1 The formalism

1. In a two dimensional Hilbert space with basis $\{|1\rangle, |2\rangle\}$ what is the matrix representation of the operator $\hat{A} = |1\rangle\langle 2|$?

2. Show that a product of unitary operators is unitary.

3. Show that Unitary operators preserve the inner product between the states they act on.

4. What is the Hermitean conjugate of an operator $\hat{A} = |\alpha\rangle\langle\beta|$?

5. Define the trace of an operator by using an orthonormal basis $|n\rangle$ as

$$\mathrm{Tr}(\hat{A}) = \sum_n \langle n| \hat{A} |n\rangle .$$

   Show that the definition is independent of the choice of basis by introducing a different orthonormal basis $|n'\rangle$ and using that both sets of basis vectors are complete.

6. If $\{|n\rangle\}$ and $\{|n'\rangle\}$ are two different sets of orthonormal basis vectors. We may define the operator

$$\hat{U} = \sum_{n'=n} |\mathrm{n}'\rangle\langle \mathrm{n}|$$

   which maps a state $|n\rangle$ in the first basis to a state $|n'\rangle$ in the second basis. Show that $\hat{U}$ is a unitary operator.

7. Show that the eigenvalues of a unitary operator are complex numbers of unit modulus.

8. Show that the eigenvectors of a unitary operator are mutually orthogonal (if no degeneracy).

9. Show that cyclicity of the trace holds $\mathrm{Tr}(\hat{A}\hat{B}) = \mathrm{Tr}(\hat{B}\hat{A})$.

10. Show that $\mathrm{Tr}(|\psi\rangle\langle\chi|) = \langle\chi|\psi\rangle$.

11. Show that $(|\psi\rangle\langle\chi|)^\dagger = |\chi\rangle\langle\psi|$.

12. By using the sesquilinearity of $(\cdot,\cdot)$, show that $(|\psi\rangle, |\chi\rangle) = (|\chi\rangle, |\psi\rangle)^\star$.

13. In a space with three basis vectors $\{|1\rangle, |2\rangle, |3\rangle\}$ we define an operator $\hat{R}$ according to its action on the basis states as

$$\hat{R}|1\rangle = |2\rangle \quad \hat{R}|2\rangle = -|1\rangle \quad \hat{R}|3\rangle = |3\rangle$$

What is the matrix representative of this operator? If we have a state $|\psi\rangle = a|1\rangle + b|2\rangle + c|3\rangle$, what is its matrix representative? How does the matrix representative of $\hat{R}$ act on the matrix representative of $|\psi\rangle$?

14. Consider the operator $\hat{D} = -i\frac{d}{dx}$, defined on the space of differentiable functions of x on the interval $a \leq x \leq b$ with the inner product defined as $(f(x), g(x)) = \int_a^b dx f^\star(x) g(x)$. We may define various subspaces of the space of differentiable functions by imposing boundary conditions.

    - What are the boundary conditions that one have to impose to make $\hat{D}$ hermitian?
    - What are the eigenfunctions and eigenvalues for the operator $\hat{D}$?
    - Are the eigenfunctions part of the space on which we define $\hat{D}$? For what boundary conditions are the eigenfunctions part of the space on which $\hat{D}$ is defined?
    - What if $a = -\infty$ and $b = \infty$?

15. What boundary conditions must be imposed on the functions $\{f(\bar{x})\}$ defined in some finite or infinite volume of space in order for the Laplace operator $\Delta = \nabla^2$ to be Hermitian?

2 Path Integrals

1. Show that if $|n\rangle$ are eigenstates of the Hamiltonian with energy E_n, the propagator can be written as $K(x,t;x',t') = \sum_n e^{-\frac{i}{\hbar}E_n(t-t')}\langle \mathrm{x}|\mathrm{n}\rangle\langle \mathrm{n}|\mathrm{x}'\rangle$.

2. In a two dimensional Hilbert space with a basis of normalized eigenstates of the hamiltonian $|1\rangle$ and $|2\rangle$ with energy eigenvalue E_1 and E_2, write the time evolution operator in terms of the states $|\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$.

3. Assume that space consists of two points, x and y. We will try to find the time evolution of the system by assuming that the probability

amplitude at each time step Δt to stay at the same point is given by $1 + i\omega\Delta t$ and the probability amplitude to change points is given by $i\beta\Delta t$ where ω and β are arbitrary real numbers. Define the probability amplitude (i.e the propagator)

$$\begin{aligned}
K_{xx}(T) &= \text{To go from x at t=0 to x at t=T} \\
K_{xy}(T) &= \text{To go from y at t=0 to x at t=T} \\
K_{yx}(T) &= \text{To go from x at t=0 to y at t=T} \\
K_{yy}(T) &= \text{To go from y at t=0 to y at t=T}
\end{aligned}$$

If we divide the time interval into N pieces so that $\Delta t = \frac{T}{N}$, show that

$$\begin{aligned}
K_{xx}(T) &= K_{xx}(T - \Delta t)(1 + i\omega\Delta t) + K_{yx}(T - \Delta t)i\beta\Delta t \\
K_{yx}(T) &= K_{yx}(T - \Delta t)(1 + i\omega\Delta t) + K_{xx}(T - \Delta t)i\beta\Delta t
\end{aligned}$$

Show that this gives a recursion relation that can be solved as

$$\begin{aligned}
K_{xx}(T) &= \frac{1}{2}\left[(1 + i(\omega + \beta)\Delta t)^N + (1 + i(\omega - \beta)\Delta t)^N\right] \\
K_{yx}(T) &= \frac{1}{2}\left[(1 + i(\omega + \beta)\Delta t)^N - (1 + i(\omega - \beta)\Delta t)^N\right]
\end{aligned}$$

which when we let $N \to \infty$ becomes

$$\begin{aligned}
K_{xx} &= e^{i\omega T}\cos(\beta T) \\
K_{yx} &= ie^{i\omega T}\sin(\beta T)
\end{aligned}$$

Is the probability conserved? What is the wavefunction at T for a particle which is localized at x when $t = 0$? What is the wavefunction at T for a particle with an initial wavefunction $\psi(x) = \frac{1}{\sqrt{2}}, \psi(y) = \frac{1}{\sqrt{2}}$?

4. A model of a moving wave-packet in 1 dimension is given by the wavefunction

$$N \int dp e^{-\frac{a}{2}(p-p_0)^2} |p\rangle$$

where a is a constant and N is the normalization factor. Determine N and use the propagator of a free particle to find how the packet moves in time. *Interpret* your result!

5. Let $|n\rangle$ be a complete set of eigenstates of the time independent Hamiltonian $\hat{H}$ where $\hat{H}|n\rangle = E_n|n\rangle$ and with configuration space representation $\psi_n(x) = \langle \mathrm{x}|\mathrm{n}\rangle$. Using these elements, write expressions for the time evolution operator in the $|k\rangle$ and $|x\rangle$ basis *i.e.* find A_{kl} and $B(x,x')$ in the expressions

$$\hat{U}(t,t') = \sum_{k,l} A_{kl}|k\rangle\langle l| = \int dx\, dx'\ B(x,x')|x\rangle\langle x'|$$

6. Calculate the propagator for a particle in a linear potential

$$S[x(t)] = \int dt(\frac{1}{2}m\dot{x}^2 - Fx)$$

using path integral methods. Here are some useful observations that you might want to use

a) In the path integral, we sum over all paths *with the prescribed boundary conditions.*

b) The sum will be the same if we shift all paths by some particular *fixed* path.

c) Define the new path $y(t)$ as the old path shifted by a solution of the equations of motion $x_{cl}(t)$ so that $y = x - x_{cl}$.

d) However, shifting a path satisfying a particluar boundary condition by a fixed path gives a new path that usually does not satisfy the same boundary condition. What boundary conditions should $y(t)$ fulfil if the classical solution x_{cl} satisfies the same boundary conditions as x?

e) Find a particular x_{cl} with the same boundary conditions as x, *i.e.* that begins at x' at time t' and ends at x at time t.

f) Show that the action $S[y(t)]$ consists of only of a kinetic term and a term dependent only on the boundary conditions. In particular there is no potential for $y(t)$.

g) The path integral over $y(t)$ can now be done using the result for the path integral of a free particle. Remeber that is is given by

$$\int \mathcal{D}x e^{\frac{i}{\hbar}S_{free}[x(t)]} = \sqrt{\frac{m}{2\pi i\hbar(t-t')}}e^{\frac{im(x-x')^2}{2\hbar(t-t')}}$$

for a path that starts at x' at time t' and ends at x at time t.

Check that your result agrees with the result of the previous problem.

3 Scattering theory

1. Analysis of the scattering of particles of mass m and energy $E = \frac{\hbar^2 k^2}{2m}$ from a fixed scattering center with characteristic length a finds the pase shifts

$$\sin \delta_l = \frac{(iak)^l}{\sqrt{(2l+1)!}}$$

   a) Derive a closed expression for the total cross section as a function of the incident energy E.

   b) At what values of E does the S-wave ($l = 0$) scattering give a good estimate of σ?

2. Using the Born approximation, obtain an expression for the total cross section for scattering of particles of mass m from the attractive Gaussian potential

$$V(r) = -V_0 e^{-\frac{r^2}{a^2}}$$

3. Consider a scattering situation in which only the $l = 0$ and $l = 1$ partial waves have appreciable phase shifts. Discuss how the contribution of the $l = 1$ wave affects the total cross section. How does it affect the angular distribution of scattered particles? What sort of measurements should be made to obtain an accurate value of δ_0 and δ_1 respectively?

4. Determine in the first Born approximation the differential cross-section for the potential

$$V = \begin{cases} 0 & \text{for} \quad r > R \\ -V_0 & \text{for} \quad r < R \end{cases}$$

   with $V_0 > 0$. Sketch the dependence (using a computer if you wish) of the cross-section on 1) the angle θ and 2) the energy.

5. Consider the scattering of a particle by a repulsive delta function shell potential

$$V(r) = \frac{\hbar^2\gamma^2}{2m}\delta(r-R),$$

   a) Set up an equation that determines the s-wave phase shift δ_0 as a function of k (remember that $E = \frac{\hbar^2 k^2}{2m}$).
   b) Assume now that γ is very large,

$$\gamma \gg \frac{1}{R}, k.$$

   Show that if $\tan kR$ is *not* close to zero, the s-wave phase shift resembles the hard-sphere result discussed in the lectures. Show also that for $\tan kR$ close to (but not exactly equal to) zero, resonance behavior is possible; that is, $\cot \delta_0$ goes through zero from the positive side as k increases. Determine approximately the positions of the resonances keeping terms of order $\frac{1}{\gamma}$.

4 Relativistic QM

1. For the Dirac equation written in the ϕ_A, ϕ_B basis used in the lecture notes, find the explicit form of the gamma-matrices and show that they satisfy the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. Find the plane wave solutions.

2. In non-relativistic physics, the transformation between two inertial systems, moving with a relative speed v, is through the Galileo transformation

$$\begin{aligned} \mathbf{x}' &= \mathbf{x} + \mathbf{v}t \\ t' &= t \end{aligned}$$

   Assume that the wave function $\psi(\mathbf{x}', t')$ is a solution to the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t'}\psi(\mathbf{x}', t') = -\frac{\hbar^2}{2m}\nabla'^2\psi(\mathbf{x}', t') + V(\mathbf{x}', t')\psi(\mathbf{x}', t')$$

Show that the wave function $\psi(\mathbf{x}+\mathbf{v}t,t)$ is not a solution of the Schrödinger equation in the unprimed system

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}+\mathbf{v}t,t) \neq -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x}+\mathbf{v}t,t)+V(\mathbf{x}+\mathbf{v}t,t)\psi(\mathbf{x}+\mathbf{v}t,t)$$

However, if we allow for a phase factor in the transformation of the wave function $\psi(\mathbf{x}',t') \to e^{\frac{i}{\hbar}f(\mathbf{x},t)}\psi(\mathbf{x}+\mathbf{v}t,t)$ find the form of f that makes it a solution. Interpret the result in the case where ψ is a plane wave.

In relativistic physics on the other hand, the transformation between two inertial systems, moving with a relative speed v (in the x-direction for simplicity), is through the Lorentz transformation

$$\begin{aligned} t' &= \gamma t + \gamma\frac{v}{c}x \\ x' &= \gamma\frac{v}{c}t + \gamma x \\ y' &= y \\ z' &= z \end{aligned}$$

where $\gamma^{-2} = 1 - \frac{v^2}{c^2}$. If $\phi(t',x',y',z')$ is a solution to the Klein-Gordon equation in the primed system, show that $\phi(\gamma t+\gamma\frac{v}{c}x, \gamma\frac{v}{c}t+\gamma x, y, z)$ is a solution to the Klein-Gordon equation in the unprimed system *without any phase factor.*

3. A plane wave solution to the Dirac equation can be written as

$$\psi(x) = u(p)e^{-\frac{i}{\hbar}p\cdot x}$$

where $u(p)$ is a spinor. Find the matrix equation that $u(p)$ has to satisfy and analogously find an equation for $\bar{u} = u^\dagger\gamma^0$. Use this to show

$$\bar{u}(p)\not{q}u(p) = \frac{p\cdot q}{mc}$$

if we chose $u(p)$ to be normalized to 1. ($\not{q} = \gamma^\mu q_\mu$).

4. How would the Dirac equation look like in 2,3,4 and 5 space-time dimensions? Find explicit representations of the gamma matrices in all these cases and show that they satisfy the appropriate Clifford algebra.

5. In three space-time dimensions, verify that one can choose the gamma matrices as following

$$\gamma^0 = \sigma^3 \quad \gamma^1 = i\sigma^1 \quad \gamma^2 = -i\sigma^2$$

i.e., verify that they satisfy the appropriate Clifford algebra. Construct the matrices

$$\begin{aligned} M^{01} &= \frac{1}{4i}\left[\gamma^0, \gamma^1\right] \\ M^{20} &= \frac{1}{4i}\left[\gamma^2, \gamma^0\right] \\ M^{12} &= \frac{1}{4i}\left[\gamma^1, \gamma^2\right] \end{aligned}$$

and show that they satisfy the $SO(1,2)$ algebra

$$\begin{aligned} \left[M^{01}, M^{20}\right] &= -iM^{12} \\ \left[M^{12}, M^{01}\right] &= iM^{20} \\ \left[M^{20}, M^{12}\right] &= iM^{02} \end{aligned}$$

which exept for the minus sign in the first row is the same as the algebra of the rotation group $SO(3)$. Show that under a rotation with angle θ in the 12-plane, the spinors transform as

$$e^{i\theta M^{12}}\psi = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

whereas under a boost in the 2 direction, the spinor transforms as

$$e^{i\alpha M^{20}}\psi = \begin{pmatrix} \cosh(\frac{\theta}{2}) & \sinh(\frac{\theta}{2}) \\ \sinh(\frac{\theta}{2}) & \cosh(\frac{\theta}{2}) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

How do $\psi^\dagger\psi$ and $\psi^\dagger\gamma^0\psi$ transform under these transformations?