarrow \alpha - \beta = 0$$

$Ch_\varphi(x)$ má v $\mathbb{C}$ alespoň jeden kořen. λ vl. číslo, $|\lambda|=1$

$\Downarrow$

$\varphi([u_1]) \subseteq [u_1]$ u_1 vl. vektor $\lambda \neq 0$

$[u_1]\oplus[u_1]^\perp = V$ $\quad \varphi([u_1]^\perp) \overset{?}{\subseteq} [u_1]^\perp$

$x\in[u_1]^\perp \Rightarrow \varphi(x)\in[u_1]^\perp$

$\langle x, u_1\rangle = 0 = \langle \varphi(x), \varphi(u_1)\rangle =$

$\langle \varphi(x), u_1\rangle \bar{\lambda}$

$$ch_\varphi(x) = ch_{\varphi_1}(x) \cdot ch_{\varphi_2}(x)$$

$$\varphi_1 = \varphi|_{[u_1]}, \quad \varphi_2 = \varphi|_{[u_1]^\perp}$$

pokračujeme indukcí až obdržíme ON bázi V

$$\varphi(x) = Ax, \quad \varphi: \mathbb{R}^n \to \mathbb{R}^n$$

$$\varphi^{\mathbb{C}}(x) = Ax, \quad x \in \mathbb{C}^n$$

$$\varphi^{\mathbb{C}}: \mathbb{C}^n \to \mathbb{C}^n$$

$$\lambda = a + ib, \quad \underset{\in \mathbb{C}^n}{u} = \underset{\in \mathbb{R}^n}{u_1} + i \underset{\in \mathbb{R}^n}{u_2}$$

$$\overline{Au} = \overline{\lambda u}$$

$$\overline{A} \cdot \overline{u} = \overline{\lambda} \cdot \overline{u} \Rightarrow A\overline{u} = \overline{\lambda}\,\overline{u}.\llcorner$$

reálná

$$\varphi^{\mathbb{C}}(u) = \lambda u, \quad u \neq 0, \quad u \in \mathbb{C}^n$$

$$u = u_1 + i u_2, \quad u_1, u_2 \in \mathbb{R}^n$$

$$\frac{\lambda \neq \pm 1}{\lambda \neq \overline{\lambda}} \qquad \varphi^{\mathbb{C}}(\overline{u}) = \overline{\lambda}\,\overline{u}$$

$$0 = \langle u_1 + i u_2, u_1 - i u_2 \rangle =$$

$$= \left(\langle u_1, u_1 \rangle_{\mathbb{R}^n} - \langle u_2, u_2 \rangle_{\mathbb{R}^n} \right) +$$

$$i \left(\langle u_2, u_1 \rangle_{\mathbb{R}^n} + \langle u_1, u_2 \rangle_{\mathbb{R}^n} \right) \Rightarrow \begin{array}{c} \|u_1\| \\ \| \\ \|u_2\| \end{array}$$

$$\langle u_1, u_2 \rangle = 0$$

$$\lambda = \cos\alpha + i\sin\alpha = a + ib$$

$$Au = \lambda u$$

$$\overline{A(u_1 + iu_2)} = \overline{(a+ib)(u_1+iu_2)}$$

u_1: $\quad Au_1 = au_1 - bu_2 \in [u_1, u_2]$

u_2: ! $\quad Au_2 = bu_1 + au_2$

$$\|u_1\| = \|u_2\| = 1 \qquad B = (u_2, u_1)$$

$$(\varphi)_B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$\varphi : \mathbb{R}^n \to \mathbb{R}^n \qquad \varphi^{\mathbb{C}} : \mathbb{C}^n \to \mathbb{C}^n$

$\varphi(x) = Ax \qquad \varphi^{\mathbb{C}}(x) = Ax$

$\pm 1 \quad \ker(\varphi - (\pm 1)\mathrm{id}_V)$

$g^1, \; g^{-1}$

$\pm 1, \; \cos\alpha + i\sin\alpha$

$\cos\alpha - i\sin\alpha$

$\pm 1 \neq \lambda \neq \mu \neq \pm 1$

$u = u_1 + i u_2 \qquad \bar{v} = \bar{v}_1 + i \bar{v}_2$

$$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & \begin{array}{c} \cos\alpha_2 + i\sin\alpha_2 \\ \cos\alpha_2 - i\sin\alpha_2 \end{array} \end{pmatrix}$$

$$[u_1, u_2] \perp [\overline{v}_1, \overline{v}_2]$$

$$\overline{v} \neq u \;,\; \overline{v} \neq \overline{u}$$

$$\langle u_1 + i u_2, \overline{v}_1 + i \overline{v}_2 \rangle = 0$$

$$\|$$

$$\left(\langle u_1, \overline{v}_1\rangle - \langle u_2, \overline{v}_2\rangle\right) + i\left(\langle u_2, \overline{v}_1\rangle + \langle u_1, \overline{v}_2\rangle\right)$$

$$0 = \langle u_1, \overline{v}_1\rangle = \pm\langle u_2, \overline{v}_2\rangle, \quad \langle u_2, \overline{v}_1\rangle = \pm\langle u_1, \overline{v}_2\rangle = 0$$

$$\langle u_1 + i u_2, \overline{v}_1 - i \overline{v}_2 \rangle = 0$$

$$\left(\langle u_1, \overline{v}_1\rangle + \langle u_2, \overline{v}_2\rangle\right) + i\left(\langle u_2, \overline{v}_1\rangle + \langle u_1, \overline{v}_2\rangle\right)$$