1 Motion and curve

We shall deal with curves in the Euclidean plane and three dimensional space. Although we shall need only these two dimensions, we shall start with the general n-dimensional Euclidean space E_n.

1.1 **1.1.** Let $I \subseteq \mathbb{R}$ be an open interval. We shall understand points of I as valus of time t. A map $f : I \to \mathbb{R}$ could be viewed as a motion whose trajectory is a curve in a "reasonable" case.

To employ calculus we shall need in particular differentibility of the map f. We recall the real function $\varphi : I \to \mathbb{R}$ is of the class C^r if it has continuous derivatives of the order $\leq r$ on I. Choosing cartesian coordinates on E_n, then $f(t) = (f_1(t), \dots, f_n(t))$ is n-tuple of real functions. We say f is of the class C^r if all fucntions $f_1, \dots, f_n$ are fo the class C^r.

One needs to show that this notion does not depend on the choice of the coordinate system. This is indeed true but a direct verification (based on a transformation from one coordinate sysytem to another one) is difficult. It is much easier to control independence only on the choice of the origin of coordinates. The choice of the origin identifies E_n with its associated vector space $Z(E_n)$ which is an n-dimensional Euclidean vector space.

1.2 **1.2.** We shall therefore focus on the n-dimensional Euclidean vector space V first. We denote by $||u||$ the norm of the vector u and by (u, v) the vector porduct of vectors u and v.

Definition. A map $v : I \to V$ is called a *vector function* on the interval I.

1.3 **1.3.** Notion of the limit of a vector function is introduced analogoulsu as the limit of a real function.

Definition. Vector function v has the *limit* v_0 at the point $t_0 \in I$ if for each $\epsilon > 0$ there exists $\delta > 0$ such that the following holds: if $t \neq t_0$ satisfies $|t - t_0| < \delta$ then $||v(t) - v_0|| < \epsilon$.

We write $v_0 = \lim_{t \to t_0} v(t)$.

If $v(t_0) = \lim_{t \to t_0} v(t)$ we say the vector function v is continuous at the point t_0.

de1.4 **1.4 Definition.** If the limit

$$\lim_{t \to t_0} \frac{v(t) - v(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{1}{t - t_0} \big(v(t) - v(t_0) \big),$$

it is called *derivative of the vector function* $v(t)$ at the piont t_0.

We shall denote this derivative by $\frac{dv(t_0)}{dt}$ or $v'(t_0)$.
Higher order derivatives are defined by the usual iteration.

1.5 **1.5.** Let $e_1, \dots, e_n$ is a basis of V. For each $t \in I$ we have

$$v(t) = v_1(t)e_1 + \ldots + v_n(t)e_n.$$

Real functions $v_i(t)$, $i = 1, \dots, n$ are called **components of the vector function** $v(t)$.

The following theorem has a simple proof which however belongs to calculus. Therefore we do not state it.

Theorem. A vector function is continuous if all its components are continuous. The vector function $v(t)$ has the derivative at the point t_0 if all components have derivative at the point t_0. Then it holds

$$\frac{dv(t_0)}{dt} = \left(\frac{dv_1(t_0)}{dt}, \dots, \frac{dv_n(t_0)}{dt}\right).$$

A similar statement holds also for limits and higher order derivatives.

diffscalar 1.6 **1.6.** Now we shall state an auxiliary result which we shall need later. Let $v(t)$ and $w(t)$ be two vector functions of the class C^1 on I. Their scalar product $\big(v(t), w(t)\big)$ is a real function of the class C^1 on I. Also scalar products $\big(\frac{dv}{dt}, w\big)$ and $\big(v, \frac{dw}{dt}\big)$ are real function on I.

Theorem. The following holds

$$\frac{d(v,w)}{dt} = \big(\frac{dv}{dt}, w\big) + \big(v, \frac{dw}{dt}\big).$$

Proof. In coordinates we have

$$\big(v(t), w(t)\big) = v_1(t)w_1(t) + \ldots + v_n(t)w_n(t).$$

Thus using the chain rule we have

$$\frac{d(v(t), w(t))}{dt} = \frac{dv_1}{dt}w_1 + v_1\frac{dw_1}{dt} + \ldots + \frac{dv_n}{dt}w_n + v_n\frac{dw_n}{dt}.$$

This is the coordinate form of our statement. □

1.7 **1.7.** Let us consider E_n with its associated vector space V . Let us choose the origin $P \in E_n$. Then the map $f : I \to E_n$ determines the vector function $\overrightarrow{Pf} : I \to V$, $\overrightarrow{Pf}(t) = \overrightarrow{Pf(t)}$ which is called *radius* (or *radious vector*) of f.

Definition. The map $f : I \to E_n$ is called *motion* in the space E_n. We say f is motion of the class C^r, if $\overrightarrow{Pf}$ is a vector function of the class C^r.

Beside the word "motion" we can equivalently say *path*. The terminology "motion" is more illustrative, "path" has a more technical nature.

The vector $\frac{d\overrightarrow{Pf}}{dt}$ does not depend on the choice of the origin. Indeed, for another point $Q \in E_n$ we have $\overrightarrow{Pf} = \overrightarrow{PQ} + \overrightarrow{Qf}$ where $\overrightarrow{Pf}$ is a constant vector. Thus $\frac{d\overrightarrow{Pf}}{dt} = \frac{d\overrightarrow{Qf}}{dt}$.

de1.8 **1.8 Definition.** The vector $\frac{d(\overrightarrow{Pf})}{dt} =: \frac{df}{dt}$ is called the **velocity vector** of the motion f.

It will be also denoted by f'.

At the second order we put $f'' = (f')'$; here we already differentiate the vector function f' (and analogously in higher orders).

If $f(t) = \big(f_1(t), \dots, f_n(t)\big)$ is the coordinate expression of the motion f, we have

$$\frac{d^k f(t)}{dt^k} = \Big(\frac{d^k f_1(t)}{dt^k}, \dots, \frac{d^k f_n(t)}{dt^k}\Big).$$

regula9 **1.9 Definition.** The motion $f : I \to E_n$ of the class is called **regular**, if $\frac{df(t)}{dt} \neq o$ for every $t \in I$. The point of the paramter t_0 at which $\frac{df(t_0)}{dt} = o$ is called **a singular point of the motion** f.

Here o denotes the zero vector of the space $V = Z(E_n)$.

We shall show two examples:

(i) In the case of constant motionu $f(t) = Q \in E_n$, it holds for every $t \in I$ that $\frac{df(t)}{dt} = o$. Thus for every vaule of time $t \in I$ we obtain a singular point.

(ii) Consider motion $x = t^2$, $y = t^3$ in E_n, $t \in (-\infty, \infty)$. This moves along so colled semicubic parabola $y^2 - x^3 = 0$. We have $f(t) = (t^2, t^3)$, $f'(t) = (2t, 3t^2)$ hence $f'(0) = o$. The singular point bod $t = 0$ is so called **edge**, see the picture. JS: missing picture

repa10 **1.10.** Consider another open interval J with the variable τ and a bijective correspondence $\varphi\colon J \to I$ (i.e. a real function) of the class C^r such that $\frac{d\varphi}{d\tau} \neq 0$ for every $\tau \in J$.

Lemma. If $f\colon I \to E_n$ is the regular motion of the class C^r then $f \circ \varphi\colon J \to E_n$ is also the regular motion of the class C^r.

Proof. We have $\frac{d(f\circ\varphi)}{d\tau} = \frac{df}{dt}\frac{d\varphi}{d\tau}$ where $\frac{d\varphi}{d\tau}$ is a scalar and $\frac{d(f\circ\varphi)}{d\tau}$ and $\frac{df}{dt}$ are vectors. Indeed, the coordinate expression of $f\circ\varphi$ is $f_1(\varphi(\tau)), \dots, f_n(\varphi(\tau))$. The differentiation with respect to τ means to differentiate, at every component, a composed function with the same inner factor $\varphi(\tau)$. Thus $\frac{d(f\circ\varphi)}{d\tau} = \left(\frac{df_1}{dt}\frac{d\varphi}{d\tau}, \dots, \frac{df_n}{dt}\frac{d\varphi}{d\tau}\right) = \frac{df}{dt}\frac{d\varphi}{d\tau}$. Since $\frac{df}{dt}$ is a nonzero vector and each $\frac{d\varphi}{dt}$ is a nonzero scalar, also every $\frac{d(f\circ\varphi)}{d\tau}$ is a nonzero vector. □

de1.11 **1.11 Definition.** Motion $f\colon I \to E_n$ is called **simple**, if f is an injective mapping, i.e. for $t_1, t_2 \in I$, $t_1 \neq t_2$ we have $f(t_1) \neq f(t_2)$.

From the geometric point of view, f is a motion without self-intersections.

simplde1r12 **1.12 Definition.** The set $C \subset E_n$ is called **a simple curve of the classs** C^r, if there is a simple regular motion $f\colon I \to E_n$ of the class C^r, such that $C = f(I)$.

The map $f\colon I \to E_n$ is called **parametrization of the simple curve** C. and the map φ from 1.10 is called **reparametrization** of the curve C.

1.13 **1.13.** Let J be another interval with the variable τ and $g\colon J \to E_n$ be another parametrization of the curve C of the class C^r. The rule $f\big(\varphi(\tau)\big) = g(\tau)$ deteremines a map $\varphi\colon J \to I$, $t = \varphi(\tau)$, see the picture.

Theorem. φ je funkce tdy C^r a plat $\frac{d\varphi}{d\tau} \neq 0$ pro vechna $\tau \in J$.

Proof. Considering coordinate epxressions $f(t) = \big(f_1(t), \dots, f_n(t)\big)$, $g(\tau) = \big(g_1(\tau), \dots,$
$g_n(\tau)\big)$, the functiion φ is determined by relations

e1 (1) $$f_i(t) = g_i(\tau)\,, \qquad i = 1, \dots, n\,.$$

Consider an arbitrary point $\tau_0 \in J$ and put $t_0 = \varphi(\tau_0) \in I$. Since $\frac{df(t_0)}{dt} \neq o$, at least one of components, say k, of this vector is nonzero. Let us write the relation $f_k(t) = g_k(\tau)$ as

e2 (2) $$f_k(t) - g_k(\tau) = 0\,.$$

The left hand side is a function fo two variables t a τ of the class C^r on the product $I \times J$. We have $\frac{df_k(t_0)}{dt} \neq 0$ hence we can apply the implicit fucntion theorem to the eqaution (2) Therefore, t is determined as a function of the class C^r with the variable τ on an open neighbourhood of the point t_0. At every point we find that $t = \varphi(\tau)$ is a function of the class C^r. Accoring to the geometric situation, this function satisfies all equations in (1), i.e. $g(\tau) = f\big(\varphi(\tau)\big)$. By differentiation we find euality of vectors $\frac{dg}{d\tau} = \frac{df}{dt}\frac{d\varphi}{d\tau}$, where $\frac{d\varphi}{d\tau}$ is a scalar. Here $\frac{d\varphi(t_0)}{d\tau} = 0$ at some point would mean $\frac{dg(\tau_0)}{d\tau} = o$, which is a contradiction with our assumptions. □

We have shown that every two of parametrizations of of the class C^r a simple curve differ by a reparametrization in the sense of 1.10 and 1.12.

globalcurve 1.14 **1.14.** Now we shall introduce the notion of a global curve.

Definition. A subset $C \subset E_n$ is called **curve of the class** C^r, if at each point $p \in C$ there is its neighbourhood U such that $C \cap U$ is a simple curve of the class C^r.

A parametrization of the intersection is called **local parametrization of the curve** C.

1.15 **1.15. Agreement.** Henceforth we shall assume the class r of the curve we consider is sufficiently high for all performed operations. This will not be usually explicitly stated.

1.16 **1.16.** We shall she several example in the plane E_2.

a) Parabola is a global simple curve. b) Circle is a curve but not a simple curve. c) This shape "quarterfoil" is a curve in our (i.e. differential geometric) definition. d) Two circles with the same center can be considered as one curve (connectivity is not assumed in the definition 1.14. It is in fact often useful to say that the border of the annulus determined by these two circles is one curve. JS: fix translation

On the other hand, the whole semicubuc parabola from 1.9, the Descart curve from in e) or the lemniscata in f) are not curves in our sense. JS: missing picture

de1.17 **1.17 Definition.** Two **parametrizations** $f(t)$ and $g(\tau)$ of a simple curve C are called **corresponding each other**, if $\frac{d\varphi}{d\tau} > 0$ where φ is the function from 1.10.

Two congrunet parametrizations determine the same orientation os a simple curve C. The choice of an orientation of C thus means to determine the "direction of motion'. This is possible to do in two ways.

de1.18 **1.18 Definition.** Let $f\colon I \to E_n$ be a local parametrization of the curve C v E_n. The line determined by the point $f(t_0)$, $t_0 \in I$ and the vector $f'(t_0)$ is called **tangent line of** C at the point $f(t_0)$.

This definition is independent on the choice of parametrization, because accoridng to 1.10, two different parametrizations determine colinear vectors. Thus the tangent line at the point $f(t_0)$ has a parametrization

$$f(t_0) + v, f'(t_0)\,, \qquad v \in \mathbb{R}\,.$$

de1.19 **1.19 Definition.** The **deviation of curves** C and $\bar{C}$ at the intersection point p is the angle of their tangent lines at this point.

cdntfa20 **1.20 Definition.** We say two curves C and $\bar{C}$ of the class C^r at the intersection point p have the **contact of the order** k, $k \le r$, if there are local parametrizations $f(t)$ and $\bar{f}(t)$ of, respectively, of C and $\bar{C}$ on the same interval I with $f(t_0) = \bar{f}(t_0) = p$, if

e3 (3)
$$\frac{d^i f(t_0)}{dt^i} = \frac{d^i \bar{f}(t_0)}{dt^i} \qquad \text{for all } i = 1, \dots, k\,.$$

That is, both curves "agree up to the order k" in such parametrization at the given point.

contacpequ21 **1.21 Remark.** It is easy to verify that the "contact of the order k of two curves" is an equivalnce relation.

ve1.22 **1.22 Theorem.** Two curves C and $\bar{C}$ have, at the intersection point p, the contact of the first order, if and only if their tangent lines at the point p are equal.

Proof. If C and $\bar{C}$ have the contact of the first order at the point p, there will exist parametrizations $f(t)$ a $\bar{f}(t)$, $f(t_0) = \bar{f}(t_0) = p$ such that $\frac{df(t_0)}{dt} = \frac{d\bar{f}(t_0)}{dt}$. Thus their tangent lines are equal. In the opposite direction, consider $\bar{C}$ with an arbitrary parametrization $\bar{f}(\bar{t})$, $f(t_0) = \bar{f}(t_0)$. If both tangent lines are equal then $\frac{d\bar{f}(t_0)}{d\bar{t}} = k\ \frac{df(t_0)}{dt}$, $k \neq 0$. Let us perform the reparametrization $\bar{t} = t_0 + \frac{1}{k}(t - t_0)$ of $\bar{C}$. Then using the new parametrization $\bar{f}\big(t_0 + \frac{1}{k}(t - t_0)\big)$ of the curve $\bar{C}$, we have $\frac{d\bar{f}(t_0)}{dt} = \frac{d\bar{f}(t_0)}{d\bar{t}} \cdot \frac{d\bar{t}}{dt} = \frac{d\bar{f}(t_0)}{d\bar{t}}\ \frac{1}{k}$. This is equal to $\frac{df(t_0)}{dt}$ according to the definition of k. Thus C a and $\bar{C}$ have the contact of the first order. $\square$

du1.23 **1.23 Corollary.** Tangent line is the only line which has the contact of the first order with the given curve.

de1.24 **1.24 Definition.** The point $p \in C$ is called **inflection point of the curve** C, if the tangent line at $p$has the contact of the 2nd order with the curve C.

inflection25 **1.25 Theorem.** Let f be a local parametrization of the curve C. Then $p = f(t_0)$ is the inflection point if and only if the vector $\frac{d^2 f(t_0)}{dt^2}$ is colinear with the vector $\frac{df(t_0)}{dt}$.

Proof. Put $v = \frac{df(t_0)}{dt}$. An arbitrary motion along the tangent line has the form $g(t) = p + h(t)v$ where h is a real function. We have $\frac{dg(t_0)}{dt} = \frac{dh(t_0)}{dt}v$, $\frac{d^2 g(t_0)}{dt^2} = \frac{d^2 h(t_0)}{dt^2}v$ which are colinear vectors. If C and its tangent line have the contact of the 2nd order, also vectors $\frac{df(t_0)}{dt}$ a $\frac{d^2 f(t_0)}{dt^2}$ are colinear. In the opposite direction, let $\frac{d^2 f(t_0)}{dt^2} = k\frac{df(t_0)}{dt}$. Consider the parametrization of the tangent line

$$g(t) = p + \big[(t - t_0) + \frac{k}{2}(t - t_0)^2\big]v\,.$$

Then $\frac{dg(t_0)}{dt} = v\frac{df(t_0)}{dt}$, $\frac{d^2 g(t_0)}{dt^2} = kv = \frac{d^2 f(t_0)}{dt^2}$. Thus the curve C has the contact of the 2nd order with the tangent line. □

arclength26 **1.26 Definition.** A parameter s of the parametrization $f\colon I \to E_n$ of the curve C is called **arc-length**, if $\big\|\frac{df}{ds}\big\| = 1$ for all $s \in I$.

Thus the arc-length denotes "motion with constant norm of the velocity" along the curve.

Let $f(t)$ be a parametrization of the curve C. We want to find a reparametrization $s = s(t)$ with the inverse function $t = t(s)$ such that s is the arc. That is,

$$1 = \Big\|\frac{df}{ds}\Big\| = \Big\|\frac{df}{dt}\Big\|\,\Big|\frac{dt}{ds}\Big|\,.$$

Thus $\big|\frac{ds}{dt}\big| = \big\|\frac{df}{dt}\big\|$. Assuming parameters s and t are corresponding each other, we have

$$\frac{ds}{dt} = \Big\|\frac{df}{ds}\Big\| = \sqrt{\Big(\frac{df_1}{dt}\Big)^2 + \cdots + \Big(\frac{df_n}{dt}\Big)^2}\,.$$

This means

e4 (4) $$ds = \sqrt{(f_1')^2 + \cdots + (f_n')^2}\,dt\,.$$

Now we shall find the arc by integration. Thus the arc-length is given on every simple curve up to an additive constant.

The display (4) shows that our notion of the arc-legth agrees with the length of the curve as introduced in calculus. It also agrees with the physical meaning in the sense that if we move along a curve with the unique velocity then the length of the curve agrees with the length of the corresponding time interval.

arcinflection ve1.27 **1.27 Theorem.** Assuming $f(s)$ is the arc-length parametrization, $f(s_0)$ is the inflection point if and only if $\frac{d^2 f(s_0)}{ds^2} = o$.

Proof. The vector $\frac{df}{ds}$ is unit which equivalentnly means

$$\left(\frac{df}{ds}, \frac{df}{ds}\right) = 1\,.$$

Accoridng to the theorem 1.6, the differentiation of this relation yields $2\left(\frac{df}{ds}, \frac{d^2 f}{ds^2}\right) = 0$. That is, the vector $\frac{d^2 f(s_0)}{ds^2}$ is perpendicular to the unit vector $\frac{df(s_0)}{ds}$. These two vector must be colinear at inflection points accoridng to the theorem 1.25. Thus $\frac{d^2 f(s_0)}{ds^2} = o$. □

ve1.28 **1.28 Theorem.** The simple curve C where all point are inflection, points is a part of a line.

Proof. Considering the arc-length parametrization $f(s)$ of the curve C, all points are inflection points if and only if $\frac{d^2 f}{ds^2} = o$. By integration we obtain $\frac{df}{ds} = a$ for a constant vector a. vektor. One more integration yields $f = as + b$ where b is another constant vector. This is a parametrization of a line. □

2 Plane curves

2.1 **2.1.** Let us fix cartesian coordinates (x, y) in E_2. Parametrization of curves has the form $f(t) = \big(f_1(t), f_2(t)\big)$, $\frac{df}{dt} \neq o$. In particularm the graph of teh function $y = f(x)$, $x \in (a, b)$ of the class C^r is a curve of the class C^r. Its parametrization is $g(t) = \big(t, f(t)\big)$, $t \in (a, b)$, thus $\frac{dg}{dt} = \big(1, \frac{df}{dt}\big) \neq o$. We term this **an explicit expression of a plane curve**.

2.2 **2.2.** Recall that a function $f\colon U \to \mathbb{R}$ of two variables defined on an open set $U \subset \mathbb{R}^2$ is of the class C^r if it has continuous partial derivatives on U of all orders $\leq r$.

Theorem. Let $U \subset \mathbb{R}^2$ be an open set and $F\colon U \to \mathbb{R}$ be a function of the class C^r. Assume the set C defined by $F(x, y) = 0$ is nonempty and satisfies $\partial F(x_0, y_0)\colon = \big(\frac{\partial F(x_0,y_0)}{\partial x}, \frac{\partial F(x_0,y_0)}{\partial y}\big) \neq o$ for all $(x_0, y_0) \in C$. Then the curve C is of the class C^r.

Proof. Let $F(x_0, y_0) = 0$ and assume that e.g. $\frac{\partial F(x_0,y_0)}{\partial y} \neq 0$. Then according to the implicit function theorem, the set C can be locally expressed in the form $y = f(x)$ where $f(x)$ is a function of the class C^r. This is a local parametrization of the curve C. If $\frac{\partial F(x_0,y_0)}{\partial x} \neq 0$ we can (again using the implicit function tehorem) express C locally in the form $x = g(y)$. □

Definition. The point (x_0, y_0) such that $F(x_0, y_0) = 0$, $\frac{\partial F(x_0,y_0)}{\partial x} = 0$, $\frac{\partial F(x_0,y_0)}{\partial y} = 0$ is called a **singular point** of the set $F(x, y) = 0$.

pr2.3 **2.3. Examples.** (i) Consider the set $x^2 + y^2 = a$, $a \in \mathbb{R}$, i.e. $F(x, y) = x^2 + y^2 - a$. The set $F(x, y) = 0$ is empty for $a < 0$ and it is a single point for $a = 0$ where both partial derivatives $\frac{\partial F}{\partial x} = 2x$, $\frac{\partial F}{\partial y} = 2y$ are zero. Of course, this point is not a curve. The case $a > 0$ corresponds to the circle with the center at the origin and the radius $\sqrt{a}$. The vector $\partial F = (2x, 2y)$ is then nonzero at all points.
(ii) Consider the Descrat list $F(x, y) = x^3 + y^3 - 3axy = 0$. We have $\partial F = (3x^2 - 3ay, 3y^2 - 3ax)$. Assuming $a = 0$, $(0, 0)$ is the unique single point. Assuming $a \neq 0$, one easily verifies that the equation $\partial F = 0$ has two solutions: $(0, 0)$ and (a, a). The point (a, a) is not on the curve thus $(0, 0)$ the unique singular point.
(iii) In the case of the semicubic parabola we have $F(x, y) = y^2 - x^3 = 0$ m me $\partial F = (-3x^2, 2y)$. Thus the origin is the unique singular point.

implicittangent **2.4 Theorem.** The tangent line to the curve $F(x,y)=0$ at the point(x_0,y_0) is given by the equation

$$\frac{\partial F(x_0,y_0)}{\partial x}(x-x_0)+\frac{\partial F(x_0,y_0)}{\partial y}(y-y_0)=0\,. \tag{1}$$

Proof. Let $\big(f_1(t),f_2(t)\big)$ be a parametrization of this curve with $\big(f_1(t_0), f_2(t_0)\big)=(x_0,y_0)$. Differentiatin of $F\big(f_1(t),f_2(t)\big)=0$ and putting $t=t_0$ we obtain

$$\frac{\partial F(x_0,y_0)}{\partial x}\frac{df_1(t_0)}{dt}+\frac{\partial F(x_0,y_0)}{\partial y}\frac{df_2(t_0)}{dt}=0\,.$$

The vector $\partial F(x_0,y_0)$ is thus perpendicular to the vector $\frac{df(t_0)}{dt}$. The equation (2.4) describes the line through the point (x_0,y_0) which is perpendicular to the vector $\partial F(x_0,y_0)$, i.e. the tangent line. □

The condition $\partial F(x_0,y_0)\neq o$ for a curve given by the equation, thus guarantees existence of the tangent line similarly as the condition $\frac{df(t_0)}{dt}\neq o$ for the parametrization of a curve. There might not be a uniue tangent line at singular points.

The line through a point on the curve which is perpendicular to the tangent line at this point is called **normal line**. The vector $\partial F(x_0,y_0)$ thus yields the direction of the normal line.

implparcont **2.5.** According to 1.20, two plane curves C and $\bar C$ have, at a intersection point p, the contact of the kth order if there exist local parametrizations $\big(f_1(t),f_2(t)\big)$ and $\big(\bar f_1(t),\bar f_2(t)\big)$ of, respectively, C and $\bar C$ on the same interval I such that

parcontact
$$\frac{d^i f_1(t_0)}{dt^i}=\frac{d^i \bar f_1(t_0)}{dt^i}\,,\ \frac{d^i f_2(t_0)}{dt^i}=\frac{d^i \bar f_2(t_0)}{dt^i}\,,\quad i=1,\dots,k, \tag{2}$$

where t_0 is the parametr of the intersection point p. The direct approach to the question whether such parametrizations do or do not exist is rather complicated in general. However, there is a very simple answer if $\bar C$ is given by the equation $F(x,y)=0$. In this case we can form the one variable function

$$\Phi(t)=F\big(f_1(t),f_2(t)\big)\,. \tag{3}$$

Theorem. Curves $C\equiv\big(f_1(t),f_2(t)\big)$ and $\bar C\equiv F(x,y)=0$ have, at a intersection point $(x_0,y_0)=\big(f_1(t_0),f_2(t_0)\big)$ the contact of the kth order if and only if

implcontact
$$\frac{d^i\Phi(t_0)}{dt^i}=0\,,\qquad i=1,\dots,k\,. \tag{4}$$

Proof. Let $\big(\bar f_1(t), \bar f_2(t)\big)$ be a local parametrization of the curve $\bar C$ such that the condition (2) for the contact is satisfied. Then

$$F\big(\bar f_1(t), \bar f_2(t)\big) = 0 \tag{5}$$

for all t, i.e. all derivatives of the composed function of the left hand side are zero. Also the function Φ is composed with outer factor $F(x,y)$ and inner factors $f_1(t)$ and $f_2(t)$. According to our assumption about the contact, the derivatives up to the order k of inner factors at the point t_0 are the same for both $\bar f_1(t)$ and $\bar f_2(t)$. Thus (4) holds.

In the opposite direct, assume (4) holds. Further assume $\frac{\partial F(x_0,y_0)}{\partial y} \neq 0$. We shall locally parametrize the curve $\bar C$ in the form $\big(f_1(t), g(t)\big)$ where the function $g(t)$ is determined by

$$F\big(f_1(t), g(t)\big) = 0\,. \tag{6}$$

This is always possible. Indeed, consider

$$G(t,y) = F\big(f_1(t), y\big)\,,$$

which is well defined on some neighbourhood V of the point (t_0, y_0). We have

$$\frac{\partial G(t_0, v_0)}{\partial y} = \frac{\partial F(x_0, y_0)}{\partial y} \neq 0\,,$$

thus we can use the implicit function theorem for the function $G(t,y) = 0$. We need to show that

f2gcontact

$$\frac{d^i f_2(t_0)}{dt^i} = \frac{d^i g(t_0)}{dt^i}\,, \qquad i = 1, \dots, k\,. \tag{7}$$

On $V \times \mathbb{R}$ we shall consider the function H of three variabls,

$$H(t,y,z) = G(t,y) - z\,.$$

It satisfies $H(t_0, y_0, 0) = 0$ and $\frac{\partial H(t_0,y_0,0)}{\partial y} = \frac{\partial G(t_0,y_0)}{\partial y} \neq 0$. Hence using the implicit function theorem once more, from $H = 0$ we can locally (and uniquelly) compute $y = K(t,z)$. Since $G\big(t, g(t)\big) = 0$ and $G\big(t, f_2(t)\big) = \Phi(t)$, we have

$$g(t) = K(t,0) \quad \text{a} \quad f_2(t) = K\big(t, \Phi(t)\big)\,.$$

Similarly as in the first part of the proof, we have th same outer factor $K(t,z)$. Derivations of the constant function $t \mapsto 0$ and the function $\Phi(t)$ up to the order k at the point t_0 agree (because they are zero). According to the chain rule, (7) implies (4) □

2.6 **2.6.** Now we shall discuss how to approximate an arbitrary plane curve C at a given point p using circles.

Definition. A circle which has the 2nd order contact with the curve C at the point $p \in C$, is called **osculating circle** at the point p.

ve2.7 **2.7 Theorem.** Assume $p \in C$ is not an inflection point. Then there is exactly one osculating curve at p.

Proof. Denote by (a, b) the center and by r the radius of the circle, i.e. its equation is

ocircle (8)
$$(x - a)^2 + (y - b)^2 - r^2 = 0\,. \tag{8}$$

Using the theorem 2.5 we shall find a condition for (8) to have the 2nd order contact with the curve given by the parametrization $\big(f_1(t), f_2(t)\big)$ at the point t_0. We have

$$\begin{aligned}
\Phi(t) &= \big(f_1(t) - a\big)^2 + \big(f_2(t) - b\big)^2 - r^2\,,\\
\Phi'(t) &= 2(f_1 - a)f_1' + 2(f_2 - b)f_2'\,,\\
\Phi''(t) &= 2(f_1')^2 + 2(f_1 - a)f_1'' + 2(f_2')^2 + 2(f_2 - b)f_2''\,.
\end{aligned}$$

Coordinates a, b are solutions of the equation $\Phi' = 0$, $\Phi'' = 0$ which are equivalent to

osystem (9)
$$\begin{aligned}
af_1' + bf_2' &= f_1 f_1' + f_2 f_2'\,,\\
af_1'' + bf_2'' &= f_1 f_1'' + f_2 f_2'' + f_1'^2 + f_2'^2\,.
\end{aligned} \tag{9}$$

Away from inflection points, vectors (f_1', f_2') and (f_1'', f_2'') are linearly independent. Hence the determinant of the system (9) is nonzero and these equations determine the unique pair (a, b). The radius r is then given by the equation $\Phi = 0$. $\square$

ve2.8 **2.8 Theorem.** The radius r of the osculationg curve satisfies

oradius (10)
$$r^2 = \frac{(f_1'^2 + f_2'^2)^3}{(f_1' f_2'' - f_2' f_1'')^2} \tag{10}$$

Proof. One computes (using e.g. the Cramer's rule) from (9) that

$$a = f_1 - \frac{f_2'(f_1'^2 + f_2'^2)}{\begin{vmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{vmatrix}}\,, \qquad b = f_2 + \frac{f_1'(f_1'^2 + f_2'^2)}{\begin{vmatrix} f_1' & f_2' \\ f_1'' & f_2'' \end{vmatrix}}\,.$$

The relation $r^2 = (f_1 - a)^2 + (f_2 - b)^2$ then yields (10). $\square$

2.9 **2.9.** Osculating curves do not exist at inflection points. The tangent line has the 2nd order contact with the curve hence this curve would have to have the contact of the 2nd order with the osculating curve accoring to 1.21. However, a simple computation reveals that a circle has the contact of the 1st order with its tangent line.

Indeed, we can choose such coordinate system such that the circle is given by the parametrization $x = r\cos t$, $y = r\sin t$. Its tangent line at the point $t = 0$ has the equation $x - r = 0$. We have $\Phi(t) = r\cos t - r$ thus $\Phi(0) = 0$, $\Phi'(0) = -r\sin 0 = 0$ but $\Phi''(0) = r\cos 0 \neq 0$. JS: missing picture

planecurvature de2.10 **2.10 Definition.** Let r be the radius of the osculating curve at the point $p \in C$ (away from inflection points). The number $\varkappa = \frac{1}{r}$ is called **curvature of the curve** C vat the point p. We define $\varkappa = 0$ at inflection points.

This terminology is motivated by the observation that a smaller radius of the osculating curve means the curve is more “curved”.

The center of the osculating curve is called **center of the curvature** of the curve C at the given point.

de2.11 **2.11 Definition.** Assume $p \in C$ is not the inflection point. If the osculating curve at $p \in C$ has the contact of the 3rd order with C the p is called **vertex of the curve**.

We shall show in 2.16 that in the case of an ellipse, our general notion of vertices agrees with the usual definition of vertices of ellipses.

The oscullating curve at vertex of a curve is called **hyperosculating**.

arccurvature 2.12 **2.12.** Consider the arc-length parameter s. Then the vector $e_1 = \frac{df}{ds}$ is unit and perpendicular to the vector $\frac{de_1}{ds} = \frac{d^2 f}{ds^2}$ according to 1.26 and the proof the theorem 1.27. Here the inflection point $f(s_0)$ is characterized by $\frac{de_1(s_0)}{ds} = o$.

At the point $f(s_0)$ (away from inflection points) we denote by $e_2(s_0)$ the unit vector parallel with $\frac{de_1(s_0)}{ds}$ in the same direction. Thus $e_1(s_0)$ and $e_2(s_0)$ is a pair of orthonormal vectors.

Theorem. We have $\left\|\frac{de_1(s_0)}{ds}\right\| = \varkappa(s_0)$ and the center of the osculating circle lies on the halfline given by the point $f(s_0)$ and the vector $e_2(s_0)$.

Proof. One can derive this from the expression for the center of the osculating curve in (10). But it will be useful for later considerations to perform the whole computation once more (with some simplifications). Since the osculating circle has the contact of the 1st order with the tangent line, its center lies on the normal line. Thus this center is of the form $f(s_0)+re_2(s_0)$ for some $r \in \mathbb{R}$. The equation of the circle with this center and the radius r can be written using the scalar product as

$$\big(z - f(s_0) - re_2(s_0), z - f(s_0) - re_2(s_0)\big) - r^2 = 0\,,$$

where $z = (x, y)$ is an arbitrary point of the plane. For the computation of the contact we shall therefore use the function

$$\Phi(s) = \big(f(s) - f(s_0) - re_2(s_0), f(s) - f(s_0) - re_2(s_0)\big) - r^2\,.$$

By the differentiation and using 1.6 we obtain

$$\frac{1}{2}\frac{d\Phi}{ds} = \big(e_1(s), f(s) - f(s_0) - re_2(s_0)\big)\,.$$

Conditions $\Phi(s_0) = 0$ and $\frac{d\Phi(s_0)}{ds} = 0$ are satisfied; geometrically this follows from the fact that we chose the center on the normal line. One more differentiaion yields

e2.11 (11)
$$\frac{1}{2}\frac{d^2\Phi}{ds^2} = \Big(\frac{de_1(s)}{ds}, f(s) - f(s_0) - re_2(s_0)\Big) + \big(e_1(s), e_1(s)\big)\,.$$

This must be zero at the point s_0 hence

e2.12 (12)
$$r\Big(\frac{de_1(s_0)}{ds}, e_2(s_0)\Big) = 1\,.$$

Since the vector $\frac{de_1(s_0)}{ds}$ is congruently parallel with the vector $e_2(s_0)$, the scalar product (12) is equal to the norm of this vector. Our statement is thus a direct conseuqence of the definition 2.10. □

e12curvature da2.13 **2.13 Corollary.** We have $\frac{de_1(s)}{ds} = \varkappa(s)e_2(s)$.

Proof. It follows from the theorem 2.12 away from inflection points and from the theorem 1.27 in inflection points. □

ve2.14 **2.14 Theorem.** We have $\frac{de_2(s)}{ds} = -\varkappa(s)e_1(s)$.

Proof. Since e_2 is a unit vector, we have $(e_2, e_2) = 1$. By differentiation we obtain $\big(e_2, \frac{de_2}{ds}\big) = 0$. Thus the vector $\frac{de_2}{ds}$ is perpendicular to e_2, i.e.

$\frac{de_2}{ds} = ce_1$. Since vectors e_1 and e_2 are perpendicular, we have $(e_1, e_2) = 0$. By differentiation we obtain

$$0 = \Big(\frac{de_1}{ds}, e_2\Big) + \Big(e_1, \frac{de_2}{ds}\Big) = \varkappa + c\,.$$

□

ve2.15 **2.15 Theorem.** The point $f(s_0)$ is the vertex if and only if $\frac{d\varkappa(s_0)}{ds} = 0$.

Proof. We shall continue in the proof of the theorem 2.12 and use also the corollary 2.13. We obtain

$$\frac{1}{2}\frac{d^2\phi}{ds^2} = \varkappa(s)\big(e_2(s), f(s) - f(s_0) - re_2(s_0)\big) + 1\,.$$

Further differentiation yields the condition of the contact of the 3rd order

$$0 = \frac{1}{2}\frac{d^3\phi(s_0)}{ds^3} = \frac{d\varkappa(s_0)}{ds}\cdot(-r) + \varkappa(s_0)\left[\big(-\varkappa(s_0)e_1(s_0), -re_2(s_0)\big)\right]\,.$$

Our statement now follows from $e_1(s_0) \perp e_2(s_0)$ and $r \neq 0$. □

vr2h16 **2.16 Corollary.** Consider an arbitrary parametrization $f(t)$ of the curve C. Then away from inflection points, the point $f(t_0)$ is vertex if and only if $\frac{d\varkappa(t_0)}{dt} = 0$.

Proof. The transformation from t to s is realized using a reparametrization $t = \varphi(s)$, $t_0 = \varphi(s_0)$. It follows form the chain rule that

$$\frac{d\varkappa(\varphi(s_0))}{ds} = \frac{d\varkappa(t_0)}{dt}\,\frac{d\varphi(s_0)}{ds}\,.$$

Here $\frac{d\varphi(s_0)}{ds} \neq 0$ since φ is a reparametrization. □

From this in particularly follows that vertex of an ellipse in the differential geometric sense are vertices of an ellipse in the classical sense since the curavture at these points achieves maximum of minimum.

onlyvert2c17 **2.17 Theorem.** A simple curve whose every point is vertex, is a part of the circle.

Proof. The center of the osculating circle is $c(s) = f(s) + \frac{1}{\varkappa}e_2(s)$. If every point is vertex, $\varkappa$ will be a constant. Thus by differentiation and using 2.13, we obtain

$$\frac{dc(s)}{ds} = e_1(s) - \frac{1}{\varkappa}\,\varkappa e_1(s) = o\,.$$

Thus $c(s)$ is a fixed point and also the radius $\frac{1}{\varkappa}$ is constant. All osculating curves thus coincide and the curve lies on a circle. □

2.18 Definition. Relations

e2.13 (13)
$$\frac{df}{ds} = e_1\,, \quad \frac{de_1}{ds} = \varkappa e_2\,, \quad \frac{de_2}{ds} = -\varkappa e_1$$

are called **Frenet formulae** of the plane curve C without inflection points. "Moving" frame $\big(f(s), e_1(s), e_2(s)\big)$ is called **Frenet frame** of the curve C.

2.19 **2.19.** Now we shall show how to use relations (13) in order to characterize congruence of plane curves.

Definition. Curves C and $\bar{C} \subset E_2$ are called **congruent** if there is an Euclidean transformation $\varphi\colon E_2 \to E_2$ such that $\varphi(C) = \bar{C}$.

ve2.20 **2.20 Theorem.** Let C, $\bar{C}$ be curves without inflection points, $f\colon I \to E_2$, $\bar{f}\colon I \to E_2$, respectively their arc-length parametrizations on the same interval I and $\varkappa(s)$, $\bar{\varkappa}(s)$, respectively be their curvatures. Then curves C and $\bar{C}$ are congruent if and only if $\varkappa = \bar{\varkappa}$ on I.

Proof. One direction is obvious: an Euclidean transformation maps arc-length to arc-length and preserves the contact thus radii of osculating circles at corresponding points must be the same. In the opposite direction, consider C resp. $\bar{C}$ with Frenet frame $\big(f(s), e_1(s), e_2(s)\big)$ resp. $\big(\bar{f}(s), \bar{e}_1(s), \bar{e}_2(s)\big)$. Thus beside (13) we have also

e2.14 (14)
$$\frac{d\bar{f}}{ds} = \bar{e}_1\,, \quad \frac{d\bar{e}_1}{ds} = \varkappa \bar{e}_2\,, \quad \frac{d\bar{e}_2}{ds} = -\varkappa \bar{e}_1$$

with teh same $\varkappa$. Thus (13) and (14) is the same system of differential equations for the 6-tuple of real functions which are components of f, e_1 and e_2. Given $s_0 \in I$, the triple of vectors $f(s_0)$, $e_1(s_0)$, $e_2(s_0)$ as well as the triple $\bar{f}(s_0)$, $\bar{e}_1(s_0)$, $\bar{e}_2(s_0)$ is formed by the point and the pair of orthonormal vectors. Hence there exists a unique Euclidean transformation $\varphi\colon E_2 \to E_2$ which transforms $f(s_0)$ to $\bar{f}(s_0)$, $e_1(s_0)$ to $\bar{e}_1(s_0)$ and $e_2(s_0)$ to $\bar{e}_2(s_0)$. Thus the parametrization $\bar{f}\colon I \to E_2$ of the curve $\bar{C}$ together with vector functions $\bar{e}_1(s)$, $\bar{e}_2(s)$ and the parametrization $\varphi \circ f\colon I \to E_2$ of the curve $\varphi(C)$ together with vector functions $\varphi \circ e_1$, $\varphi \circ e_2$ satisfy the same system of differential equations with the same initial conditions. According to the theorem of the unique existence of a solution of a the system of differential equations, we have $\bar{f} = \varphi \circ f$, $\bar{e}_1 = \varphi \circ e_1$, $\bar{e}_2 = \varphi \circ e_2$. The first relation $\bar{f} = \varphi \circ f$ implies $\bar{C} = \varphi(C)$. □

pr2.21 **2.21 Example.** The assumption that curves C and $\bar{C}$ are without inflection points, is essential. Consider curves given explicitly as $y = x^3$ and

$y = |x|^3$, $x \in (-\infty, \infty)$. Both curves are of the class C^2 and have the same curvature as a function of the arc-length. But they are not congruent. JS: missing picture

2.22 **2.22.** The next statement can be briefly rephrased as that we can prescribe the curvature arbitrarily.

Theorem. Let $\varkappa \colon I \to \mathbb{R}$ be a positive function. Then locally there exists a curve C parametrized by arc-length on I such that $\varkappa$ is its curvature.

Idea of the proof: We solve the system of equations (13). □

Remark. Globally, this curve might not be simple. For example, if $\varkappa = \frac{1}{r}$ is a constant, the solution of teh corresponding system of differential equations is the circle $x = r\cos\frac{s}{r}$, $y = r\sin\frac{s}{r}$. Then for $s \in (-\infty, \infty)$ one goes along the circle repeatedly.

2.23 **2.23.** We shall finish this section with a global result about plane curves. Recall the subset in E_2 is called bounded if it lies inside a circle.

Definition. The plane curve C of the class C^r is called **oval** of the class C^r if it is the border of a bounded convex set in E_2.

Examples: (i) (ii) JS: missing picture

ve2.24 **2.24. Four vertex theorem.** Each oval of the class C^3 without points of inflection, has at least four vertices.

Proof. Consider C parametrizad by the arc-length $f(s) = \big(f_1(s), f_2(s)\big)$ on the interval $s \in [0, a]$ such that for $s = a$ the oval closes, i.e. $f(0) = f(a)$. Thus the curvature $\varkappa$ is in fact defined on the closed interval hence it reaches its maximum and minimum. This yields two vertices of the oval C. We can assume that $\varkappa$ ha minimum at $s = 0$ and it has maximum at some point $b \in (0, a)$. Choose $f(0)$ to be the origin, $f(b)$ on the x-axis and the orientation of the y-axis such that $f_2(s) > 0$ for $s \in (0, b)$. (If this holds for one point, it holds for all points by the convexity). Then $f_2(s) < 0$ for $s \in (b, a)$ again using the convexity. The case $\varkappa(0) = \varkappa(b)$ i.e. $\varkappa$ equal to a constant, is the circle (according to Theorem 2.17) and we can exclude this case. JS: missing picture

Assume now that $f(0)$ and $f(b)$ are unique vertices. Then $\frac{d\varkappa}{ds} > 0$ on $(0, b)$ and $\frac{d\varkappa}{ds} < 0$ on (b, a). The integration by parts now yields

$$0 < \int_0^a \frac{d\varkappa}{ds} f_2 ds = \big[\varkappa f_2\big]_0^a - \int_0^a \varkappa \frac{df_2}{ds}\, ds\,.$$

But $\left[\varkappa f_2\right]_0^a = 0$ because $f(0) = f(a)$ and $\varkappa(0) = \varkappa(a)$.

Let us expand the relation vztah $\frac{de_1}{ds} = \varkappa e_2$. We have $e_1 = \left(\frac{df_1}{ds}, \frac{df_2}{ds}\right)$. Since e_2 is the unit vector perpendicular to e_1, we have $e_2 = \pm\left(\frac{df_2}{ds}, \frac{df_1}{ds}\right)$ hence $\frac{d^2 f_1}{ds^2} = \pm\varkappa \frac{df_2}{ds}$. Thus

$$0 < -\int_0^a \varkappa \frac{df_2}{ds}\, ds = \pm \int_0^a \frac{d^2 f_1}{ds^2}\, ds = \pm\left[\frac{df_1}{ds}\right]_0^a = 0\,,$$

because $\frac{df_1(0)}{ds} = \frac{df_1(a)}{ds}$ according to the periodicity of the parametrization of the oval. This is a contradiction.

In fact we have shown there exists another point where $\frac{d\varkappa}{ds}$ changes the sign, i.e. $\varkappa$ has either minimum or maximum at this point. But minima and maxima appear in pairs. Thus the existence of the fourth vertex follows. $\square$

3 Envelope of a family of plane curves

3.1 **3.1.** Consider the one-parameter family of plane curves given by the equations

e3.1 (1)
$$F(x, y, t) = 0\,,$$

$t \in I$ where $F(x, y, t)$ is a function of the class C^1 defined on an open set $U \subset \mathbb{R}^3$. Let us denote by C_{t_0}, $t_0 \in I$ the curve of the equation $F(x, y, t_0) = 0$, i.e. we consider (1) as the system of plane curves (C_t).

3.2 **3.2.** Intersection points of curves C_t and C_s, $t \neq s$ are determined by the pair of equations

$$F(x, y, t) = 0\,, \quad F(x, y, s) = 0\,.$$

This system of equations is oviously equivalent with the system

$$F(x, y, t) = 0\,, \quad \frac{F(x, y, s) - F(x, y, t)}{s - t} = 0\,.$$

Considering a fixed t, we obtain the following equation in the limit $s \to t$,

e3.2 (2)
$$F(x, y, t) = 0\,, \quad \frac{\partial F(x, y, t)}{\partial t} = 0\,.$$

Definition. Points determined by the equation (2) are called **characteristic points on the curve** C_t. The set of such points for all $t \in I$ is called **charakteristic set of the system** (C_t).

From the computational point of view, we have two basic possibilities how o express the characteristic set. If we eliminate the parametr t from (2), we express the characteristic set in the form of an equation $G(x,y)=0$. If we compute x and y from (2) as function of t, we obtain a parametrization of the charatcteristic set.

3.3 **3.3.** We say **two curves touch each other in the intersection point** if they have the contact of the 1st order, i.e. the same tangent line.

Definition. The curve D with a parametrization $f(t)$, $t \in (a,b) \subset I$ is called **envelope** of the family (C_t) if D touches the curve C_{t_0} at the point $f(t_0)$ for all $t_0 \in (a,b)$.

ve3.4 **3.4 Theorem.** Each envelope of the family (C_t) is a subset of its characteristic set.

Proof. The condition that each point of the envelope $f(t) = \big(f_1(t), f_2(t)\big)$ lies on the curve C_t, is

e3.3 (3)
$$F\big(f_1(t), f_2(t), t\big) = 0\,.$$

The condition that tangent lines for D and C_t coincide at the point $f(t)$, has the form

e3.4 (4)
$$\frac{\partial F(f_1(t), f_2(t), t)}{\partial x}\,\frac{df_1(t)}{dt} + \frac{\partial F(f_1(t), f_2(t), t)}{\partial y}\,\frac{df_2(t)}{dt} = 0\,.$$

By differentiation of (3) we obtain

e3.5 (5)
$$\frac{\partial F(f_1(t), f_2(t), t)}{\partial x}\,\frac{df_1(t)}{dt} + \frac{\partial F(f_1(t), f_2(t), t)}{\partial y}\,\frac{df_2(t)}{dt} + \frac{\partial F(f_1(t), f_2(t), t)}{\partial t} = 0\,.$$

Subtracting (4) from (5) yields

e3.6 (6)
$$\frac{\partial F(f_1(t), f_2(t), t)}{\partial t} = 0\,.$$

Thus every envelope is a part of the characteristic set. □

3.5 **3.5.** Consider a very simple case of the family of circles centered on the x-axis and the constant radius r. That is, $F(x,y,t) = (x-t)^2 + y^2 - r^2 = 0$. Then $\frac{\partial F}{\partial t} = -2(x-t) = 0$. Putting $x = t$ to the first equation, we get $y = \pm r$. Of course, both these lines are envelopes.

3.6 **3.6.** In the opposite direction, we have the following:

Theorem. If the curve $f(t)$ is a solution of (2), it is an envelope of the family (C_t).

Proof. The curve $f(t)$ satisfies (3), hence $f(t) \in C_t$. By differentiation we obtain (5). Further we have (6) and subtracting (6) from (5), we obtain (4). Thus $f(t)$ touches the curve C_t. □

3.7 **3.7.** Having a pair of functions $x = f_1(t)$, $y = f_2(t)$ which is a solution of (2), then it is an envelope of the family (C_t) assuming further that $f(t) = \big(f_1(t), f_2(t)\big)$ is a curve. In particular $\frac{df}{dt} \neq o$. The picture shows first the family of curves centered on a cirle of the radius r with constant radius $\varrho < r$, where the inner and outer envelopes are cicrles. The second case is $\varrho = r$, where the inner envelopes "degenerates" to a point.

JS: missing picture

3.8 **3.8.** Normal lines of an arbitrary plane curves C form a one-parametr family of curves.

Definition. The characteristic set of the family of normal lines of the curve C is called **evolute of the curve** C.

Theorem. Evolute of the curve C without inflection points coincides with the set of centers of their osculating curves.

Proof. Let $z = (x, y)$ be an arbitrary point in the plane. We shall parametrize C by the arc-length and consider its Frener frame $e_1(s)$, $e_2(s)$ at the point $f(s)$. The equation of the normal line at the point $f(s)$ then is

e3.7 (7) $$F(x, y, s) = \big(e_1(s), z - f(s)\big) = 0\,.$$

Using Frenet formulae. we find the condition

e3.8 (8) $$\frac{\partial F}{\partial s} = \big(\varkappa(s) e_2(s), z - f(s)\big) - \big(e_1(s), e_1(s)\big) = 0\,.$$

The characteristic set is solution of equations (7) and (8), which we shall find geometrically. It follows from (7) that plyne

$$z = f(s) + c(s) e_2(s)\,.$$

Putting this into (8), we get $\varkappa(s)c(s) - 1 = 0$, thus $c(s) = \frac{1}{\varkappa(s)}$. This is the center of the osculating circle. □

3.9. Above we found the parametric expression of the evolute,

$$z(s) = f(s) + \frac{1}{\varkappa(s)} e_2(s) .$$

Thus $\frac{dz}{ds} = e_1(s) - \frac{\varkappa'(s)}{\varkappa s^2} e_2(s) - e_1(s)$. If $\varkappa'(s) \neq 0$, this vector is nonzero. This means, that in some neighbourhood of the point, which is not a vertex, is evolute a cirve.

The picture shows the evolute of an ellipse. Their edges correspond to vertices of the ellipse. JS: missing picture

4 Spacial curves and surfaces

Beside a parametric expression, a spacial curve can be also given as an intersection of two surfaces. In the study of spacial curves, we shall use their contact with certain auxiuliary surfaces. Now we shall give a general definition of surfaces in E_3.

4.1 **4.1.** We shall need notion of vector functions of two variables. To simplify the rpesentation, we shall work only with 3-dimensional Euclidean vector space V.

Coordinates of the point $u \in \mathbb{R}^2$ will be denoted by (u_1, u_2). Let $D \subset \mathbb{R}^2$ be an open set. The mapping $w \colon D \to V$ will be called **vector function of two variables**. If e_1, e_2, e_3 is a basis of V, we have $w(u) = w(u_1, u_2) = w_1(u_1, u_2)e_1 + w_2(u_1, u_2)e_2 + w_3(u_1, u_2)e_3$. Real function w_1, w_2, w_3 are called **components of the vector function** w and we write

e4.1 (1) $$w(u_1, u_2) = \big(w_1(u_1, u_2), w_2(u_1, u_2), w_3(u_1, u_2)\big) .$$

The limit and continuity of vector functions of the vector function w are defined similarly as in 1.3. We say w has the limit $v \in V$ at the point $u_0 = (u_1^0, u_2^0)$, if for each $\varepsilon > 0$ there is $\delta > 0$ such that $|u_1 - u_1^0| < \delta$, $|u_2 - u_2^0| < \delta$, $(u_1, u_2) \neq (u_1^0, u_2^0)$ implies $\|w(u_1, u_2) - v\| < \varepsilon$. We write $\lim\limits_{u \to u_0} w(u) = v$. Further, w is continuous at the point u_0 if $\lim\limits_{u \to u_0} w(u) = w(u_0)$.

4.2 **4.2.** Partial derivatives of the vector fucntion w are defined by

$$\frac{\partial w(u_0)}{\partial u_1} = \lim_{u_1 \to u_1^0} \frac{w(u_1, u_2^0) - w(u_1^0, u_2^0)}{u_1 - u_1^0} , \qquad \frac{\partial w(u_0)}{\partial u_2} = \lim_{u_2 \to u_2^0} \frac{w(u_1^0, u_2) - w(u_1^0, u_2^0)}{u_2 - u_2^0}$$

Higher order partial derivatives $\frac{\partial^k w}{\partial u_1^i \partial u_2^j}$, $i + j = k$, are defined by the usual iteration.

As in 1.5, a vector function is continuous if and only if all its components are continuos. An anlogousl statement holds also for limits and partial derivatives. In particular, following (1) we have

$$\partial_1 w\colon = \frac{\partial w}{\partial u_1} = \Big(\frac{\partial w_1}{\partial u_1}, \frac{\partial w_2}{\partial u_1}, \frac{\partial w_3}{\partial u_1}\Big)\,, \quad \partial_2 w\colon = \frac{\partial w}{\partial u_2} = \Big(\frac{\partial w_1}{\partial u_2}, \frac{\partial w_2}{\partial u_2}, \frac{\partial w_3}{\partial u_2}\Big)$$

and similarly for higher order partial derivatives.

We say the function $w\colon D \to V$ is **of the class** C^r, if it has continuous partial derivatives of the order $\leq r$ at the point D.

4.3 **4.3.** Consider V as the associate vector space of E_3. Choose an auxiliary origin $P \in E_3$. Then the mapping $f\colon D \to E_3$ determines **radius vector** which is the vector function $\overrightarrow{Pf}\colon D \to V$, $\overrightarrow{Pf}(u) = \overrightarrow{Pf(u)}$. We put

e4.2 (2) $$\partial_1 f = \frac{\partial f}{\partial u_1} = \frac{\partial(\overrightarrow{Pf})}{\partial u_1}\,, \quad \partial_2 f = \frac{\partial f}{\partial u_2} = \frac{\partial(\overrightarrow{Pf})}{\partial u_2}\,.$$

Similarly as in 1.7, this does not depend on the choice of the origin P.

Here (2) are vector functions of two variables. By iteration we have

e4.3 (3) $$\partial_{11} f = \frac{\partial^2 f}{\partial u_1 \partial u_1}\,, \quad \partial_{12} f = \frac{\partial^2 f}{\partial u_1 \partial u_2}\,, \quad \partial_{22} f = \frac{\partial^2 f}{\partial u_2 \partial u_2}$$

and similarly for higher orders.

de4.4 **4.4 Definition.** The set $S \subset E_3$ is called **simple surface of the class** C^r, if there is ana open set $D \subset \mathbb{R}^2$ and an injective mapping $f\colon D \to E_3$ of the class C^r such that $S = f(D)$ and vectors $\partial_1 f$ a $\partial_2 f$ are linearly independent at each point of the set D.

JS: missing picture

We say f is **parametrization of the surface** S and D is **parameter space**.

The condition that vectors $\partial_1 f$ and $\partial_2 f$ are linearly indepedent shall be written in the form $\partial_1 f \times \partial_2 f \neq o$ where $\times$ denotes the vector product. We shall illustrate the meaning of this condition on a parametrization of the plane E_3. Consider $D = \mathbb{R}^2$ and put

$$f = P + u_1 a \ + u_2 b\,, \quad P \in E_3\,, \quad a, b \in V,\ u_1, u_2 \in \mathbb{R}\,.$$

In coordinates (x, y, z) on E_3 we have

$$x = p_1 + u_1 a_1 + u_2 b_1\,, \quad y = p_2 + u_1 a_2 + u_2 b_2\,, \quad z = p_3 + u_1 a_3 + u_2 b_3\,.$$

Then $\partial_1 f = a$ a $\partial_2 f = b$. From the analytic geometry we know that f determines the plane if and only if vectors a and b are linearly independent. If these vector are linearly dependent, we obtain only a line, and the case $a = b = o$ yiedls only a point.

4.5 **4.5.** Given the fucntion $z = f(x, y)$ tdy C^r of two variables on $D \subset \mathbb{R}^2$ then its graph $\bar{f}(x, y) = \big(x, y, f(x)\big)$, $\bar{f} \colon D \to \mathbb{R}^3$ is a simple surface of the class C^r. The reason is that $\partial_1 \bar{f} = \big(1, 0, \frac{\partial f}{\partial x}\big)$, $\partial_2 \bar{f} = \big(0, 1, \frac{\partial f}{\partial y}\big)$ and these vectors are linearly independent everywhere. We say that $f(x, y)$ is an explicit description of the surface.

de4.6 **4.6 Definition.** The subset $S \subset E_3$ is called **surface of the class** C^r if for each $p \in S$ there is its neighbourhood U such that $U \cap S$ is a simple surface of the class C^r.

Examples. JS: missing picture

a) Rotational paraboloid is globally a simple surface. b) The sphere JS: check the translation
is a surface which is not simple. c) Anuloid is the surface given by the rotation of the circle around the axis which lies in the same plane and has empty intersection with the cicrle. The physical model is “pneumatika”. JS: check the translation
d) Also “an v lec” is an interesting global example of a surface.

4.7 **4.7. Agreement.** Further we shall assume that the class r of the the suface or function under consideration is high enough for required constructions and this will not be usually explicitly stated.

4.8 **4.8.** A curve on a surface will be usually given in the parameter space D, i.e. $u = u(t)$, tj. $u_1 = u_1(t)$, $u_2 = u_2(t)$, $t \in I$. On the surface $S = f(D)$ then we have the curve $f\big(u(t)\big) = f\big(u_1(t), u_2(t)\big)$.

Theorem. Tangent lines of all curves on the surface S at the point $pinS$ fill the plane which is called **tangent plane of the surface** S at the point p.

Proof. Let $p = f(u_0)$. The velocity vector of the motion $f\big(u(t)\big)$, $u(t_0) = u_0$ is given by the differentiation of the composed function

e4.4 (4)
$$\frac{df(u_1(t_0), u_2(t_0))}{dt} = \frac{\partial f(u_0)}{\partial u_1}\frac{du_1(t_0)}{dt} + \frac{\partial f(u_0)}{\partial u_2}\frac{du_2(t_0)}{dt}\,.$$

Hence this is a linear combination of vectors $\partial_1 f(u_0)$ and $\partial_2 f(u_0)$. In the opposite direction, for arbitrary vector $a = a_1 \partial_1 f(u_0) + a_2 \partial_2 f(u_0)$ we have the motion $u(t) = \big(u_1(t), u_2(t)\big)$ such that $\frac{du_1(t_0)}{dt} = a_1$, $\frac{du_2(t_0)}{dt} = a_2$. Considered tangent lines this fill the whole plane given by the point p and vectors $\partial_1 f(u_0)$ and $\partial_2 f(u_0)$. □

The tangent plane of the surafce S at the point p will be denoted by $\tau_p S$ and its associated vector space $T_p S$ is called **tangent vector space** of S at the point p.

The previous theorem shows the geometric meaning of the condition of linear independence of vectors $\partial_1 f$ a $\partial_2 f$ which guarantees existence of the tangent plane.

4.9 **4.9.** Henceforth we fix the coordinate system (x, y, z), i.e. $E_3 \approx \mathbb{R}^3$.

Theorem. Let $U \subset \mathbb{R}^3$ be an open set and $F : U \to \mathbb{R}$ is a function of the class C^r such that the set S given by the equation $F(x, y, z) = 0$ is nonempty and

$$\partial F(x_0, y_0, z_0) \colon = \Big(\frac{\partial F(x_0, y_0, z_0)}{\partial x}, \frac{\partial F(x_0, y_0, z_0)}{\partial y}, \frac{\partial F(x_0, y_0, z_0)}{\partial z}\Big) \neq o$$

for each $(x_0, y_0, z_0) \in S$. Then S is a surface of the class C^r.

Proof. Let $F(x_0, y_0, z_0) = 0$ and e.g. $\frac{\partial F(x_0,y_0,z_0)}{\partial z} \neq 0$. According to the implicit function theorem the equation $F(x, y, z) = 0$ locally yiedls $z = f(x, y)$ for a function f of the class C^r. Locally this is an explicit description of the surfaces S. If $\frac{\partial F(x_0,y_0,z_0)}{\partial y} \neq 0$ resp. $\frac{\partial F(x_0,y_0,z_0)}{\partial x} \neq 0$, one can locally compute $y = g(x, z)$ resp. $x = h(y, z)$. □

The point (x_0, y_0, z_0) at which $\partial F(x_0, y_0, z_0) = o$ is called **singular point of the set** $F(x, y, z) = 0$.

4.10 **4.10. Examples.** (i) The case $F(x, y, z) = x^2 + y^2 + z^2 - a$ is similar as in **??**. The set $F(x, y, z) = 0$ is empty for $a < 0$. If $a = 0$, the equation is satisfied only by the origin which is the unique singular point. If $a > 0$, we have the sphere centered at the origin wit the radius $\sqrt{a}$. The vector $\partial F = (2x, 2y, 2z)$ is nonzero at all points.

(ii) Consider "rotacni kuzel" $F(x, y, z) = z^2 - x^2 - y^2 = 0$. The point $(0, 0, 0)$ is the unique singular point. Observe the tangent plane does not exist at this point. JS: missing picture

ve4.11 **4.11 Theorem.** The equation of the tangent plane of the surface S given by the equation $F(x, y, z) = 0$ at $(x_0, y_0, z_0) \in S$ is

e4.5 (5)

$$\frac{\partial F(x_0, y_0, z_0)}{\partial x}(x - x_0) + \frac{\partial F(x_0, y_0, z_0)}{\partial y}(y - y_0) + \frac{\partial F(x_0, y_0, z_0)}{\partial z}(z - z_0) = 0\,.$$

Proof. Let the curve $\big(f_1(t), f_2(t), f_3(t)\big)$ lie on S and goes through the point bodem $(x_0, y_0, z_0) \in S$ for $t = t_0$. Then

$$F\big(f_1(t), f_2(t), f_3(t)\big) = 0\,.$$

Differentiating of the composed function and putting $t = t_0$, we obtain

e4.6 (6)

$$\frac{\partial F(x_0,y_0,z_0)}{\partial x}\,\frac{df_1(t_0)}{dt} + \frac{\partial F(x_0,y_0,z_0)}{\partial y}\,\frac{df_2(t_0)}{dt} + \frac{\partial F(x_0,y_0,z_0)}{\partial z}\,\frac{df_3(t_0)}{dt} = 0\,.$$

Thus the normal vector of the plane (5) is perpendicular to the tangent vector of any curve on S hence (5) is the tangent plane. □

de4.12 **4.12 Definition.** Consider the surface S. The line N_pS through the point $p \in S$ and perpendicular to the tangent plane τ_pS, is called **normal line of the surface** S at the point p.

Thus the vector $\partial F(x_0, y_0, z_0)$ is the directional vector of the normal line of the surface $F(x, y, z) = 0$ at its point (x_0, y_0, z_0). The condition $\partial F \neq o$ geometrically guarantees existence of the tangent plane as well as the condition $\partial_1 f \times \partial_2 f \neq o$ in the case of a parametric description of the surface.

4.13 **4.13.** We shall study the question when the intersection of two surfaces

e4.7 (7)

$$F(x, y, z) = 0\,, \qquad G(x, y, z) = 0$$

is a curve

Theorem. Let $U \subset \mathbb{R}^3$ be an open set and F, $G\colon U \to \mathbb{R}$ be functions of the class C^r such that the set C given by the equation (7) is nonempty and vectors $\partial F(x_0, y_0, z_0)$ and $\partial G(x_0, y_0, z_0)$ are linearly independent for each $(x_0, y_0, z_0) \in C$. Then C is a curve of the class C^r.

Proof. Since vector ∂F and ∂G are linearly independent, there is at least one nonzero subdeterminant of the order 2 in the matrix

e4.8 (8)

$$\begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{pmatrix}.$$

If this is the subdetereminant

$$\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix},$$

it follows from the generalized implicit function theorem that from (7) one can locally compute $y = f(x)$ a $z = g(x)$. Here f and g are again function of the class C^r. (This theorem can be found at 2.6 of the textbook "Úvod do globální analýzy" which is stated as [5] in the list of references.) Thus $\big(t, f(t), g(t)\big)$ is locally a parametrization of the curve given by equations $F = 0$ and $G = 0$. If another subdeterminant of the order 2 is nonzero, we can locally express x and z as fucntions of y or x and y as functions of z. □

4.14 **4.14.** We shall illustrate the generalized implicit function theorem on tje simplest example of two linear equations

$$
\begin{aligned}
F(x,y,z) &= a_1x + a_2y + a_3z = 0\,,\\
G(x,y,z) &= b_1x + b_2y + b_3z \; = 0\,.
\end{aligned}
$$

In this case we have

$$
\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}
$$

and if this determinant is nonzer, one can use the Cramer's rule to compute y and z.

4.15 **4.15.** Geometrically, the theorem 4.13 says that the intersection of two surfaces S_1 and S_2 is locally a curve in a neighbourhood of a point $p \in S_1 \cap S_2$ where the tangent planes $\tau_p S_1$ and $\tau_p S_2$ are different.

A simple examples of two touching spheres (which intersect in a single point) shows that this condition is necessary.

An interesting example is so called Viviani curve which is the intersection of the sphere and "válce" with half radius which goes through the center of the sphere, see the view "from above" in a). The tangent planes of both surfaces are different surfaces with the exception of the point A. Indeed, here the intersection of both surfaces is not locally a curve in our sense, see the "front" point of view in b) and the general picture in c). JS: add translation JS: missing picture

4.16 **4.16.** The definition of the contact of a curve with a plane reduces to contact of two curves.

Definition. We say the curve C and the surface S have the **contact** of the **k-th order** at the intersection point p, if there exists a curve $\bar{C}$ on S such that C and $\bar{C}$ have the contact of the kth order at the point p.

One can easily see that C and S have the contact of the 1st order if and only if the tangent line of the curve lies in the tangent plane of the surface.

4.17 **4.17.** The following simple condition to determine the contact of a curve with a surafce is similar to the theorem 2.5. Let S be given by the equation $F(x, y, z) = 0$ and C is given by the parametrization $\big(f_1(t), f_2(t), f_3(t)\big)$.

Theorem. Let $f(t_0) = (x_0, y_0, z_0)$ be an intersection point of the curve C and the surface S. Consider the function $\Phi(t) = F\big(f_1(t), f_2(t), f_3(t)\big)$. Then C and S have the contact of the order k if and only if

e4.9 (9)
$$\frac{d^i\Phi(t_0)}{dt^i} = 0\,, \qquad i = 1, \dots, k\,.$$

Proof. Let $\bar{f}(t)$ be a parametrization of the curve $\bar{C}$ on the surface S such that derivatives of $f(t)$ and $\bar{f}(t)$ coincide up to the order k at $t = t_0$. Since $\bar{C}$ lies on S, we have

e4.10 (10)
$$F\big(\bar{f}_1(t), \bar{f}_2(t), \bar{f}_3(t)\big) = 0\,,$$

hence all derivatives with respect to t of the left hand side are zero. The function $\Phi(t)$ and (10) have the same outer factor $F(x, y, z)$ and derivatives of inner factors up to the order k coincide according to the condition of contact. Thus (9) holds.

In the opposite direction, let e.g. $\frac{\partial F(x_0,y_0,z_0)}{\partial z} \neq 0$. According to the implicit function theorem, the equation

$$F\big(f_1(t), f_2(t), z\big) = 0$$

locally determines the function $z = g(t)$ and the curve $\bar{C} \equiv \big(f_1(t), f_2(t), g(t)\big)$ lies on S. It is sufficient to show

e4.11 (11)
$$\frac{d^i f_3(t_0)}{dt^i} = \frac{d^i g(t_0)}{dt^i}\,, \qquad i = 1, \dots, k\,.$$

Put $G(t, z) = F\big(f_1(t), f_2(t), z\big)$. This function is defined on some neighborhood V of the point (t_0, z_0). Consider the function of three variables

e4.12 (12)
$$H(t, z, w) = G(t, z) - w\,.$$

on $V \times \mathbb{R}$. According to the implicit function theorem, the equation $H(t, z, w) = 0$ locally allows to compute $z = K(t, w)$. Since $G\big(t, g(t)\big) = 0$ and $G\big(t, f_3(t)\big) = \Phi(t)$, we have

$$g(t) = K(t, 0) \quad \text{a} \quad f_3(t) = K\big(t, \Phi(t)\big)\,.$$

As in the proof of the theorem 2.5, we obtain (11). □

5 Frenet frame of spacial curves

Consider a curve $C \subset E_3$.

ve5.1 **5.1 Theorem.** There exists a unique plane ω at non-inflection $p \in C$ which has the contact of the 2nd order with C. Then ω is called **osculating plane** of the curve C at the point p.

Proof. Consider a parametrization $f(t) = \big(f_1(t), f_2(t), f_3(t)\big)$ of the curve C and an arbitrary plane $ax + by + cz + d = 0$. According to 4.17 we consider the function

$$\Phi(t) = af_1(t) + bf_2(t) + cf_3(t) + d\,.$$

For the 2nd order contact we have conditions $\Phi(t_0) = 0$ and

$$\begin{aligned} \frac{d\Phi(t_0)}{dt} &= a\frac{df_1(t_0)}{dt} + b\frac{df_2(t_0)}{dt} + c\frac{df_3(t_0)}{dt} = 0\,, \\ \frac{d^2\Phi(t_0)}{dt^2} &= a\frac{d^2f_1(t_0)}{dt^2} + b\frac{d^2f_2(t_0)}{dt^2} + c\frac{d^2f_3(t_0)}{dt^2} = 0\,. \end{aligned}$$

These conditions mean that the normal vector (a, b, c) of the required plane is tangent to vectors $\frac{df(t_0)}{dt}$ and $\frac{d^2f(t_0)}{dt^2}$. Since these two vectors are linearly independent, the reuired plane is unique. □

Similarly as the osculating circle of a plane curve, the condition of the 2nd order contact of the osculating plane with the spacial curve means that the osculating plane approximates the curve in the best way (among all possible planes).

du5.2 **5.2 Corollary.** At a non-inflection point $f(t_0)$, the associated vector space of the osculating plane is given by vectors $\frac{df(t_0)}{dt}$ a $\frac{d^2f(t_0)}{dt^2}$.

Thus its equation can be epxressed in the form

$$\begin{vmatrix} x - f_1(t_0), & y - f_2(t_0), & z - f_3(t_0) \\ f_1'(t_0), & f_2'(t_0), & f_3'(t_0) \\ f_1''(t_0), & f_2''(t_0), & f_3''(t_0) \end{vmatrix} = 0\,.$$

5.3 **5.3.** Now we can define the following objects at a non-inflection points $p \in C$: objekty:

(i) The plane ν through the point p perpendicular to the tangent line is called **normal plane**.

(ii) The intersection $n = \nu \cap \omega$ of the normal and osculating planes is called **principal normal line**.

(iii) The line b through the point p perpendicular to the osculating plane is called **binormal line**.

(iv) The plane ϱ determined by the tangent and binormal lines is called **rectifying plane**.

JS: missing picture

de5.4 **5.4 Definition.** Non-inflection point $p \in C$ is called **planar point** if the osculating plane at this point has the 3rd order contact with the curve C.

ve5.5 **5.5 Theorem.** The non-inflection point $f(t_0)$ is planar if and only if the vector $\frac{d^3 f(t_0)}{dt^3}$ is linearly independent on vectors $\frac{df(t_0)}{dt}$ and $\frac{d^2 f(t_0)}{dt^2}$.

Proof. Consider the function $\Phi(t)$ from the proof of the theorem 5.1 and two conditions in this proof for the first and second derivatives of $\Phi(t)$. Considering the 3rd order contact, we have moreover

e5.1 (1)
$$\frac{d^3\Phi(t_0)}{dt^3} = a\frac{d^3 f_1(t_0)}{dt^3} + b\frac{d^3 f_2(t_0)}{dt^3} + c\frac{d^3 f_3(t_0)}{dt^3} = 0\,.$$

Thus the vector $\frac{d^3 f(t_0)}{dt^3}$ lies in the associated vector space of the osculating plane therefore it is linearly dependent on vectors $\frac{df(t_0)}{dt}$ and $\frac{d^2 f(t_0)}{dt^2}$. In the opposite direction, if this linear dependence holds then the equation (1) is a consequence of two equations from the proof of the theorem 5.1. Thus C has the 3rd order contact with the osculating plane. □

5.6 **5.6.** Let us further consider the arc-length s. We have $e_1(s) = \frac{df(s)}{ds}$ which is the unit vector. Considering a non-inflection point $f(s)$, we denote by $e_2(s)$ the unit collinear with $\frac{de_1(s)}{ds}$ in the same direction. Thus

e5.2 (2)
$$\frac{de_1(s)}{ds} = \varkappa(s) e_2(s)\,, \quad \varkappa(s) > 0\,.$$

and the vector $e_2(s)$ lies in the osculating plane. According to 1.26, the vector $e_2(s)$ is perpendicular to $e_1(s)$. Thus $e_2(s)$ yields the direction of the principal normal line at the point $f(s)$.

5.7 **5.7.** Assume further the space E_3 is oriented. By $e_3(s)$ we denote the unit vector perpendicular to $e_1(s)$ and $e_2(s)$ such that the basis $\big(e_1(s), e_2(s), e_3(s)\big)$ is positive. Thus the vector $e_3(s)$ yields the direction of the binormal.

Definition. The frame $\big(f(s_0), e_1(s_0), e_2(s_0), e_3(s_0)\big)$ is called **Frenet frame of the curve** C in the non-inflection point $f(s_0)$.

The argument s will be further usually omitted.

5.8 **5.8.** Since the vector e_2 is unit, by differentiating of the relation $(e_2, e_2) = 1$ we obtain $\left(e_2, \frac{de_2}{ds}\right) = 0$. Thus

$$\frac{de_2}{ds} = ce_1 + \tau e_3 \,.$$

Similarly by differentiating $(e_1, e_2) = 0$ we get $\left(\frac{de_1}{ds}, e_2\right) + \left(e_1, \frac{de_2}{ds}\right) = 0$ i.e. $\varkappa + c = 0$. Thus

e5.3 (3)
$$\frac{de_2(s)}{ds} = -\varkappa(s)e_1(s) + \tau(s)e_3(s) \,.$$

By differentiating $(e_3, e_3) = 1$ we see the vectpr $\frac{de_3}{ds}$ is perpendicular to e_3. By differentiating of the relation $(e_1, e_3) = 0$ we obtain

$$\left(\frac{de_1}{ds}, e_3\right) + \left(e_1, \frac{de_3}{ds}\right) = 0 \,.$$

But $\frac{de_1}{ds} = \varkappa e_2$ thus the first scalar product is zero, i.e. $\frac{de_3}{ds} = ke_2$. Finally by differentiating $(e_2, e_3) = 0$ we get $\left(\frac{de_2}{ds}, e_3\right) + \left(e_2, \frac{de_3}{ds}\right) = 0$. From this it follows $\tau + k = 0$ hence

e5.4 (4)
$$\frac{de_3(s)}{ds} = -\tau(s)e_2(s) \,.$$

We have shown

Theorem (Frenet formulae). The curve $f(s)$ without inflection points satisfies

e5.5 (5)
$$\begin{aligned}
\tfrac{df}{ds} &= e_1, \\
\tfrac{de_1}{ds} &= \qquad\quad \varkappa e_2, \\
\tfrac{de_2}{ds} &= -\varkappa e_1 \qquad\qquad +\tau e_3, \\
\tfrac{de_3}{ds} &= \qquad\quad -\tau e_2 \,.
\end{aligned}$$

de5.9 **5.9 Definition.** The number $\varkappa(s_0) > 0$ is called **curvature** and the number $\tau(s_0)$ is called **torsion** of the spacial curve $f(s)$ in the non-inflection point $f(s_0)$.

5.10 **5.10.** Frenet formulae of the curve $f(s)$ yield

e5.6 (6)
$$\frac{df}{ds} = e_1 \,, \quad \frac{d^2 f}{ds^2} = \varkappa e_2 \,, \quad \frac{d^3 f}{ds^3} = \frac{d\varkappa}{ds} e_2 + \varkappa(-\varkappa e_1 + \tau e_3) \,.$$

Assume $0 \in I$ for the curve $f(s)$, $f : I \to E_3$. Thus at $s = 0$ we have the Taylor expansion

e5.7

$$f(s) = f(0) + s\, e_1(0) + \frac{\varkappa(0)s^2}{2} e_2(0) + \frac{s^3}{6}\Big[\frac{d\varkappa(0)}{ds} e_2(0) - \varkappa^2(0) e_1(0) + \varkappa(0)\,\tau(0)\, e_3(0)\Big] + \nu(s)\,, \tag{7}$$

where $\nu(s)$ is the vector function whose value and first three derivatives are zero at the origin. In the other words, we have:

Theorem. Let x, y, z be coordinates with respect to the Frenet frame $\big(f(0), e_1(0), e_2(0), e_3(0)\big)$. Then the curve $f(s)$ in a neighbourhood of the point $f(0)$ is given by

e5.8

$$\begin{aligned} x &= s - \frac{\varkappa^2(0)}{6} s^3 + \xi(s)\,, \\ y &= \frac{\varkappa(0)}{2} s^2 + \frac{1}{6}\frac{d\varkappa(0)}{ds} s^3 + \eta(s)\,, \\ z &= \frac{\varkappa(0)\tau(0)}{6} s^3 + \zeta(s)\,, \end{aligned} \tag{8}$$

where real functions $\xi(s)$, $\eta(s)$ and $\zeta(s)$ have zero value and first three derivatives at the origin.

Relations (8) yield so called **local expansion of the curve** $f(s)$ **with respect to its Frenet frame**. Using this, we shall study orthogonal projections of the curve to its three basic planes of the Frenet frame.

5.11. A geometric meaning of the curvature of a plane curve is given by the definition 2.10. The curve $C \equiv f(s)$ in E_3 satisfies

Theorem. Considering a non-inflection point $p \in C$, the curvature of the curve C is equal to the curvature of its orthogonal projections C_p to the osculating plane.

Proof. We can assume $p = f(0)$. Accoridng to (7), C_p has parametrization (or rather its Taylor expansion)

e5.9

$$x = s + \alpha(s)\,, \quad y = \frac{\varkappa(0)}{2} s^2 + \beta(s)\,, \tag{9}$$

where $\alpha(s)$ and $\beta(s)$ have zero values and first two derivatives at the origin. The circle

e5.10

$$x^2 + \Big(y - \frac{1}{\varkappa(0)}\Big)^2 = \Big(\frac{1}{\varkappa(0)}\Big)^2, \quad \text{tj.} \quad x^2 + y^2 - \frac{2}{\varkappa(0)} y = 0 \tag{10}$$

has the second order contact with C_p at the origin. Indeed, putting (9) into (10) we get JS: missing picture

$$s^2 - s^2 + \gamma(s) = 0\,,$$

where $\gamma(s)$ is a function which has zero value and first two derivatives at the origin. □

5.12. Similarly, a parametrization of the orthogonal projection of the curve C to the normal plane os

$$y = \frac{\varkappa(0)}{2}s^2 + \frac{1}{6}\frac{d\varkappa(0)}{ds}s^3 + \eta(s)\,, \quad z = \frac{\varkappa(0)\tau(0)}{6}s^3 + \zeta(s)\,.$$

Denoting this vector valued function by $g(s)$, we have $\frac{dg(0)}{ds} = o$. Thus in a sense, the origin is an edge of the type of semicubic parabola.

JS: missing picture

5.13. The orthogonal projection of C to its rectifying plane is

$$x = s - \frac{\varkappa^2(0)}{6}s^3 + \xi(s)\,, \quad z = \frac{\varkappa(0)\tau(0)}{6}s^3 + \zeta(s)\,.$$

Denoting by $h(s)$ this vector valued function, we have $\frac{dh(0)}{ds} = (1,0)$, $\frac{d^2h(0)}{ds^2} = (0,0)$. Hence the origin is an inflection point.

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Comparision of all three projections yields a better understanding how the curve C moves "through" its Frenet frame.

5.14. It follows from Frenet formulae that planar points of the spacial curve C are easily characterized in terms of the torsion.

Theorem. The point $f(s_0)$ is planar if and only if $\tau(s_0) = 0$.

Proof. It follows from (6) that the vector $\frac{d^3f(s_0)}{ds^3}$ is a linear combination of $e_1(s_0)$ a $e_2(s_0)$ if and only if $\tau(s_0) = 0$. □

5.15. The curve $C \subset E_3$ is called **planar** if it lies in some plane $\varrho \subset E_3$. Since C lies in ϱ lies, every point of C is planar hence the torsion of a plane curve is zero. The opposite direction follows from Frenet formulae.

Theorem. A simple curve where all points are planar, is a part of a plane.

Proof. The condition $\tau = 0$ yileds $\frac{de_3}{ds} = o$ i.e. e_3 is a constant vector. Consider the plane through the point $f(s_0)$ perpendicular the vector $e_3(s_0)$. Its equation is $\big(e_3(s_0), w - f(s_0)\big) = 0$ where $w = (x, y, z)$ is an arbitrary point in E_3. Consider the function $\varphi(s) = \big(e_3(s_0), f(s) - f(s_0)\big)$. We have $\frac{d\varphi}{ds} = \big(e_3(s_0), e_1(s)\big) = 0$ since $e_3(s_0) = e_3(s)$. Thus φ is a constant function. Further $\varphi(s_0) = 0$ thus thus the function φ is identically zero. The whole curve lies in the considered plane. □

5.16. The basic geometrical meaning of the torsion follows firectly from (5).

Theorem. It holds $|\tau| = \left\| \frac{de_3}{ds} \right\|$. □

Therefore we can say that torsion is the velocity of rotation of the binormal vector. Zero rotation of course indicates plane curves. GEnerally one can say that greater absolute value of torison, more the curve diverges from a plane curve.

5.17 **5.17.** We shall find a formula for the curvature $\varkappa$ with respect to an arbitrary parametrization $f(t)$ of the curve C. It follows from (6) that $\varkappa = \left\| \frac{df}{ds} \times \frac{d^2f}{ds^2} \right\|$. Put $t = t(s)$. The chain rule yields

e5.11 (11)
$$\frac{df}{ds} = \frac{df}{dt}\frac{dt}{ds}, \quad \frac{d^2f}{ds^2} = \frac{d^2f}{dt^2}\Big(\frac{dt}{ds}\Big)^2 + \frac{df}{dt}\frac{d^2t}{ds^2}.$$

Further we know $\left|\frac{dt}{ds}\right| = 1/\left\|\frac{df}{dt}\right\|$. Since the vector product of two colinear vectors is zero, we have

$$\frac{df}{ds} \times \frac{d^2f}{ds^2} = \Big(\frac{df}{dt}\frac{dt}{ds}\Big) \times \Big(\frac{d^2f}{dt^2}\Big(\frac{dt}{ds}\Big)^2 + \frac{df}{dt}\frac{d^2t}{ds^2}\Big) = \Big(\frac{df}{dt} \times \frac{d^2f}{dt^2}\Big)\Big(\frac{dt}{ds}\Big)^3.$$

Thus we have proved

Theorem. It holds

e5.12 (12)
$$\varkappa = \frac{\left\| \frac{df}{dt} \times \frac{d^2f}{dt^2} \right\|}{\left\| \frac{df}{dt} \right\|^3}$$

5.18. We shall derive a formula for torsion τ with respect to an arbitrary parametrization $f(t)$ of the curve C. Recall three vectors $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$ in the oriented three-dimensional Euclidean vector space determine the exterior product

$$[u, v, w] = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Theorem. It holds

e5.13 (13)
$$\tau = \frac{\left[\frac{df}{dt}, \frac{d^2f}{dt^2}, \frac{d^3f}{dt^3}\right]}{\left\|\frac{df}{dt} \times \frac{d^2f}{dt^2}\right\|^2}\,.$$

Proof. First we observe the exterior product satisfies

$$[u, v + au, w + bu + cv] = [u, v, w]\,.$$

It follows from (6) that

$$\left[\frac{df}{ds}, \frac{d^2f}{ds^2}, \frac{d^3f}{ds^3}\right] = \varkappa^2\tau[e_1, e_2, e_3] = \varkappa^2\tau\,,$$

as e_1, e_2, e_3 is a positive basis. We shall rewrite (11) (obtained in the proof of the theorem 5.17) in the form

e5.14 (14)
$$\frac{df}{ds} = \frac{df}{dt}\frac{dt}{ds}\,, \qquad \frac{d^2f}{ds^2} = \frac{d^2f}{dt^2}\Big(\frac{dt}{ds}\Big)^2 + g\frac{df}{dt}\,,$$

where $g = \frac{d^2t}{ds^2}$ but we shall not need this fact. Further differentiation yields

e5.15 (15)
$$\frac{d^3f}{ds^3} = \frac{d^3f}{dt^3}\Big(\frac{dt}{ds}\Big)^3 + h\frac{d^2f}{dt^2} + k\frac{df}{dt}\,,$$

where we shall not need coefficients h and k explicitly. Thus we have

$$\varkappa^2\tau = \left[\frac{df}{ds}, \frac{d^2f}{ds^2}, \frac{d^3f}{ds^3}\right] = \left[\frac{dt}{ds}\frac{df}{dt}, \Big(\frac{dt}{ds}\Big)^2\frac{d^2f}{dt^2}, \Big(\frac{dt}{ds}\Big)^3\frac{d^3f}{dt^3}\right] =$$
$$= \Big(\frac{dt}{ds}\Big)^6\left[\frac{df}{dt}, \frac{d^2f}{dt^2}, \frac{d^3f}{dt^3}\right].$$

Using (12) for $\varkappa$ and the relation $\left|\frac{dt}{ds}\right| = 1/\left\|\frac{df}{dt}\right\|$, (13) follows. □

5.19 Example. We shall find curvature and torsion of the screw line. This curve is given by the trajectory of the uniform screw motion. Its parametrization therefore is JS: missing picture

$$f(t) = (a\cos t, a\sin t, bt)\,, \qquad t \in (-\infty, \infty),\ a > 0\,.$$

The number a is the radius of the circular cylinder on which the screw line lies. The number b is called **slope** of the **screw line**.

Consecutive differentiation yields

$$\begin{aligned} f' &= (-a\sin t, a\cos t, b)\,, \\ f'' &= (-a\cos t, -a\sin t, 0)\,, \\ f''' &= (a\sin t, -a\cos t, 0)\,. \end{aligned}$$

Thus $f' \times f'' = (ab\sin t, -ab\cos t, a^2)$, $\|f' \times f''\| = a\sqrt{a^2+b^2}$. Further $\|f'\| = \sqrt{a^2+b^2}$. Podle (12) m me $\varkappa = \frac{a}{a^2+b^2}$. To deteremine torison we compute the determinant

$$[f', f'', f'''] = \begin{vmatrix} -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix} = ba^2\,.$$

According (13) we have $\tau = \frac{ba^2}{a^2(a^2+b^2)} = \frac{b}{a^2+b^2}$.

Thus the screw line has constant curvature and torsion.

5.20 **5.20.** Recall **proper Euclidean transformation in an oriented space** E_3 is such Euclidean transformation $\varphi\colon E_3 \to E_3$ which preserves orientation. Similarly as in a plane, we call two curves C, $\bar{C} \subset E_3$ congruent if there exists such proper Euclidean transformation φ such that $\varphi(C) = \bar{C}$. As in the plane, we shall assume that C and $\bar{C}$ are simple and both are parametried by the arc-length on the same interval I.

Theorem. Let curves C and $\bar{C}$ are without inflection points, $f\colon I \to E_3$ and $\bar{f}\colon I \to E_3$ are their arc-length parametrizationson the same interval I and $\varkappa(s)$, $\bar{\varkappa}(s)$ and $\tau(s)$, $\bar{\tau}(s)$ are tehir curactures and torsions, respectively. Then curves C and $\bar{C}$ are conguent if and only if $\varkappa = \bar{\varkappa}$ and $\tau = \bar{\tau}$ on I.

Proof. On one side, it directly follows from the geometric construction of the Frenet frame that given two congruent curves, their curvatures and torsions are the same functions of the arc-length. In the opposite direction, consider C and $\bar{C}$ with Frenet frames $\big(f(s), e_1(s), e_2(s), e_3(s)\big)$ and $\big(\bar{f}(s), \bar{e}_1(s), \bar{e}_2(s), \bar{e}_3(s)\big)$, respectively. Beside (5), we have also

e5.16 (16) $$\frac{d\bar{f}}{ds} = \bar{e}_1\,, \quad \frac{d\bar{e}_1}{ds} = \varkappa\bar{e}_2\,, \quad \frac{d\bar{e}_2}{ds} = -\varkappa\bar{e}_1 + \tau\bar{e}_3\,, \quad \frac{d\bar{e}_3}{ds} = -\tau\bar{e}_2$$

with the same $\varkappa$ and τ. Thus (5) and (16) is the same system of differencial equations for twelve real functions which are components of f, e_1, e_2 and e_3. Given $s_0 \in I$, both $f(s_0)$, $e_1(s_0)$, $e_2(s_0)$, $e_3(s_0)$ as well as $\bar{f}(s_0)$, $\bar{e}_1(s_0)$,

$\bar{e}_2(s_0)$, $\bar{e}_3(s_0)$ are the point and a positive orthonormal frame. Hence there is a unique proper Euclidean motion $\varphi\colon E_3 \to E_3$ which transforms the first of these 4-tuples to the second one. Then parametrization $\bar{f}\colon I \to E_3$ of the curve $\bar{C}$ together with vector functions $\bar{e}_1$, $\bar{e}_2$ and $\bar{e}_3$ and parametrizations $\varphi \circ f\colon I \to E_3$ of the curve $\varphi(C)$ together with vector functions $\varphi \circ e_1$, $\varphi \circ e_2$ and $\varphi \circ e_3$ satisfy the same system of differential equations with the same initial conditions. According to the theorem about unique existence of solution of a system of differential equations, we in particularly have $\bar{f} = \varphi \circ f$. Thus $\bar{C} = \varphi(C)$. □

5.21 **5.21.** Similarly as in the plane we also have the opposite direction.

Theorem. Let $\varkappa$, $\tau\colon I \to \mathbb{R}$ be real functions, $\varkappa > 0$. Then locally there exists a curve C parametrized by the arc-length on I such that $\varkappa$ is its curvature and τ is its torsion,

5.22 Example. We shall show the screw lines are unique curves with constant curvature and torsion (Zero torsion corresponds to a circle as the screw line with zero slope.) Indeed, we computed in (19) that

e5.17 (17)
$$\varkappa = \frac{a}{a^2+b^2}\,, \qquad \tau = \frac{b}{a^2+b^2}\,.$$

for screw lines. Let $\varkappa > 0$ and τ be given. Then we compute from (17) that $\frac{\tau}{\varkappa} = \frac{b}{a}$ thus $a = k\varkappa$, $b = k\tau$ for some $k > 0$. Putting this to the formula for $\varkappa$ w eget $\varkappa = \frac{k\varkappa}{k^2(\varkappa^2+\tau^2)}$, i.e. $k = \frac{1}{\varkappa^2+\tau^2}$. It follows from theorems 5.20 and 5.21 that parts of the screw lines with values $a = \frac{\varkappa}{\varkappa^2+\tau^2}$ and $b = \frac{\tau}{\varkappa^2+\tau^2}$ are unique curves with given constant $\varkappa$ and τ.

5.23 Remark. Another interesting (however less imporant in practice) geometrical object determined by the curve $C \equiv f(s)$ is its **osculating sphere**. Assuming $f(s_0)$ is a non-planar point then there exists a unique sphere S which has the 3rd order contact with the curve C at $f(s_0)$. We shall only sketch its construction. It follows from the 1st order contact that the tangent line of the curve at the point $f(s_0)$ is also tangent to S hence the center of the osculating sphere lies on the normal line. Let $f(s_0) + ae_2(s_0) + be_3(s_0)$ is this center. We shall write the equation of the sphere S in the form of scalar product

$$\big(w - f(s_0) - ae_2(s_0) - be_3(s_0), w - f(s_0) - ae_2(s_0) + be_3(s_0)\big) = a^2 + b^2$$

where $w = (x, y, z)$ is an arbitrary point in E_3. To determine the contact of C and S we use the function

$$\Phi(s) = \big(f(s) - f(s_0) - ae_2(s_0) - be_3(s_0), f(s) - f(s_0) - ae_2(s_0) - be_3(s_0) - a^2 - b^2\,.$$

Relations $\Phi(s_0) = 0$ and $\frac{d\Phi(s_0)}{ds} = 0$ hold by construction. Conditions $\frac{d^2\Phi(s_0)}{ds^2} = 0$ and $\frac{d^3\Phi(s_0)}{ds^3} = 0$ yield

e5.18 (18)
$$a = \frac{1}{\varkappa(s_0)}\,, \quad b = -\frac{\varkappa'(s_0)}{\varkappa^2(s_0)\tau(s_0)}\,, \quad \varkappa'(s) = \frac{d\varkappa}{ds}\,.$$

The radius $r = \sqrt{a^2 + b^2}$ of the usculating sphere thus is

e5.19 (19)
$$r = \frac{1}{\varkappa^2|\tau|}\sqrt{\varkappa^2\tau^2 + \left(\frac{d\varkappa}{ds}\right)^2}\,.$$

Display relations (18) and (19) illustrate an interesting general akt. According to theorems 5.20 and 5.21, the curve C is geometrically determined by its curvature and torsion. Thus also other geometrical objects deteremined by the curve and its invariants are expressed using $\varkappa$ and τ and their derivatives with respect to the arc-length.

6 The first fundamental form of the surface

Now we start a systematic study of surfaces in E_3.

6.1 **6.1.** Consider a surface S with a local parametric expression $f(u_1, u_2)$, $(u_1, u_2) \in D$, see 4.4. We shall use the abbreviated notation $f_1 = \partial_1 f$, $f_2 = \partial_2 f$. Thus $f_1(u_0)$, $f_2(u_0)$ form a basis of the tangent space T_pS of the surface S at the point $p = f(u_0)$.

Consider vectors A, $B \in T_pS$, $A = a_1 f_1 + a_2 f_2$, $B = b_1 f_1 + b_2 f_2$. Their scalar product is given by

e6.1 (1) $$(A, B) = (a_1 f_1 + a_2 f_2, b_1 f_1 + b_2 f_2)\,.$$

Put

e6.2 (2) $$g_{11} = (f_1, f_1)\,, \quad g_{12} = (f_1, f_2)\,, \quad g_{22} = (f_2, f_2)\,.$$

That is, g_{ij}, $i, j = 1, 2$ are functions on D. Then (1) can be written in the form

e6.3 (3) $$(A, B) = g_{11}a_1b_1 + g_{12}(a_1b_2 + a_2b_1) + g_{22}a_2b_2\,.$$

This is a bilinear form on T_pS. The corresponding quadratic form deteremines the length of the vector A,

$$\|A\| = \sqrt{g_{11}a_1^2 + 2g_{12}a_1a_2 + g_{22}a_2^2}\,.$$

The angle φ of vector A, B satisfies

(4) $$\cos\varphi = \frac{g_{11}a_1b_1 + g_{12}(a_1b_2 + a_2b_1) + g_{22}a_2b_2}{\sqrt{g_{11}a_1^2 + 2g_{12}a_1a_2 + g_{22}a_2^2}\;\sqrt{g_{11}b_1^2 + 2g_{12}b_1b_2 + g_{22}b_2^2}}$$

6.2. Consider a curve $u(t) = \big(u_1(t), u_2(t)\big)$ on S. We have $\frac{df}{dt} = f_1\frac{du_1}{dt} + f_2\frac{du_2}{dt}$ hence

$$\Big\|\frac{df}{dt}\Big\| = \sqrt{g_{11}\Big(\frac{du_1}{dt}\Big)^2 + 2g_{12}\frac{du_1}{dt}\frac{du_2}{dt} + g_{22}\Big(\frac{du_2}{dt}\Big)^2}$$

From the formula for length of an arc of a space curve we have

Theorem. Length s of the arc of the curve $u(t)$ on the surface $f(u)$ between points with parameters t_1 and t_2 is

(5) $$s = \int_{t_1}^{t_2} \sqrt{g_{11}\Big(\frac{du_1}{dt}\Big)^2 + 2g_{12}\frac{du_1}{dt}\frac{du_2}{dt} + g_{22}\Big(\frac{du_2}{dt}\Big)^2}\,dt\,.$$

Thus the differential ds is given by the expression which follows the symbolem of integral. Its square

e6.6 (6) $$(ds)^2 = g_{11}(du_1)^2 + 2g_{12}du_1du_2 + g_{22}(du_2)^2$$

is a quadratic form corresponding to the bilinear form (3).

6.3 Definition. The quadratic form (6) is called **first fundamental form of the suface**. It is denoted by Φ_1 or $(ds)^2$.

We shall use the same symbol Φ_1 also for the bilinear form determined by this quadratic form.

6.4 **6.4 Example.** Consider the sphere S centered at the origin se with the radius r. Given a point $p \in S$ away from the z-axis, consider its projection q to the plane (x, y). We shall denote by u_1 the angle of the radius vector of the point q with the positive half x-axis, i.e. $u_1 \in [0, 2\pi)$. We denote by u_2 the angle of the radius vector of the point p with the plane (x, y), i.e. $u_2 \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$. Thus $z = r\sin u_2$ and length of the radius vector of q is $r\cos u_2$. Situation in the plane (x, y) corresponds to polar coordinates, i.e. $x = r\cos u_2 \cos u_1$, $y = r\cos u_2 \sin u_1$. Summarizing, we have JS: missing picture

(7) $$f(u_1, u_2) = (r\cos u_1 \cos u_2, r\sin u_1 \cos u_2, r\sin u_2)\,, \qquad u_1 \in (0, 2\pi),\ u_2 \in \big(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\big)\,.$$

The sphere is a not a simple surface hence our parametrization does not cover half-circle which is intersection of the sphere with the half-plane $x \geq 0$ in the (x, z)-plane. However, this incompletness is usually not a problem.

We shall find the first fundamental form of the sphere. We have

$$f_1 = r(-\sin u_1 \cos u_2, \cos u_1 \cos u_2, 0)\,,$$
$$f_2 = r(-\cos u_1 \sin u_2, -\sin u_1 \sin u_2, \cos u_2)\,.$$

Thus $g_{11} = (f_1, f_1) = r^2\cos^2 u_2$, $g_{12} = (f_1, f_2) = 0$, $g_{22} = r^2$. The first fundamental form of the sphere is of the form

(8) $$\Phi_1 = r^2\big[\cos^2 u_2 (du_1)^2 + (du_2)^2\big]\,.$$

6.5 **6.5.** We shall compute the first fundamental form of an explicitly given surface $z = f(x, y)$, $(x, y) \in D$, see 4.5. Its parametrization is given by $\bar{f}(x, y) = \big(x, y, f(x, y)\big)$. Hence $\bar{f}_1 = (1, 0, f_x)$, $\bar{f}_2 = (0, 1, f_y)$ where f_x and f_y are partial derivatives of f with respect to x and y, respectively. Computing scalar products (2), we obtain

(9) $$\Phi_1 = (1 + f_y^2)\,(dx)^2 + 2f_xf_y\,dx\,dy + (1 + f_y^2)\,(dy)^2\,.$$

6.6 Definition. System of curves on a simple surface S is a 1-parameter family $\mathscr{L}$ of curves on S such that there is a unique curve from $\mathscr{L}$ through every point of the surface S.

First consider system $\mathscr{L}$ in the parameter space D of the plane (u_1, u_2). Assume that tangent lines of curves of the system are not parallel with the u_2-axis. Then system $L(u_1, u_2)$ of tangent lines of the system $\mathscr{L}$ satisfy

e6.10 (10)
$$\frac{du_2}{du_1} = L(u_1, u_2)\,.$$

We say (10) is **differential equation of the system $\mathscr{L}$.**

Vector field on the space D is the rule which assignes a vector in the tangent space T_pD for every point $p \in D$. Having a nowhere vanishing vector field $\big(F_1(u_1, u_2), F_2(u_1, u_2)\big)$ on D tangent to the system $\mathscr{L}$, non-parllelity with the u_2-axis mean $F_1(u_1, u_2) \neq 0$. Then the differential equation of the system $\mathscr{L}$ is

(11)
$$\frac{du_2}{du_1} = \frac{F_2(u_1, u_2)}{F_1(u_1, u_2)}\,.$$

The system $\mathscr{L}$ on a surface S is usually given on the parameter space. **Vector field on the surface** S is a rule which assignes a vector in the tangent space T_pS to every point $p \in S$.

6.7 Definition. Orthogonal trajectories of the system $\mathscr{L}$ on the surface S is a system $\mathscr{L}$' na S such that curves of $\mathscr{L}$ and $\mathscr{L}$' are perpendicular at every point.

Theorem. If $\big(F_1(u_1, u_2), F_2(u_,u_2)\big)$ is a coordinate expression of a vector field tangent to the system $\mathscr{L}$ then the differential system of its orthogonal trajectories is

e6.12 (12)
$$\frac{du_2}{du_1} = -\frac{g_{11}F_1 + g_{12}F_2}{g_{12}F_1 + g_{22}F_2}\,.$$

Proof. Let (du_1, du_2) is a tangent vector to the required system $\mathscr{L}$'. Following (3), the orthogonality of both systems means

$$g_{11}F_1du_1 + g_{12}(F_1du_2 + F_2du_1) + g_{22}F_2du_2 = 0\,.$$

Now (12) follows by an algebraic manipulation. □

Remark. If there zero in the denominator of the right hand side of (12) at some point of the surface, it means orthogonal trajectories through this point are, considered at the parameter space, tangent to the axis u_2. Then we should consider the differential equation of the system with interchanged axes u_1 and u_2.

6.8 Example. We shall find orthogonal trajectories for the system $u_1 + u_2 = \text{konst}$ on the sphere from 6.4. By differentiation be find $du_1 + du_2 = 0$, i.e. the differential equation of this system is $\frac{du_2}{du_1} = -1$. Thus we cna put $F_1 = 1$ and $F_2 = -1$. We found $g_{11} = r^2 \cos^2 u_2$, $g_{12} = 0$, $g_{22} = r^2$ an 6.4. Following (2), the differential equation of orthogonal trajectories is

e6.13 (13)
$$\frac{du_2}{du_1} = \cos^2 u_2\,.$$

Separating variables in (13) and integrating, we obtain the equation of orthogonal trajectories in the form

$$\operatorname{tg} u_2 = u_1 + \text{konst}.$$

6.9 Definition. Net on the surface S are two systems $\mathscr{L}_1$, $\mathscr{L}_2$ whose curves have nonzero angle at every point. The net is called **orthogonal** if this is the right angle at every point.

A simple example is the parametric coordinate net formed by curves $u_1 = \text{konst.}$ and $u_2 = \text{konst.}$ given by the parametrization $f(u_1, u_2)$ of the surface S. The condition $f_1 \times f_2 \neq o$ guarantees that angle of curves of two parametric systems is nonzero.

The following statement will be useful in many specific case.

Theorem. The parametric net is orthogonal if and only if $g_{12} = 0$.

Proof. Vectors f_1 and f_2 are tangent to parametric systems and $g_{12} = (f_1, f_2)$. □

6.10 **6.10 Lemma.** It holds $g_{11}g_{22} - g_{12}^2 > 0$.

Proof. The Cauchy inequality says that vectors a, b satisfy $|(a,b)| \leq \|a\|\,\|b\|$, i.e. $(a,b)^2 \leq \|a\|^2\,\|b\|^2$. Here the equality holds only if these vectors are collinear. Putting $a = f_1$, $b = f_2$, we have $\|a\|^2 = g_{11}$, $\|b\|^2 = g_{22}$, $(a,b) = g_{12}$. Since vectors f_1 and f_2 are not collinear, the lemma follows. □

6.11. A standard result in calculus shows that the area of the surface given explicitly as $z = f(x, y)$, $(x, y) \in D$ (where f is a bounded function on a bounded space D) is given by the double integral

e6.14 (14)
$$\iint_D \sqrt{1 + f_x^2 + f_y^2}\, dx\, dy$$

6.12. The map $f\colon D \to E_3$ is **bounded** if the set $f(D)$ lies in some ball.

Theorem. Let the surface S is given by a bounded map $f(u_1, u_2)$ on a bounded space $D \subset \mathbb{R}^2$. Then its area is given by

e6.15 (15)
$$\iint_D \sqrt{g_{11}g_{22} - g_{12}^2}\, du_1\, du_2\,.$$

Proof. We know every surface cen be given explicitly in a neighbourhood of every point and let $z = f(x, y)$ be such parametrization. We have shown $g_{11} = 1+f_x^2$, $g_{12} = f_x f_y$, $g_{22} = 1+fy^2$ in 6.5, hence $g_{11}g_{22}-g_{12}^2 = 1+f_x^2+f_y^2$. Then (15) localy reduces to the usual expresssion redukuje to the usual expression (14). The global version follows from the additivity of area of the surface. □

The expression $dV\colon = \sqrt{g_{11}g_{22} - g_{12}^2}\, du_1\, du_2$ is also called **volume element of the surface** S. The formula of the area of the surface thus has the form $V = \iint_D dV$.

6.13 Example. We shall find area V of the so called spherical cap on the sursface with the radius r with the angle α, see the picture. Thus $D = (0, 2\pi) \times \left(\frac{\pi}{2} - \alpha, \frac{\pi}{2}\right)$. We found $g_{11} = r^2 \cos^2 u_2$, $g_{12} = r^2$ in 6.4. Thus JS: missing picture

$$\begin{aligned} V &= \iint_D r^2 \cos u_2\, du_1\, du_2 = r^2 \int_0^{2\pi} du_1 \int_{\pi/2-\alpha}^{\pi/2} \cos u_2 du_2 \\ &= 2\pi r^2 \big[\sin u_2\big]_{\pi/2-\alpha}^{\pi/2} = 2\pi r^2 (1 - \cos\alpha)\,. \end{aligned}$$

The case $\alpha = \frac{\pi}{2}$ gives the area $2\pi r^2$ of the half of the sphere.

6.14. Summarizing, the first fundamental form Φ_1, determined by scalar products at each tangent space of the surface, is used mainly for computation of length of curves on surfaces, angle of these curves and area of the surface. A fundamental theoretical meaning of Φ_1 will be discussed later.

2fundamental

7 The second fundamental form of the surface

7.1 **7.1.** Consuider the normal line N_pS of the surface S at the point p. We have two unit vectors on the normal line, a choice of one of them yields orientation of the line N_pS.

Definition. Orientation of the surface S is a choice of orientation of every normal line in a continuous way.

One can orient every simple surface. Given a parametrization $f(u_1, u_2)$, we can chose the direction of the normal as the direction of the vector product $f_1 \times f_2$.

7.2. The case of Möbius strip shows that there surface which cannot be oriented.

Definition. The surface S, which can be oriented, is called **orientable**. An orientable surface together with a choice of orientation is called **oriented**.

We shall denote the unit vector of the oriented normal line by n. Dependence on parameters is expressed if we write $n(u_1, u_2)$. In the case of orientation determined by the parametrization $f(u)$, we have

e7.1 (1)
$$n = \frac{f_1 \times f_2}{\|f_1 \times f_2\|}\,.$$

The orthogonality of the normal line and the surface is expressed by equations

e7.2 (2)
$$(n, f_1) = 0\,, \quad (n, f_2) = 0\,.$$

7.3 **7.3.** Further we assume the surface S is oriented.

Given an arbitrary motion $\gamma(t)$ in the space E_3, the vector $\frac{d^2\gamma}{dt^2}$ is called accelaration. Consider a motion on the surafce S with a local parametrizaiton $f(u)$ given by $\big(u_1(t), u_2(t)\big)$ in the parameter space D. This is the motion $\gamma(t) = f\big(u_1(t), u_2(t)\big)$ in E_3. Let us compute teh acceleration. The first derivative of the composed function is given by the expression

$$\frac{d\gamma}{dt} = f_1\big(u_1(t), u_2(t)\big)\frac{du_1}{dt} + f_2\big(u_1(t), u_2(t)\big)\frac{du_2}{dt}\,.$$

We shall use an abbreviation

e7.3 (3)
$$f_{11} = \partial_{11} f\,, \quad f_{12} = \partial_{12} f\,, \quad f_{22} = \partial_{22} f\,.$$

to compute the second derivative. We obtain

e7.4 (4) $$\frac{d^2\gamma}{dt^2} = f_{11}\Big(\frac{du_1}{dt}\Big)^2 + 2f_{12}\frac{du_1}{dt}\frac{du_2}{dt} + f_{22}\Big(\frac{du_2}{dt}\Big)^2 + f_1\frac{d^2u_1}{dt^2} + f_2\frac{d^2u_2}{dt^2}\,.$$

Following (2), the scalar product of n and $\frac{d^2\gamma}{dt^2}$ depends only on $\frac{d\gamma}{dt}$.

Definition. The scalar product $\big(n, \frac{d^2\gamma}{dt^2}\big)$ is called **normal acceleration** corresponding to the vector $\frac{d\gamma}{dt} \in T_pS$. In the case of $\|\frac{d\gamma}{dt}\|$=1, this is termed **normal curvature of the oriented surface** S in the direction of this vector.

That is, the sign of the normal acceleration depends on orientation fo the surface.

7.4 **7.4.** Consider scalar products

e7.5 (5) $$h_{11} = (n, f_{11})\,, \quad h_{12} = (n, f_{12})\,, \quad h_{22} = (n, f_{22})\,,$$

which are functions on the space D. Following (4), we obtain the rule which which associates the normal acceleration to every tangent vector $(du_1, du_2) \in T_pS$. This is the quadratic form on T_pS on the space

e7.6 (6) $$h_{11}(du_1)^2 + 2h_{12}du_1\,du_2 + h_{22}(du_2)^2\,.$$

Definition. The quadratic form (6) is called **second fundamental form of the surface** S and will be denoted by Φ_2.

That is, the second fundamental form of the oriented surface S is a rule which maps every vetor $A \in T_pS$ to the number $\Phi_2(A)$ which he obtained as follows. We consider a motion $\gamma(t)$ on the surface S such that $A = \frac{d\gamma(t_0)}{dt}$. We compute its acceleration $\frac{d^2\gamma(t_0)}{dt^2}$. The number $\Phi_2(A)$ is then equal to the scalar product $\big(n(\gamma(t_0)), \frac{d^2\gamma(t_0)}{dt^2}\big)$ where $n(\gamma(t_0))$ is the oriented vector of the normal line at the point $\gamma(t_0)$.

7.5 **7.5.** Consider the direction of the nonzero vector A in the tangent space T_pS. The section of the surface S by the plane determined by the normal line N_pS and the direction A is the curve which we call **normal section of the surface in direction** A.

The basic geometrical meaningof the form Φ_2 is given by

Theorem. The absolute value of the normal curvature in the direction of the vector A is equal to the curvature of the normal section in this direction.

Proof. Consider the parametrization $\gamma(s)$ of this section by the arc-length, $\gamma(s_0) = p$. Then $\frac{d\gamma}{ds}$ is a unit vector and $\frac{d^2\gamma(s_0)}{ds^2}$ is perpendicular to this vector. We know (form theory of curves) that the norm of $\frac{d^2\gamma(s_0)}{ds^2}$ is equal to the curvature of the normal section. Vectors $n(p)$ and $\frac{d^2\gamma(s_0)}{ds^2}$ are thus colinear. Since $n(p)$ is a unit vector, the absolute value of the scalar product $\big(n(p), \frac{d^2\gamma(s_0)}{ds^2}\big)$ is equal to the norm of the second vector. □

7.6. The normal curvature $\varkappa$ in the direction of the vector $A = (du_1, du_2)$ satisfies

e7.7 (7)
$$\varkappa = \frac{h_{11}(du_1)^2 + 2h_{12}du_1\, du_2 + h_{22}(du_2)^2}{g_{11}(du_1)^2 + 2g_{12}du_1\, du_2 + g_{22}(du_2)^2}\,. \tag{7}$$

Indeed, the unit vector in this direction is $\frac{1}{\|A\|}(du_1, du_2)$ where $\|A\|^2 = g_{11}(du_1)^2 + 2g_{12}du_1\, du_2 + g_{22}(du_2)^2$. Substituting this into (6), the display (7) follows.

7.7 Definition. The point $f(u_0) \in S$ is called **planar point** if the form $\Phi_2(u_0)$ is nonzero, i.e. $h_{11}(u_0) = 0$, $h_{12}(u_0) = 0$, $h_{22}(u_0) = 0$.

7.8 Definition. The surface S is called **connected** if every two its point can be connected by a motion which lies in S.

7.9 **7.9 Theorem.** A simple connected surface S with all points planar is a part of a plane.

Proof. Pur $n_1 = \partial_1 n$, $n_2 = \partial_2 n$. Differentiating (2) with respect to u_1 and u_2 yields

e7.8 (8)
$$\begin{aligned} &(n_1, f_1) + (n, f_{11}) = 0\,, \quad (n_1, f_2) + (n, f_{12}) = 0\,,\\ &(n_2, f_1) + (n, f_{12}) = 0\,, \quad (n_2, f_2) + (n, f_{22}) = 0\,. \end{aligned} \tag{8}$$

In particular, this shows that all points on a plane (with constant normal vector) are planar. Further we use the fact that n is a unit vector. By differentiating the relation $(n, n) = 1$, we obtain

e7.9 (9)
$$(n, n_1) = 0\,, \quad (n, n_2) = 0\,. \tag{9}$$

If every point of the surface S is planar, the second term in every relation of (8) is zero, cf. (5). Then first two equations of (8) and the first equation in (9) mean the vector n_1 is perpendicular to three lienarly independent vectors n, f_1, f_2. Thus n_1 is the zero vector. Analogously, it follows from remaining equations of (8) and (9) that n_2 is the zero vector. Thus the normal vector is constant, $n = a$. Consider the function

$$\varphi(u_1, u_2) = \big(a, f(u_1, u_2) - f(u_1^0, u_2^0)\big) .$$

We have $\frac{\partial\varphi}{\partial u_1} = (a, f_1) = 0$, $\frac{\partial\varphi}{\partial u_2}(a, f_2) = 0$ hence φ is a constant function. Moreover, $\varphi(u_1^0, u_2^0) = 0$, i.e. $\varphi(u) = 0$ for all u. This means that the whole surface S lies on its tangent plane through the point $f(u_0)$. □

7.10 Definition. The point $f(u_0) \in S$ is called **spherical point** if the form $\Phi_2(u_0)$ is a constant multiple of the form $\Phi_1(u_0)$.

Thus the spherical point $f(u_0)$ is characterized by the condition

e7.10 (10)

$$h_{11}(u_0) = cg_{11}(u_0)\,,\ h_{12}(u_0) = cg_{12}(u_0)\,,\ h_{22}(u_0) = cg_{22}(u_0)\,, \quad 0 \neq c \in \mathbb{R}\,.$$

The normal vector $n(u)$ of the sphere centered at the origin with raidus r satisfies $n(u) = \frac{1}{r} f(u)$. Equations (8) show that all points on the sphere are spherical.

7.11 Theorem. A simple connected surface S where all points are spherical, is a part of a sphere.

Proof. Assume (10) holds. That is,

e7.11 (11) $$(n, f_{11}) = c(f_1, f_1)\,, \quad (n, f_{12}) = c(f_1, f_2)\,, \quad (n, f_{22}) = c(f_2, f_2)\,.$$

It follows from (8) and (11) thast

e7.12 (12)

$$(f_1, n_1{+}cf_1) = 0\,, \quad (f_1, n_2{+}cf_1) = 0\,, \quad (f_2, n_1{+}cf_2) = 0\,, \quad (f_2, n_2{+}cf_2) = 0\,.$$

Further, it follows from (2) and (9) that

e7.13 (13) $$(n, n_1 + cf_1) = 0\,, \quad (n, n_2 + cf_2) = 0\,.$$

Analogously as in the proof of theorem 7.9, it follows from these relations that

e7.14 (14) $$n_1 + cf_1 = o\,, \quad n_2 + cf_2 = o\,.$$

By differentiation of the first equation with respect to u_2 and the second equation with resoect to u_1, we obtain

(15) $$n_{12} + \frac{\partial c}{\partial u_2} f_1 + cf_{12} = 0\,, \quad n_{12} + \frac{\partial c}{\partial u_1} f_2 + cf_{12} = 0\,,$$

where $n_{12} = \frac{\partial^2 n}{\partial u_1 \partial u_2}$. Thus we have the difference

$$\frac{\partial c}{\partial u_2} f_1 - \frac{\partial c}{\partial u_1} f_2 = o\,.$$

Since vectors f_1 and f_2 are linearly independent, we have $\frac{\partial c}{\partial u_1} = 0$, $\frac{\partial c}{\partial u_2} = 0$, i.e. c is a constant. Following (14), the point $f + \frac{1}{c}\, n$ is fixed. The distance of every point on the surface form this point is constant and equal to $\frac{1}{|c|}$, i.e. S is a part of the corresponding sphere. □

de7.12 **7.12 Definition.** A direction in the tangent plane of the surafce is called **asymptotic direction**, if its normal curvature is zero. The tangent line in this direction is called **asymptotic tangent line**.

Thus the equation of asymptotic directions is

e7.16 (16)
$$h_{11}(du_1)^2 + 2h_{12} du_1\, du_2 + h_{22}(du_2)^2 = 0$$

Every direction is asymptoic in planar points.

Assuming the direction $du_2 = 0$ is not asymptotic, i.e. $h_{11} \neq 0$, put $\varrho = \frac{du_1}{du_2}$. Then (16) yields the quadratic equation for asymptotic directions

e17 (17)
$$h_{11}\varrho^2 + 2h_{12}\varrho + h_{22} = 0\,.$$

Its roots satisfy $\varrho_{1,2} = \frac{-h_{12} \pm \sqrt{h_{12}^2 - h_{11}h_{22}}}{h_{11}}$. Put

e7.18 (18)
$$h = \begin{vmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{vmatrix} = h_{11}h_{22} - h_{12}^2\,.$$

Thus there are two (real) asymptotic directions for $h < 0$, both directions coincide for $h = 0$ and there are imaginary roots for $h > 0$. If $h_{11} = 0$ and $h_{22} \neq 0$, (16) yields the quadratic equation for the fraction $\frac{du_2}{du_1}$ and the situation is similar. If $h_{11} = 0$ and $h_{22} = 0$, we have $h_{12} \neq 0$ in a non-planar point, i.e. asymptotic directions are $du_1 = 0$ and $du_2 = 0$.

de7.13 **7.13 Definition.** The non-planar point is called **hyperbolic** or **parabolic** or **elliptic**, if $h < 0$ or $h = 0$ or $h > 0$, respectively.

In spherical points, the inequality 6.10 means $h > 0$ hence this is a special case of an elliptic point.

7.14 Definition. The curve C on the surface S is called **asymptotic** if its tangent line ate every point is asymptotic tangent line.

Thus we have two systems of asymptotic curves on surfaces with only hyperbolic points, one system of asymptotic curves on surfaces with only parabolic points and there is no system of asymptotic curves on surfaces with only elliptic points.

7.15 Theorem. A line in the tangent plane $\tau_p S$ is an asymptoticc line if and only if it has contact of the second order with the surface.

Proof. If a direction is asymptotic then the normal section in this direction has zero curvature at the point p. Thus p is inflection point of the normal section, i.e. the its tangent line has contact of the second order with this section. In the opposite direction, if a tangent line at the point $p \in S$ has contact of the 2nd order with some curve $\gamma(t)$ on S, $\gamma(t_0) = p$, it is the inflection point of this curve. Thus the vector $\frac{d^2\gamma(t)}{dt^2}$ is colinear with vector $\frac{d\gamma(t_0)}{dt}$ which is perpendicular to the normal vector $n(p)$, i.e.

e7.19 (19)
$$\Big(n(p), \frac{d^2\gamma(t_0)}{dt^2}\Big) = 0\,.$$

□

7.16. Recall the osculating plane of a spacial curve is not determined in its inflection points,

Theorem. A curve C on the surface S is asymptotic if and only if, at each its point, the osculating plane coincides with the tangent plane of the surface or is not determined.

Proof. The osculating plane of the curve $C \equiv \gamma(t)$ at the point $p = \gamma(t_0)$ is determined by vectors $\frac{d\gamma(t_0)}{dt}$, $\frac{d^2\gamma(t_0)}{dt^2}$ if these are linearly independent. Here $\frac{d\gamma(t_0)}{dt}$ llies in the tangent plane of the surface. Thus the tangent plane of S coincides with the osculating plane of the curve C if and only if teh normal vector $n(s)$ is perpendicular to $\frac{d^2\gamma(t_0)}{dt^2}$, i.e. (19) holds. If this is an inflection point, the vector $\frac{d^2\gamma(t_0)}{dt^2}$ is colinear with $\frac{d\gamma(t_0)}{dt}$ and (19) holds as well. In the opposite direction, if $\Phi_2\big(\frac{d\gamma(t_0)}{dt}\big) = 0$ then (19) holds and similarly as in the first part of the proof, one verifies this is one of two cases obtained above. □

7.17. We know from 1.28 that a (part of a) line is characterized by the fact that all its poins are inflection. Thus if there is a (part of a) line on the surface, it is an asymptotic curve. This yields e.g. asymptotic directions and curves on regular ruled surfaces, i.e. on the hyperboloid of one sheet and hyperbolic paraboloid.

7.18. Considering hyperbolic points, asymptotic directions divide directions in the tangent plane to two parts The sign of the normal curvature is positive in one of them and negative in the other one. Thus in the positive part, normal sections lie locally above the tangent plane in the direction of the oriented normal line and in the negative part, normal section lie on the opposite part of the tangent plane. Thus the surface lies on both sides of the tangent plane. A prominent example is the surface $z = xy$. Axes x and y lie on this surface, i.e. they are asymptotic curves. The tangent plane at the origin is $z = 0$. Assuming $x > 0$, $y > 0$ or $x < 0$, $y < 0$, the surface lies above the tangent plane, assuming $x > 0$, $y < 0$ nebo $x < 0$, $y > 0$, the surface lies under the tangent plane. JS: missing picture

The sign of the curvature in an elliptic point is the same in all directions hence the whole surface locally lies on one side of the tangent plane. The simples cases are sphere and ellipsoid.

Another interesting example is the anuloid. On the "out part of the tire", the surface lies on one side of the tangent plane, i.e. all points are elliptic there. The the whole inner part of the anuloid lies locally lies on both sides of the tangent plane, i.e. these are hyperbolic points. "Bottom and top" circles are formed by parabolic points.

7.19 Remark. Finally we shall show how one can characterize planar and spherical point using the notionof **contact of surfaces**.

Let p be a common point of S and $\bar{S}$. We say **surfaces** S and $\bar{S}$ **have contact of the order** k **at the point** p if for every curve $C \subset S$ through the point p there exists a curve $\bar{C} \subset \bar{S}$ such that curves C and $\bar{C}$ contact of the kth order at the point p. It is shown in [5] that this is an equivalence relation and also a computational criterion (similar to the case of curves and surfaces in 2.5 and 4.7) is derived.

Assuming the surface S is given by a parametrization $f(u)$ and the surface $\bar{S}$ is given by an equation $F(x, y, z) = 0$ then we shall form a function of two variables

$$\Phi(u_1, u_2) = F\big(f_1(u_1, u_2), f_2(u_1, u_2), f_3(u_1, u_2)\big)\,.$$

It holds that if surfaces S and $\bar{S}$ have contact of the kth order at the joint point $p = f(u_0)$ if and only if all partial derivatives of the function Φ at the point $u_0 = (u_1^0, u_2^0)$ up to the order $\leq k$ are zero. The case $k = 1$ then means two surfaces have contact of the 1st order at a joint pont if and only if they have the same tangent line at this point.

Considering the surface $\bar{S}$,

$$ax + by + cz + d = 0\,,$$

we have

$$\Phi(u_1, u_2) = af_1(u_1, u_2), bf_2(u_1, u_2), cf_3(u_1, u_2) + d\,.$$

Conditions for the contact of the first order

$$a\partial_1 f_1(u_0) + b\partial_1 f_2(u_0) + c\partial_1 f_3(u_0) = 0\,,\ a\partial_2 f_1(u_0) + b\partial_2 f_2(u_0) + c\partial_2 f_3(u_0) = 0$$

mean that the vector (a, b, c) is colinear with the normal vectorem $n(u_0)$ of the surface S at the point $f(u_0)$. The condition for contact of the 2nd order is

$$\big(n(u_0), \partial_{11} f(u_0)\big) = 0\,, \quad \big(n(u_0), \partial_{12} f(u_0)\big) = 0\,, \quad \big(n(u_0), \partial_{22} f(u_0)\big) = 0\,.$$

Thus the point $p \in S$ is planar if and only if the tangent plane at this point has contact of the 2nd with the surface.

A similar computation shows that the point $f(u_0) \in S$ is spherical if and only if there exists a sphere Q such that S and Q have contact of the 2nd order at the point $f(u_0)$.

8 Principal curves

principal

8.1. Values of the normal curvature of the surface $S \equiv f(u)$ at its non-planar point p can be visualize in the folloing way. We consider the segment of the length $\frac{1}{\sqrt{|\varkappa|}}$ on the tangent line (in both directions) in a nonasymptotical direction. Here $\frac{1}{\sqrt{|\varkappa|}}$ denotes the normal curvature in this direction. If $a_1 f_1(p) + a_2 f_2(p)$ is such a vector, square of its norm is $\frac{1}{|\varkappa|}$, i.e.

e8.1 (1)
$$g_{11}a_1^2 + 2g_{12}a_1a_2 + g_{22}a_2^2 = \frac{1}{|\varkappa|}\,.$$

But $\varkappa$ is given by 7.(7) hence (1) is equivalent to rovnici

e8.2 (2)
$$|h_{11}a_1^2 + 2h_{12}a_1a_2 + h_{22}a_2^2| = 1\,.$$

Definition. The curve (2) is called **Dupin indikatrix** at a non-planar point of the surface.

8.2 **8.2.** The curve (2) is an ellipse in elliptic points. Considering the equation of the unit circle in our affine coordinates in our tangent plane is

$$g_{11}a_1^2 + 2g_{12}a_1a_2 + g_{22}a_2^2 = 1$$

then it follows from 7.(10) that the ellipse is a circle precisely in spherical points of the surface.

Considering a hyperbolic point and changing suitable coordinates, we can transform the equation $h_{11}a_1^2 + 2h_{12}a_1a_2 + h_{22}a_2^2 = 1$ to the form

e8.3 (3)
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,.$$

Thus (2) corresponds to a pair of so called conjugated hyperbolas which JS: missing picture
is formed by (3) and the hyperbola $\frac{x^2}{a^2} - \frac{z^2}{b^2} = -1$.

Considering a parabolic point, (2) is a pair of parallel lines in the tangent plane with the point of the surface in the middle. Indeed, we have $h_{11}h_{22} = h_{12}^2$ in this case. Considering $h_{11} > 0$, $h_{12} > 0$, we have $h_{12} = \pm\sqrt{h_{11}}\sqrt{h_{22}}$. Consider the case of the positive sign first. Then the equation (2) has the form

e8.4 (4)
$$1 = h_{11}a_1^2 + 2\sqrt{h_{11}}\sqrt{h_{22}}a_1a_2 + h_{22}a_2^2 = \left(\sqrt{h_{11}}a_1 + \sqrt{h_{22}}a_2\right)^2.$$

This is an equation of the pair of parallel lines

(5)
$$1 = \sqrt{h_{11}}a_1 + \sqrt{h_{22}}a_2\,, \quad -1 = \sqrt{h_{11}}a_1 + \sqrt{h_{22}}a_2$$

with the origin in the middle. The case of the negative sign yields the sema result. A similar computation yields the same result for $h_{11} < 0$, $h_{22} < 0$.

8.3. Assuming a non-spherical point, we define **axes of the Dupin indikatrix** as axes of the ellipse or common axes of the pair of conjugated hyperbolas or as the axis of the pair of parallel lines together with the parpendicular line through the origin. JS: missing picture

Definition. Directions of the Dupin indikatrix are called **principal directions of the surface** S at a given point. A curve on S which touches a principal direction at every point is called **principal curve**.

Principal directions are not defined in planar and spherical points.

We thus have the net of principal curves on surfaces without planar and spherical points. This net is orthogonal.

8.4 **8.4.** Since Φ_2 is a quadratic form, it deteremines the polar bilinear form denoted by the same symbol. Given two vectors $A = (a_1, a_2)$, $B = (b_1, b_2) \in T_pS$, we have

e8.6 (6) $$\Phi_2(A, B) = h_{11}(p)a_1b_1 + h_{12}(p)(a_1b_2 + a_2b_1) + h_{22}(p)a_2b_2 \,.$$

The condition $\Phi_2(A, B) = 0$ depends only on directions of the vectors A, B. This is the condition of polar conjugation with respect to $\Phi_2(p)$.

Definition. Directions in the tangent plane determined by nonzero vectors $A, B \in T_pS$ are called **conjugated** if they are polar conjugated with respect to $\Phi_2(p)$.

From the computational point of view, the condition of conjugation is given by (6) beeing zero.

ve8.5 **8.5 Theorem.** Principal directions of the surface are directions which are in the same time conjugated and perpendicular.

Proof. We know from analytic geometry that this characterizes axes ellipses and hyperbolas. The case of parallel lines can be computed analogously. □

8.6 **8.6.** Beside (6) beeing zero, principal directions satisfy also the condition of orthogonality

e8.7 (7) $$\Phi_1(A, B) = g_{11}a_1b_1 + g_{12}(a_1b_2 + a_2b_1) + g_{22}a_2b_2 = 0 \,.$$

If (b_1, b_2) is a nonzero direction satisfying (7) and for which (6) is zero, we have a system of two homogeneous linear equations with a nonzero solution. The determinant of the system is zero,

e8.8 (8)
$$\begin{vmatrix} g_{11}a_1 + g_{12}a_2, & g_{12}a_1 + g_{22}a_2 \\ h_{11}a_1 + h_{12}a_2, & h_{12}a_1 + h_{22}a_2 \end{vmatrix} = 0\,.$$

Paasing to the diferentials $du_1 = a_1$, $du_2 = a_2$, we get

Theorem. The differential equation of the net of principal curves is

e8.9 (9)
$$\begin{vmatrix} g_{11}du_1 + g_{12}du_2, & g_{12}du_1 + g_{22}du_2 \\ h_{11}du_1 + h_{12}du_2, & h_{12}du_1 + h_{22}du_2 \end{vmatrix} = 0\,.$$

Since (9) is generally a quadratic equation for th fraction $\frac{du_2}{du_1}$. Its two solutions $\frac{du_2}{du_1} = F_1(u_1, u_2)$, $\frac{du_2}{du_1} = F_2(u_1, u_2)$ are diferential equations of these two systems of principal curves.

de8.7 **8.7 Definition.** Normal curvatures $\varkappa_1$, $\varkappa_2$ in principal directions are called **principal curvatures of the surface**. The sum $H = \varkappa_1 + \varkappa_2$ of principal curvatures is called **mean curvature**, the product $K = \varkappa_1\varkappa_2$ is called **Gauss** (or **total**) **curvature**.

Considering spherical points, the normal curvature has the same value $\varkappa$. Here we define $H = 2\varkappa$, $K = \varkappa^2$. Considering planar points, all normal curvatures are zero. Here we put $H = 0$, $K = 0$.

A change of orientation of the surface S changes the sign of the normal curvature. The sign of the mean curvature H thus depeneds on the orientation of the surface, however the sign of the Gauss curvature K is independent on the orientation of the surface.

8.8 **8.8.** The Dupin indikatrix shows that the normal curvature has extremals at principal directions. We shall use that to derive a formula for principal curvatures. The following computation concerns only the "generic" case but one can show that the result holds in all cases. Considering the direction $\varrho = \frac{du_1}{du_2}$ then according to 7.(7), the normal curvature $\varkappa(\varrho)$ in this direction satisfies

$$\varkappa(\varrho) = \frac{h_{11}\varrho^2 + 2h_{12}\varrho + h_{22}}{g_{11}\varrho^2 + 2g_{12}\varrho + g_{22}}\,.$$

To simplify the computation, we shall write this in the form

e8.10 (10)
$$\varkappa(g_{11}\varrho^2 + 2g_{12}\varrho + g_{22}) - (h_{11}\varrho^2 + 2h_{12}\varrho + h_{22}) = 0\,.$$

Differentiating this with respect to ϱ and using the condition fors extremals $\frac{d\varkappa}{d\varrho} = 0$, we get

e8.11 (11) $$\varkappa(g_{11}\varrho + g_{12}) - (h_{11}\varrho + h_{12} = 0\,.$$

Multiplying this by to $-\varrho$ and adding the result to (10), we odtain

e8.12 (12) $$\varkappa(g_{12}\varrho + g_{22}) - (h_{12}\varrho + h_{22}) = 0$$

Putting back $\varrho = \frac{du_1}{du_2}$ and using an algebraic manipulation, (11) and (12) have the form

e8.13 (13) $$\begin{aligned}(\varkappa g_{11} - h_{11})\,du_1 + (\varkappa g_{12} - h_{12})\,du_2 = 0\\ (\varkappa g_{12} - h_{12})\,du_1 + (\varkappa g_{22} - h_{22})\,du_2 = 0\,.\end{aligned}$$

Here (du_1, du_2) is a nonzero direction which realizes the extremal. Thus the determinant of the system of two linear equations (13) must be zero. Therefore

Theorem. Principal curvatures $\varkappa_1$, $\varkappa_2$ are roots of the quadratic equation

e8.14 (14) $$\begin{vmatrix}\varkappa g_{11} - h_{11}, & \varkappa g_{12} - h_{12}\\ \varkappa g_{12} - h_{12}, & \varkappa g_{22} - h_{22}\end{vmatrix} = 0\,.$$

8.9 **8.9.** A simple corollary of (14) is

Theorem. The mean and the Gauss curvature satisfy

e8.15 (15) $$H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{g_{11}g_{22} - g_{12}^2}\,, \quad K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}\,.$$

Proof. It follows from (14) that

$$\varkappa^2(g_{11}g_{22} - g_{12}^2) - \varkappa(g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}) + (h_{11}h_{22} - h_{12}^2) = 0\,.$$

The sum $H = \varkappa_1 + \varkappa_2$ and the product $K = \varkappa_1\varkappa_2$ of roots have the form (15) using a well known properties of of quadratic equations. kvadratick rovnice. □

Further we shall show that (15) holds also in spherical and planar points. According to 7.(7), a spherical point satisfies $h_{ij} = \varkappa g_{ij}$, $i = 1, 2$ where $\varkappa$ is a common value of the normal curvature in all directions. Then (15) means $H = 2\varkappa$, $K = \varkappa^2$. We have $h_{ij} = 0$ in planar points, i.e. $H = 0$ and $K = 0$.

8.10 **8.10.** Since $g_{11}g_{22} - g_{12}^2 > 0$ and $h_{11}h_{22} - h_{12}^2$ is the expression used in the definition 7.13, we obtained a new insight in this definition.

Corollary. Elliptic, parabolic and hyperbolic pointsare characterized by the condition $K > 0$, $K = 0$ and $K < 0$, respectively.

Remark. Since $K = 0$ in planar points, planar points are sometimes considered as parabolic points.

pr8.11 **8.11 Example.** The Gauss curvatureof the sphere with radius r is $\frac{1}{r^2}$. Indeed, all its points are spherical and the normal section in every direction is a circle with radius r. Thus $K = \frac{1}{r^2}$.

8.12 **8.12.** The following formula nicely describes the normal curvature in an arbitrary direction using principal curvatures.

Theorem (Euler formula)**.** Let σ_1 and σ_2 are principal directions at the point p of the surface S, let $\varkappa_1$ and $\varkappa_2$ are corresponding principal curvatures and let s be the direction which has the angle σ_1 with φ. Then the normal curvature $\varkappa_s$ in this direction satisfies

e8.16 (16)
$$\varkappa_s = \varkappa_1 \cos^2 \varphi + \varkappa_2 \sin^2 \varphi \,.$$

Proof. Let e_1, e_2 be unit vectors in directions σ_1, σ_2, respectively. We can consider parameters u_1, u_2 on S such that e_1 and e_2 are tangent vectors of the parametric net, i.e. $e_1 = (du_1, 0)$, $e_2 = (0, du_2)$. Then $g_{11}(p) = g_{22}(p) = 1$, $g_{12}(p) = 0$ and conjugacy of directions σ_1 and σ_2 yields $h_{12}(p) = 0$. It follows from the general formula 7.(7) for $\varkappa$ that $\varkappa_1 = h_{11}(p)$, $\varkappa_2 = h_{22}(p)$. The unit vector in the direction s has the form $e_1 \cos\varphi + e_2 \sin\varphi$. Putting this vector into 7.(7), we get $\varkappa_s = \varkappa_1 \cos^2 \varphi + \varkappa_2 \sin^2 \varphi$. □

8.13 **8.13.** We shall discuss one more geometric property which directly characterizes principal curves. Given a curve $\gamma(t)$ on the surface S, we denote by n_γ the 1-parameter system of normal vectors along γ.

Theorem. The curve $\gamma(t)$ is a principal curve of the surface S if and only if the vector $\frac{dn_\gamma}{dt}$ is colinear with the vector $\frac{d\gamma}{dt}$ for all t.

Proof. Let S be given by the parametrization $f(u)$ and γ is given by $\big(u_1(t), u_2(t)\big)$ in the parameter space. That is,

e8.17 (17)
$$\frac{d\gamma}{dt} = f_1 \frac{du_1}{dt} + f_2 \frac{du_2}{dt} \,.$$

Assume the vector

e8.18 (18)
$$a_1 f_1 + a_2 f_2$$

is perpendicular to (17). Similarly we have $n_\gamma(t) = n\big(u_1(t), u_2(t)\big)$ hence

e8.19 (19)
$$\frac{dn_\gamma}{dt} = n_1 \frac{du_1}{dt} + n_2 \frac{du_2}{dt}\,.$$

This vector lies in the tangent plane since vectors n_1 and n_2 are perpendicular to n, see 7.(9). Vectors (17) and (19) are colinear if and only if vectors (18) and (19) are perpendicular. Using the formula 7.(8), we get

e8.20 (20)
$$\begin{aligned} 0 &= \Big(f_1 a_1 + f_2 a_2, n_1 \frac{du_1}{dt} + n_2 \frac{du_2}{dt}\Big) \\ &= -\Big[h_{11} a_1 \frac{du_1}{dt} + h_{12}\Big(a_2 \frac{du_1}{dt} + a_1 \frac{du_2}{dt}\Big) + h_{22} a_2 \frac{du_2}{dt}\Big]\,. \end{aligned}$$

Thus directions $\big(\frac{du_1}{dt}, \frac{du_2}{dt}\big)$ and (a_1, a_2) are orthogonal and conjugate, i.e. they are principal directions. Thus $\gamma(t)$ is a principal curve. In the opposite direction, if $\gamma(t)$ is a principal curve then the vector $\frac{d\gamma}{dt}$ will be conjugated with the perpendicular vector, i.e. the second equation in (20) holds. Then the first equation (20) implies the vector $\frac{dn_\gamma}{dt}$ is colinear with the vector $\frac{d\gamma}{dt}$ for all t. □

8.14 **8.14 Example.** Consider a **surface of revolution** S given by rotation of a planar curve C around an axis in the same plane and does not intersect the curve. Similarly as in the earth, **circles of latitude** on S are formed by rotation of a particular point of the curve C, wheres **meridians** on S are positions of the curve C in particular moments of the rotation. We shall show that circles of latitude and meridians are principal curves of the surface of revolution S. Consider an arbitrary meridian of the surface S which we identify with the curve C. Thus normal vectors $n_C(t)$ of the plane curve C are in the same time normal vectors of the surface. All vectors $n_C(t)$ are unit thus $\big(n_C(t), n_C(t)\big) = 1$ shows, by differentiation, that vectors $\frac{dn_C(t)}{dt}$ are perpendicular to $n_C(t)$. Thus vectors $\frac{dn_C(t)}{dt}$ are colinear with tangent vectors of the curve C. Thus meridians are principal curves according to the theorem 8.13. Circles of latitude are perpendicualr to meridians hence theu are also principal curves (because the net of principal curves is orthogonal).

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8.15 **8.15.** One can derive the previous result also by computation. We prescribe the curve C in the plane (x, z) locally by the parametrization $x = g(t)$,

$z = h(t)$, $t \in I$, i.e. the two dimensional vector $\big(g'(t), h'(t)\big)$ is nonzero for each $t \in I$. We can assume values of the parameter t are positive. We shall rotate along the z-axis and the assumption that C does not intersect the axis of rotation, means that C lies in the half-plane $x > 0$, i.e. $g(t) > 0$ for all $t \in I$. We denote by v the angle between the projection of the rotation plane to the (x, y)-plane with the positive x-half-axis. The parameter space D can be viewed as the space between two circles in $\mathbb{R}^2$ which, in polar coordinates, is characterized by the length of the radius in the intervalu I with arbitrary polar angle. In this sense, we can write $v \in [0, 2\pi)$. JS: missing picture

The point with the x-coordinate $g(t)$ moves along the circle $x = g(t)\cos v$, $y = g(t)\sin v$ in the plane $z = h(t)$. Thus the parametric description of the surface of revolution is

$$f(t, v) = \big(g(t)\cos v, g(t)\sin v, h(t)\big)\,, \quad t \in I, v \in [0, 2\pi)\,.$$

From the point if view of the general theory, t and v play the role of parameters u_1 and u_2, respectively.

Partial derivatives with respect to t and v are

$$f_1 = (g'\cos v, g'\sin v, h')\,, \quad f_2 = g(-\sin v, \cos v, 0)\,.$$

Coefficients of the first fundamental form hence are

$$g_{11} = g'^2 + h'^2\,, \quad g_{12} = 0\,, \quad g_{22} = g^2\,.$$

Further we have

$$f_1 \times f_2 = g(-h'\cos v, -h'\sin v, g')\,, \quad n = \frac{1}{\sqrt{g'^2 + h'^2}}(-h'\cos v, -h'\sin v, g')\,.$$

In the second order we have partial derivatives

$$\begin{aligned} f_{11} &= (g''\cos v, g''\sin v, h'') \\ f_{12} &= g'(-\sin v, \cos v, 0)\,, \\ f_{22} &= g(-\cos v, -\sin v, 0)\,. \end{aligned}$$

Following 7. (5), coefficients of the second fundamenta form are

$$h_{11} = \frac{g'h'' - h'g''}{\sqrt{g'^2 + h'^2}}\,, \quad h_{12} = 0\,, \quad h_{22} = \frac{h'g}{\sqrt{g'^2 + h'^2}}\,.$$

Generally, already conditions $g_{12} = 0$, $h_{12} = 0$ simplify the differential equation of principal curves (9) to the form

$$\begin{vmatrix} g_{11} & g_{12} \\ h_{11} & h_{22} \end{vmatrix} du_1\, du_2 = 0\,.$$

The determinant is zero if and only if $h_{11} = cg_{11}$, $h_{22} = cg_{22}$, i.e. this is a spherical or planar point which is excluded from consideration leading to spherical curves. The equation $du_1 du_2 = 0$ then characterizes the parametric net $u_1 =$ konst. and $u_2 =$ konst. In our case of the surface of revoution, these are meridians and circles of revolution.

8.16 **8.16.** We shall describe a relation between the curvature of an arbitrary plane section of the surface S and the curvature of the normal section in the same direction. Let ϱ be an arbitrary plane through the point $p \in S$ different form the tangent plane $\tau_p S$.

Theorem (Meusnierova)**.** Let $\varkappa_n$ be the normal curvature of the surface S in the direction of the line $\varrho \cap \tau_p S$ and $0 \le \alpha < \frac{\pi}{2}$ be the angle between the normal line $N_p S$ and the plane ϱ. Then the curvature $\varkappa_\varrho$ of the section of the surface S by the plane ϱ at the point p satisfies

$$\varkappa_n = \varkappa_\varrho \cos\alpha\,.$$

Proof. Let $\gamma(s)$ be the arc-length parametrization of the curve $\varrho \cap S$, $\gamma(0) = p$. Following the theorem 7.5, we have

$$\varkappa_n = \left| \left(n, \frac{d^2\gamma(0)}{ds^2} \right) \right|.$$

It follows from the theory of planar curves that $\frac{d^2\gamma(0)}{ds^2} = \varkappa_\varrho e_2$ where e_2 is a unit vector in the plane ϱ. in our case, we have $|(n, e_2)| = \cos\alpha$. □

8.17 **8.17.** Consider a non-asymptotic direction A in the tangent plane. Denote by c_n the center of curvature of the normal section and by c_ϱ the center of curvature of the sectionby the plane ϱ, see the picture which shows the section by the plane perpendicular to teh direction A. It follows form the Meusnier therem that $\cos\alpha = \frac{\varkappa}{\varkappa_\varrho}$, hence the triangle $p\, c_n\, c_\varrho$ has the right angle at the vertex c_ϱ. Geometrically this means that centers of the curvature of all plane sections of the surface S in the direction A lie on a circle for which the segment pc_n is the diameter. Therefore given a fixed A, the normal section has the smalles curvature and curvatures of all remaining plane sections increases in the way described in the Meusnier theorem. An example if the sphere where these sections are circles with the radius which decreases in this way.

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8.18 **8.18.** Finally we consider a a class of surfaces which is interested both geometrically and from a point if view of applications.

Definition. The surface S is called **minimal** if its mean curvature H is zero in all points.

A nontrivial example of a minimal surface is the **helicoid** which we shall study in sections 10.7 and 10.9.

The adjective “minimal” is based on the variational calculus. One of important variational problems is the problem to “span” a surface with a minimal are to the given curve in E_3. Under rather general conditions, solutions of this problem are surfaces with minimal mean curvature.

9 Envelope of a family of surfaces

10 Ruled surfaces

10.1. One-parameter system of lines in E_3 is a mapping which each $t \in I$ maps to a line $p(t)$ where I is an open interval. The line $p(t)$ is called **generating line of the surface**. This line can be described using a point $g(t) \in p(t)$ and a nonzero vector $h(t)$ in the direction of $p(t)$. An arbitrary point of the line $p(t)$ then has the form

e10.1 (1) $$f(t,v) = g(t) + vh(t)\,, \qquad v \in \mathbb{R}\,.$$

Similarly as in agreement 1.15 or 4.7, we shall further assume $g(t)$ and $h(t)$ are functions of the class C^r where r is big enough for our consideration. Then $f\colon I \times \mathbb{R} \to E_3$ is map of the class C^r. Thus, in a sense, this is a two parametric movement. The condition from the definition of surfaces requires, beside injectivity of f, also vectors $\frac{\partial f}{\partial t} := f_t = g' + vh'$ and $\frac{\partial f}{\partial v} := f_v = h$ to be lienarly independent at every point. We shall illustrate this on examples.

10.2 **10.2.** Let the point $g(t) = a$ be fixed. Then we have **generalized cone**

e10.2 (2) $$f(t,v) = a + vh(t)\,.$$

Therefore $f_t = vh'$, $f_v = h$, $f_t \times f_v = v(h' \times h)$. Tha value $v = 0$ corersponds JS: missing picture
to the vertex of the cone which is obviously a singular point. Given $v \neq 0$, we need $h' \times h \neq o$ for all $t \in I$. If this is satisfied then, assuming injectivity of f, we get a surface. If $h' \times h = 0$ everuwhere, we have

e10.3 (3) $$\frac{\partial h}{\partial t} = k(t)\, h(t)$$

where $k(t)$ is a reak function. If we consider a real function $z(t)$ instead of $z(t)$ then our differential equation, by separation of variables, has the general solution $z = l(t)c$ where $l(t) = e^{Sk(t)\,dt}$. This holds for each component of our vector valued function $h(t)$ hence $h(t) = l(t)b$ where b is a constant vector. That is, this is the case of a two parametric movement along a line a not a surface.

10.3 **10.3.** Let $h(t) = a$ is a fixed nonzero vector. Then we get **generalized cylinder** JS: missing picture

e10.4 (4) $$f(t,v) = g(t) + va\,, \qquad t \in I,\ v \in \mathbb{R}\,.$$

Here $f_t = g'$, $f_v = a$. If $g'(t) \times a \neq o$ for all t and f is injective, it is a surface. If the tangent vector $g'(t)$ is colinear with the vector a at some point, this point is singular.

10.4 **10.4.** Consider a curve $C \equiv g(t)$ given parametrically and consider the tangent line at every point $g(t)$. This one parametric family of curves is called **tangent developable** of the curve C. Its parametric expression has the form

e10.5 (5) $$f(t,v) = g(t) + v\, g'(t)\,.$$

We have $f_t = g'(t) + v\, g''(t)$, $f_v = g'(t)$ hence

e10.6 (6) $$f_t \times f_v = -v\big(g'(t) \times g''(t))\big)\,.$$

Further we assume that C does not hav inflection points Then the vector (6) is zero if and only if $v = 0$ which is the original point on the curve.

Consider the normal plane $\nu(t,v)$ at the point $g(t_0)$ of the curve C. Using scalar product, this is given by

e10.7 (7) $$\big(g'(t_0), w - g(t_0)\big) = 0\,, \quad w(x,y,z) \in E_3\,.$$

The tangent line at $g(t)$ is given parametrically as

$$g(t) + v\, g'(t)\,.$$

Denote by $v(t)$ the parameter of its intersection with $\nu(t_0)$. That is,

e10.8 (8) $$\big(g'(t_0), g(t) + v(t)\, g'(t) - g(t_0)\big) = 0\,.$$

Such intersection is a movement in the normal plane $\nu(t_0, 0)$ given by the parametrization

e10.9 (9) $$h(t) = g(t) + v(t) g'(t)\,, \qquad v(t_0) = 0\,.$$

We shall sho that $h'(t_0) = o$. We have

$$h'(t_0) = g'(t_0) + v'(t_0)\, g'(t_0) + v(t_0)\, g''(t_0)\,.$$

Since $v(t_0) = 0$, it is enough to prove that $v'(t_0) = -1$. By differentation of (8) we get

$$\big(g'(t_0), g'(t) + v'(t)\, g'(t) + v(t)\, g''(t)\big) = 0\,.$$

Putting $t = t_0$, we obtain

$$\big(g'(t_0), g'(t_0)\big) + v'(t_0)\big(g'(t_0), g'(t_0)\big) = 0\,.$$

Since $\big(g'(t_0), g'(t_0)\big) \neq 0$, we have $v'(t_0) = -1$.

Considering the movement $h(t)$, the condition $h'(t_0) = o$ means that $h(t_0)$ is a singular point. Generally, it is an edge, cf. 1.9. It is easier to think about the tangent developable of the screw line whose projection in the direction of the axes of the screw line is on the picture. This geometrically shows that the tangent developable is not a curve in the sense od the definition 4.6 in a neighbourhood of the generating curve. JS: missing picture

10.5 **10.5.** If one wants to obey the definition 4.6 also in the case of one-parameter systems of lines, the following approach should be used:

Definition. A surface $S \subset E_3$ is called **ruled surface** if S is a part of a one-parameter system of lines.

We shall talk about a generating line of the surface S in such case although it can be only a part of a line.

10.6. We shall further discuss two examples. First consider 3 skew lines q_1, q_2, q_3 (i.e. 3 lines in general position). We consider the transversal of skew lines q_1, q_2 through every point $a \in q_3$ (this is the intersection of planes given by the point p and either the line q_1 or q_2). It is shown in theory of the projective geometry that this construction gives rise to a regular ruled quadric. Starting with 3 lines of the system of lines we have just constructed and repeating this construction, we obtain the second one-parameter system of lines on the same regular ruled quadric. JS: missing picture

10.7 **10.7.** Given a ruled surface which is neither a regular ruled quadric nor a part of a plane, we have shown there can be, beside generating lines, at most two more lines. This is related to the followng general construction. Chose two skew lines q_1, q_2 and a curve C. We consider the transversal of skew lines q_1, q_2 through every point $a \in C$. This yields a one-parameter system of lines JS: missing picture

An important example (for technical applications) is the case when one of skew lines is an improper line of some plane ϱ. Thus we have the plane ϱ, a line q and a curve C. We consider lines intersecting q and C which are parallel with ϱ. This yields a one-parameter system of lines which is called **conoid**. If the line q is perpendicular to the plane ϱ, we obtain **right conoid**. JS: missing picture

Example. IF C is the screw line, q its axis and ϱ a plane perpendicular to q, the we obtain so called **right screw conoid** or **helikoid**. We mentioned JS: missing picture this surface in 8.18. Consider q to be z-axis and C to lie on the the cylinder with he unit radius around the z-axis. Parametric description of C then is $(\cos t, \sin t, bt)$, $b \neq 0$, $t \in \mathbb{R}$. The helicoid has the form

e10.10 (10) $$f(t, v) = (v \cos t, v \sin t, bt), \qquad b \neq 0, v \in \mathbb{R}.$$

It is easy to see that kinematically this surface is built by screwing a generating line perpendicular to and intersecting the axis q, in the direction of this axis.

de10.8 **10.8 Definition. A ruled surface** is called **developable** if tangent planes at all ponits of a fixed generating line are the same.

We say the tangent plane is tangent along generating lines.

Geometrically it is clear (and the related computation is easy) that this is a property of generalized cylinders and cones. We shall show that also in also in the case of tangent developable of the curve C, the tangent line of the surface is the same at all point of a generating line. Following 10.4, the tangent plane of the surface $g(t) + vg'(t)$ at the point $g(t_0) + v_0 g'(t_0)$, $v_0 \neq 0$ is determined by this point and vectors $g'(t_0)$ and $g''(t_0)$. For a fixed t_0 and each $v_0 \neq 0$, this plane coincides with the osculating plane of the curve C at the point $g(t_0) + vg(t_0)$.

On the other hand, the tangent plane along a generating line of a regular quadric is not fixed. Here the tangent plane is determined by the given generating line and by a line of the other system through a given point. However, the other system is formed by trasnversal lines which cannot lie in a fixed plane. Thus considering the helicoid from 10.7, the tangent plane changes along a generating line. It follows from (10) that

e10.11 (11) $$f_t = (-v \sin t, v \cos t, b), \quad f_v = (\cos t, \sin t, 0)$$

thus the unit normal vector is

$$\frac{f_t \times f_v}{\|f_t \times f_v\|} = \frac{1}{\sqrt{v^2 + b^2}}(-b \sin t, b \cos t, -v)$$

which changes depending on v (for a fixed $t = t_0$).

10.9 **10.9.** Further we shall need **to express coefficients of the second fundamental form using the exteriorproduct** We know the exterior product $[a, b, c]$ of three vectors in the euclidean oriented three-dimensional space is formed id equal to the scalar product of the vector product of first vectors with the third vector, i.e.

$$[a, b, c] = (a \times b, c).$$

Using this to formulae 7.(1) and 7.(5), we obtain

e10.12 (12)
$$h_{11} = \frac{1}{\|f_1 \times f_2\|} [f_1, f_2, f_{11}] ,$$
$$h_{12} = \frac{1}{\|f_1 \times f_2\|} [f_1, f_2, f_{12}] ,$$
$$h_{22} = \frac{1}{\|f_1 \times f_2\|} [f_1, f_2, f_{22}] .$$

Example. We shall show the helicoid from (10) is a minimal surface in the sense of 8.18, i.e. that $H = 0$. Differentiating of (11) we obtain $f_{tt} = (-v\cos t, -v\sin t, 0)$, $f_{tv} = (-\sin t, \cos t, 0)$ and $f_{vv} = o$. Thus $h_{11} = 0$ and $h_{22} = 0$. Since also $g_{12} = 0$, the formula 8.(15) yields $H = 0$.

10.10 **10.10.** Since every line on a surface is asymptotic curve, all points on ruled surfaces are hyperbolic, parabolic or planar.

Theorem. A developable ruled surfaces S has zero Gaussian curvature.

Proof. Given a parametrization $f(t, v) = g(t) + v\, h(t)$ of the surface plochy S, we have $f_1 = g'(t) + v\, h'(t)$, $f_2 = h(t)$. For a fixed t, the vector $h(t)$ lies in the associated vector space of the tangent plane for all v if for the vectors $h(t)$ and $g'(t) + v_1 h'(t)$, $g'(t) + v_2 h'(t)$ are complanar for all $v_1 \neq v_2$. By a lienar combination of these vectors we obtain $g'(t)$ and $h'(t)$ hence the condition for a fixed tangent plane is

e10.13 (13)
$$\big[g'(t),\, h(t),\, h'(t)\big] = 0 .$$

A further computation yiedls $f_{11} = g''(t) + v\, h''(t)$ and $f_{12} = h'(t)$, $f_{22} = o$. Following (12) we obtain $h_{22} = 0$,

$$h_{12} = \frac{1}{\|f_1 \times f_2\|} \big[g'(t) + v\, h'(t), h(t), h'(t)\big]$$

and also $h_{12} = 0$ according to (13). It follows from 8.(15) that $K = 0$ independently on h_{11}. □

10.11 **10.11.** In the opposite direction we have the following statement.

Theorem. If every point of the surface S is parabolic then S is locally a developable ruled surface.

Proof. Given a parabolic surface S, we chose local paramatrization such that the net of asymptotic curves is given by $u_1 =$ konst.. Then $0 = h_{12} = (n, f_{12})$ and $0 = h_{22} = (n, f_{22})$. We know that $(n, f_1) = 0$, $(n, f_2) = 0$, $(n, n) = 1$. Differentiating this with respect to u_2 yields – similarly as in 7.(8) – that

$$(n_2, f_1) = 0\,, \quad (n_2, f_2) = 0\,, \quad (n_2, n) = 0\,.$$

Thus $n_2 = o$, i.e.. the normal vector along the curve $u_1 =$ konst. is fixed. Followign Podle 7.(8), it follows from $h_{12} = 0$ that (n_1, f_2)=0. Differenting this with respect to u_2 and using $u_{12} = o$, we obtain $(n_1, f_{22} = 0)$. Hence vectors f_2 and f_{22} are perpendicular to vectors n and n_1 which are linearly independent. Indeed, the vector n_1 is perpendicular to n and is nonzero. Indeed, the first of equations in 7.(8) means $(n_1, f_1) + (n, f_{11}) = 0$. Thus $n_1 = o$ would mean $h_{11} = 0$ which, together with $h_{12} = 0$ and $h_{22} = 0$, would identify a planar point which we exclude. Sicne vectors f_2 and f_{22} are perpendicular to two linearly independent vectors, they are colinear. Hence every point of the curve $u_1 =$ konst. is inflection, i.e. this curve is a line. The tangent line along this generating line is constant thus S is locally a developable ruled surface. □

de10.12 **10.12 Definition. Generating line** $g(t_0) + vh(t_0)$ of a ruled surface (1) is called **cylindric** if vectors $h'(t_0)$ and $h(t_0)$ are colinear.

Theorem. A ruled surface with all generating lines cylindric is a generalized cylinder.

Proof. We have shown in 10.2 that it follows from $\frac{dh}{dt} = k(t)\,h(t)$ that $h(t) = l(t)b$ where b is a constant vector. Thus we have

$$f(t, v) = g(t) + v\,l(t)b$$

which is a cylinder with a different parametrization of generating lines. □

10.13 **10.13.** Similarly, we have a direct geometrical characterization of a cylindrical line. Given a parametrization $p(t) \equiv g(t) + v\,h(t)$, we can assume that the vector $h(t)$ is unit. Then $h(t)$ is a motion along a unit sphere which is called **spherical image of a ruled surface**.

Differentiating $(h, h) = 1$ we obtain $(h, h') = 0$ thus the vector $h'(t)$ is perpendicular to $h(t)$ for eveery t. Assuming further that $h'(t_0)$ is colinear with $h(t_0)$, we obtain $h'(t_0) = o$. Therefore

Theorem. Generating line $p(t_0)$ of a ruled surface S is cylindric if and only if t_0 is a singular point of the sphercial image of the surface S.

10.14 **10.14.** The meaning of the word "generally" in the following statement is explained during the proof.

Theorem. A ruled surface without cylindrical lines is generally either a tangent developable or a cone.

Proof. □

10.15 **10.15.** Our results from 10.12 and 10.14 are sometimes summarized as **surface with zero Gaussian curvature is generally either a tangent developable or a generalized cylinder or a generalized cone**.

11 Isometric mappings

11.1 **11.1.** Let D, $\bar{D} \subset \mathbb{R}^2$ be open sets. A map $\varphi\colon D \to \bar{D}$ is given by a pair of functions φ_1, $\varphi_2\colon D \to \mathbb{R}$ which are called components of the map φ. Denoting u_1, u_2 coordinates in D and v_1, v_2 coordinates in $\bar{D}$, we have $v_1 = \varphi_1(u_1, u_2)$, $v_2 = \varphi_2(u_1, u_2)$.

Definition. We say φ is **mapping of the class** C^r if φ_1 and φ_2 are functions of the class C^r.

Henceforth we assume φ is a mapping of the class C^r, $r \geq 1$.

de11.2 **11.2 Definition.** The determinant $J(\varphi) = \begin{vmatrix} \frac{\partial\varphi_1}{\partial u_1}, & \frac{\partial\varphi_1}{\partial u_2} \\ \frac{\partial\varphi_2}{\partial u_1}, & \frac{\partial\varphi_2}{\partial u_2} \end{vmatrix}$ is called **Jacobian of the mapping** φ.

11.3 **11.3.** We shall study mapppings $g\colon S \to \bar{S}$ between two simple surfaces of the class C^r. Assume S and $\bar{S}$ are given by parametrizations $f(u_1, u_2)$, JS: missing picture
$(u_1, u_2) \in D$ and $\bar{f}(v_1, v_2)$, $(v_1, v_2) \in \bar{D}$. The map g determines a unique map $\psi\colon D \to \bar{D}$ such that $g \circ f = \bar{f} \circ \psi$. Thus ψ expresses g in the parameter space. In the opposite direction, ψ deteremines the mapping g.

Definition. We say that $g\colon S \to \bar{S}$ is **mapping of the class** C^r if the corresponding mapping $\psi\colon D \to \bar{D}$ is of the class C^r.

11.4 **11.4.** A choice of parametrizations of surfaces S and $\bar{S}$ is used in the previous definition. We shall show that the class of differentiability of the mapping g is independent on the choice of parametrizations f and $\bar{f}$. First we discuss a change of the parametrization $f(u_1, u_2)$ of the surface S, $(u_1, u_2) \in D$.

Consider a bijective mapping $\varphi\colon \bar{D} \to D$ of the class C^r where $\varphi = \big(\varphi_1(v_1, v_2), \varphi_2(v_1, v_2)\big)$, $(v_1, v_2) \in \bar{D}$. Then $\bar{f} = f \circ \varphi$ is also of the class C^r. In the case of parametrziation of a surface we have further the condition $f_1 \times f_2 \neq o$. Using the notation $\bar{f}_1 = \partial_1(f \circ \varphi)$, $\bar{f}_2 = \partial_2(f \circ \varphi)$ and using the chain rule, we obtain

e11.1 (1) $$\bar{f}_1 = f_1 \frac{\partial\varphi_1}{\partial v_1} + f_2 \frac{\partial\varphi_2}{\partial v_1}\,, \quad \bar{f}_2 = f_1 \frac{\partial\varphi_1}{\partial v_2} + f_2 \frac{\partial\varphi_2}{\partial v_2}\,.$$

Thus

$$\bar{f}_1 \times \bar{f}_2 = \Big(\frac{\partial\varphi_1}{\partial v_1}\frac{\partial\varphi_2}{\partial v_2} - \frac{\partial\varphi_2}{\partial v_1}\frac{\partial\varphi_1}{\partial v_2}\Big)\, f_1 \times f_2 = J(\varphi)\, f_1 \times f_2\,.$$

Definition. A bijective map $\varphi\colon \bar{D} \to D$ of the class C^r is called **reparametrization** if $J(\varphi) \neq 0$for all $(v_1, v_2) \in \bar{D}$.

11.5 **11.5.** Now it is clear that the definition 11.3 does not depend on the choice of parametrization. Consider a reparametrization $\varphi\colon D_1 \to D$ on S and a reparametrizaci $\bar{\varphi}\colon \bar{D}_1 \to \bar{D}$ on $\bar{S}$. Then $\bar{\varphi}^{-1}\colon \bar{D} \to \bar{D}_1$ is also a map of the class C^r (this follows immediately from the generalized implicit function theorem, cf. e.g. the textbook [5]). Thus we have $\bar{\varphi}^{-1} \circ \psi \circ \varphi$ instead of the mapping ψ in the definition 11.3 which is also of the class C^r.

11.6 **11.6.** Further we shall consider **motion** on surfaces.

First consider the map $\psi\colon D \to \bar{D}$, $v_1 = \psi_1(u_1, u_2)$, $v_2 = \psi_2(u_1, u_2)$. Then D itself is a surface (as a part of plane) hence we have the tangent plane T_uD for every $u \in D$ which coincides with $\mathbb{R}^2$. Its elements are tangent vectors to motions $h(t)$, $h\colon I \to D$ at the point $h(0) = u$ where we assume $0 \in I$. Coordinates of the tangent vector are $\frac{dh_1(0)}{dt}$, $\frac{dh_2(0)}{dt}$. Consider the motion $\psi \circ h\colon I \to \bar{D}$. Coordinates of its tangent vector at $t = 0$, which one findss applying the chain rule to $\psi_1\big(h_1(t)h_2(t)\big)$ and $\psi_2\big(h_1(t), h_2(t)\big)$, are

e11.2 (2)
$$\frac{\partial\psi_1}{\partial u_1}\frac{dh_1(0)}{dt} + \frac{\partial\psi_1}{\partial u_2}\frac{dh_2(0)}{dt}\,, \quad \frac{\partial\psi_2}{\partial u_1}\frac{dh_1(0)}{dt} + \frac{\partial\psi_2}{\partial u_2}\frac{dh_2(0)}{dt}\,.$$

That is, the tangent vector $\frac{d(\psi\circ h)(0)}{dt}$ is determined by the vector $\frac{dh(0)}{dt}$. Considering the linearity of (2), we have

Theorem. The rule $\frac{dh(0)}{dt} \mapsto \frac{d(\psi\circ h)(0)}{dt}$ deteremines a linear mapping $T_u\psi\colon T_uD \to T_{\psi(u)}\bar{D}$ for each $u \in D$.

Definition. This mapping is called tangent mappping ψ at the point u.

Denoting by (du_1, du_2) coordinates at T_uD by (dv_1, dv_2) coordinates in $T_{\psi(u)}\bar{D}$ then (2) has the form

e11.3 (3)
$$dv_1 = \frac{\partial\psi_1}{\partial u_1}\,du_1 + \frac{\partial\psi_1}{\partial u_2}\,du_2\,, \quad dv_2 = \frac{\partial\psi_2}{\partial u_1}\,du_1 + \frac{\partial\psi_2}{\partial u_2}\,du_2\,.$$

Thus these are differentials of functions ψ_1 and ψ_2.

11.7 **11.7.** Consider the original map $g\colon S \to \bar{S}$ and put $p = f(u) \in S$. Consider the tangent space T_pS and a vector vektor $A \in TpS$ which is tangent to the motion $\gamma(t)$ on S, $A = \frac{d\gamma(0)}{dt}$. Then $g \circ \gamma$ is a motion on $\bar{S}$ and the tangent vector $\frac{d(g\circ\gamma)(0)}{dt}$ to this motion depends only on A. Indeed, it is given (2) in terms of the parametrization. Thus we have shown

Theorem. The rule $\frac{d\gamma(0)}{dt} \mapsto \frac{d(g\circ\gamma)(0)}{dt}$ determines a linear map $T_pg\colon T_pS \to T_{g(p)}\bar{S}$.

Definition. The map T_pg is called **tangent map to** g **at the point** p.

de11.8 **11.8 Definition.** We say the mapping $g\colon S \to \bar{S}$ is **isometric** if each tangent map $T_pg\colon T_pS \to T_p\bar{S}$, $p \in S$ preserves the scalar product.

That is, $(A, B) = \big(T_pg(A), T_pg(B)\big)$ for each $A,\ B \in T_pS$.
If g is bijective, it is **isometry** of S and $\bar{S}$.

11.9 **11.9.** The bijiective map $g\colon S \to \bar{S}$ can be realised in sucha way we cosnider a common parameter space D and corresponding points $f(u_1, u_2) \in S$ and $\bar{f}(u_1, u_2) \in \bar{S}$. In this case we say the map g **is given by equality of parameters**. JS: missing picture

Theorem. The bijection $g\colon S \to \bar{S}$ given by equality of parameters is isometry if and only if first fundamental forms Φ_1 and $\bar{\Phi}_1$ of surfaces S and $\bar{S}$, respectively are coincide.

Proof. Vectors $f_1,\ f_2$ form a basis in T_pS, vectors $\bar{f}_1,\ \bar{f}_2$ form a basis in $T_{g(p)}\bar{S}$ and the linear map T_pg is given by $du_1 = du_1$, $du_2 = du_2$. Following 6.1, scalar products of tangent vectors to S and $\bar{S}$ are given by the first fundamental form. □

The condition $\Phi_1 = \bar{\Phi}_1$ explicitly means $g_{11} = \bar{g}_{11}$, $g_{12} = \bar{g}_{12}$, $g_{22} = \bar{g}_{22}$ where bared quantities are computed on the surface $\bar{S}$ depending on the same parameters (u_1, u_2).

11.10 **11.10.** The following statement justifies the terminology of isometry.

Theorem. The bijection $g\colon S \to \bar{S}$ is isometry if and only if it preserves length of curves.

Proof. The bijection g can be given by equality of parameters. Since the length of the curve $\big(u_1(t), u_2(t)\big)$, $t \in [a, b]$ is

e11.4 (4)
$$s = \int_a^b \sqrt{g_{11}\Big(\frac{du_1}{dt}\Big)^2 + 2g_{12}\frac{du_1}{dt}\frac{du_2}{dt} + g_{22}\Big(\frac{du_2}{dt}\Big)^2}\, dt\,,$$

it follows from the theorem 11.9 that length of curves corresponding to ismoetries are the same. In the opposite direction, if both curves with the same parametric expression have the same length, also their tangent vectors have the same length according to (4). Thus the linear map T_pg for each $p \in S$ preserves length of vectors. Now it follows from linear algebra theory that such mappresrves also the scalar product. □

11.11 **11.11.** Geometrically it is obvious that the cylinder $f(u) = (r\cos u_1, r\sin u_1, u_2)$, $u_1 \in (0, 2\pi)$, $u_2 \in \mathbb{R}$ is isometric with a plane strip with the same coordinates. Physically, this isometry is given by unfolding the cylinder into the plane. We verify this also by computation using the theorem 11.9. We have $f_1 = (-r\sin u_1, r\cos u_1, 0)$, $f_2 = (0, 0, 1)$ hence $g_{11} = r^2$, $g_{12} = 0$, $g_{22} = 1$. The plane $z = 0$ can be parametrized as $\bar{f}(u) = (ru_1, u_2, 0)$. Thus $\bar{f}_1 = (r, 0, 0)$, $\bar{f}_2 = (0, 1, 0)$ and $\bar{g}_{11} = r^2$, $\bar{g}_{12} = 0$, $\bar{g}_{22} = 1$ which is the same as in the case of cylinder.

de11.12 **11.12 Definition.** By **inner geometry of the surface** S we mean properties preserved by isometries.

It follows from the theorem 11.9 that inner geometry of the surface is formed by properties which can be derived from the first fundamental form. We say that properties of S which essentially depend on the second fundamental form, belong to **outer geometry of the surface**.

11.13 **11.13.** The notion of inner geometry of the surface originates in Gauss' work. He derived the following its most important statement. It is a deep result which will be proved in the next section in 12.14. Gauss called this Teorema egregium in latin (the transaltion: remarkable theorem). Let K_S be the Gaussian curvature of the surface S $K_{\bar{S}}$ the Gaussian curvature of the surface $\bar{S}$.

Teorema egregium (Gauss). If $g\colon S \to \bar{S}$ is an isometry then $K_S(p) = K_{\bar{S}}\big(g(p)\big)$ for all $p \in S$.

One should emphasize that both principal curvatures do not belong to inner geometry of the surface (the second fundamental form is essential for their computatin) whereas their product does.

11.14 **11.14 Example.** An open set in the plane cannot be isometric to an open set on the sphere. Indeed, The Gaussian curvature of the plane is zero whereas the Gaussian curvature of the sphere of the radius r is $\frac{1}{r^2}$.

11.15 **11.15.** Recall by neighbourhood on the surface S of the point $p \in S$ we mean intersection of a neighbourhood of p in E_3 with the surface S.

Definition. The surface S is called **developable** if each point $p \in S$ has a neighbourhood isometric to an open set in the plane.

This isometry is understood as “unfoldability” of the corresponding part of the surface into the plane, see the example 11.11 which justifies this terminology. On the other hand, we introduced the notion of a developable ruled surface at 10.8. The same terminology is based on the fact that both notions coincide as we shall show. If necessary to distinguish both definitions, the latter one will be referred as developability in a metric sense.

11.16 **11.16.** Consider a generalized cylinder. Assume the z-axis is parallel with generating lines and the curve g is the section of the cylinder by the plane $z = 0$ parametrized by the arc-length. The the cylinder is given by parametrization

$$f(s, v) = \big(g_1(s), g_2(s), v\big)\,.$$

Thus $f_1 = (g_1', g_2', 0)$, $f_2 = (0, 0, 1)$, $g_{11} = 1$, $g_{12} = 0$, $g_{22} = 1$. Using the parametrization $\bar{f}(s, v) = (s, v, 0)$ of the plane, we obtain the same coefficients $\bar{g}_{11} = 1$, $\bar{g}_{12} = 0$, $\bar{g}_{22} = 1$. Hence equality of parameters yields an isometry of both surfaces.

11.17 **11.17.** Consider a generalized cone with the vertex at the origin. Thus $f(t, v) = g(t)\, v$. Here we can assume the curve g is parametrized by arc-length s and $\|g(s)\| = 1$ for all s. Then $f_1 = g'(s)\, v$, $f_2 = g(s)$, i.e. $g_{11} = v^2$, $g_{22} = 1$ and $g_{12} = 0$ since vectors $g(s)$ and $g'(s)$ are perpendicular. If we choose g as the unit circle $k(s)$ in the plane we can parametrized the plane JS: missing picture
locally in the form $\bar{f}(s, v) = k(s)\, v$. Hence $\bar{g}_{11} = v^2$, $\bar{g}_{12} = 0$, $\bar{g}_{22} = 1$ which shows a local isometry of the plane and the cone.

11.18 **11.18.** Consider the tangent developable $g(t) + vg'(t)$ of the curve C. We assume C is parametrized by arc-length. Using Frenet formuale, we can parametrize the surface as

$$f(s, v) = g(s) + ve_1(s)\,.$$

Thus $f_1 = e_1(s) + v\varkappa(s)\, e_2(s)$, $f_2 = e_1(s)$, i.e. $g_{11} = 1 + v^2\, \varkappa^2(s)$, $g_{12} = 1$, $g_{22} = 1$. Further consider a curve $\bar{g}(s)$ in the plane which has locally the same curvature as the spacial curve C and parametrize the plane locally as $\bar{f}(s, v) = \bar{g}(s) + v\, \bar{g}'(s)$. This can be written also in the form

$$\bar{f}(s, v) = \bar{g}(s) + v\, \bar{e}_1(s)\,.$$

We have $\bar{f}_1 = \bar{e}_1(s) + v\, \varkappa(s)\, \bar{e}_2(s)$, $\bar{f}_2 = \bar{e}_1(s)$. Also in this case we have $\bar{g}_{11} = 1 + v^2\varkappa^2(s)$, $\bar{g}_{12} = 1$, $\bar{g}_{22} = 1$. Thus we obtained a local isometry of teh tangent developable with the plane.

11.19 **11.19.** We have shown in 10.11 that a surface with zero Gaussian curvature without planar points is locally a developable ruled surface. We know from 10.15 that a developable ruled surface is generally either tangent developable orgeneralized cylinder or generalized conev. We have shown these surfaces are developable in the metric sense. Using an alternative approach (which we shall not discuss here) can be shown the following statement (cf. the theorem 14.6).

Theorem. A surface S is locally isometric to the plane if and only if it has zero Gaussian curvature.

12 Parallel transport of vectors on a surface

12transport

12.1 **12.1.** Consider a surface $S \subset E_3$ and a motion $\gamma\colon I \to S$. We denote by V the associated vector space of E_3.

Definition. The mapping $A\colon I \to V$ is callede **tangent vector field on** S **aling the motion** γ if $A(t) \in T_{\gamma(t)}S$ for all $t \in I$.

Zero tangent vectors along γ form the field denoted by 0_γ. If A is a tangent vector field along γ and $g\colon I \to \mathbb{R}$ is a function then $g(t)A(t)$ is also a tangent vector field along γ. If A_1 and A_2 are two tangent vector fields along γ then also $A_1(t) + A_2(t)$ is a tangent vector field along γ.

12.2 **12.2.** Recall N_pS is the normal vector space of the surface S at the point p. The following definition is essential for the differential geometry of surfaces.

Definition. We say that **tangent vectior field** A **along the motion** γ **is formed by vectors parallel on** S if $\frac{dA}{dt} \in N_{\gamma(t)}S$ for all $t \in I$.

12.3 **12.3.** The vector $\frac{dA}{dt}$ decomposes into a direction in the tangent plane $T_{\gamma(t)}S$ and the normal line $N_{\gamma(t)}S$ at each point of the motion $\gamma(t)$. TTangent components form again a tangent vector field on S along γ.

Definition. Tangent vectir field $\frac{\nabla A}{dt}$ on S along γ formed by tangent components of vectors $\frac{dA}{dt}$ is called textbfcovariant derivative of the tangent vector field A on S along the motion γ.

Thus the field A is formed by vectors parallel on S if and only if $\frac{\nabla A}{dt}$ is the zero field along γ.

12.4 **12.4.** We shall find a parametric expression for $\frac{\nabla A}{dt}$. Since vectors f_1, f_2, n are linearly independent at each point of the surface, we can decompose second partial derivatives as

e12.1

$$
\begin{aligned}
f_{11} &= \Gamma^1_{11}(u_1,u_2)\, f_1 + \Gamma^2_{11}(u_1,u_2)\, f_2 + h_{11}(u_1,u_2)\, n\,,\\
f_{12} &= \Gamma^1_{12}(u_1,u_2)\, f_1 + \Gamma^2_{12}(u_1,u_2)\, f_2 + h_{12}(u_1,u_2)\, n\,, \qquad (1)\\
f_{22} &= \Gamma^1_{22}(u_1,u_2)\, f_1 + \Gamma^2_{22}(u_1,u_2)\, f_2 + h_{22}(u_1,u_2)\, n\,.
\end{aligned}
$$

Taking scalar product of each equation with the unit vector n perpendicular to f_1 and f_2 shows that coefficients at n are indeed coefficients of the second fundamental form as the notation indicates.

Definition. The function Γ^i_{jk}, $i,j,k=1,2$, $\Gamma^i_{21}=\Gamma^i_{12}$ are called **Christoffel symbols of the surface** S corresponding to the parametrization $f(u_1,u_2)$.

12.5 **12.5.** Let the motion $\gamma(t)$ is given by the parametrization $\big(u_1(t),u_2(t)\big)$ and $A(t)=\big(U_1(t),U_2(t)\big)$. Then

e12.2 (2)
$$A(t)=U_1(t)\,f_1\big(u_1(t),u_2(t)\big)+U_2(t)\,f_2\big(u_1(t),u_2(t)\big)\,. \tag{2}$$

From this one directly computes using (1) that

$$\begin{aligned}\frac{dA}{dt}&=\frac{dU_1}{dt}\,f_1+U_1\Big(f_{11}\frac{du_1}{dt}+f_{12}\frac{du_2}{dt}\Big)+\frac{dU_2}{dt}\,f_2+U_2\Big(f_{12}\frac{du_1}{dt}+f_{22}\frac{du_2}{dt}\Big)\\&=\left[\frac{dU_1}{dt}+\Gamma^1_{11}U_1\frac{du_1}{dt}+\Gamma^1_{12}\Big(U_1\frac{du_2}{dt}+U_2\frac{du_1}{dt}\Big)+\Gamma^1_{22}U_2\frac{du_2}{dt}\right]\,f_1\\&\quad+\left[\frac{dU_2}{dt}+\Gamma^2_{11}U_1\frac{du_1}{dt}+\Gamma^2_{12}\Big(U_1\frac{du_2}{dt}+U_2\frac{du_1}{dt}\Big)+\Gamma^2_{22}U_2\frac{du_2}{dt}\right]\,f_2+(\dots)n\,.\end{aligned}$$

where an expression at n is not important for us. Denoting $\frac{\nabla U_1}{dt}$, $\frac{\nabla U_2}{dt}$ coordinates of the vector field $\frac{\nabla A}{dt}$ along γ, we have

e12.3 (3)
$$\begin{aligned}\frac{\nabla U_1}{dt}&=\frac{dU_1}{dt}+\sum_{i,j=1}^{2}\Gamma^1_{ij}\big(u(t)\big)U_i\frac{du_j}{dt}\,,\\\frac{\nabla U_2}{dt}&=\frac{dU_2}{dt}+\sum_{i,j=1}^{2}\Gamma^2_{ij}\big(u(t)\big)U_i\frac{du_j}{dt}\,.\end{aligned} \tag{3}$$

12.6 **12.6.** As the first property of formulae (3) we shal derive th following:

Theorem. Tangent vector fields A, B on S along the motion γ and the function $g\colon I\to\mathbb{R}$ satisfy

e12.5 (4)
$$\frac{\nabla(A+B)}{dt}=\frac{\nabla A}{dt}+\frac{\nabla B}{dt}\,,\qquad\frac{\nabla(gA)}{dt}=\frac{dg}{dt}A+g\frac{\nabla A}{dt}\,. \tag{4}$$

Proof. This follows directly from (3). □

12.7 **12.7.** The following statement shows that the parallel transport along a given motion has similar properties as the parallel transport of vectors in the plane.

Theorem. For each motion $\gamma\colon I\ \to S$, each $t_0\in I$ and each vector $A_0\in T_{\gamma(t_0)}S$ exists unique vectir field along γ satisfying $A(t_0)=A_0$ and formed by vectors parallel on S.

Proof. Given the motion γ, the condition that right hand sides of expressions of (3) foms a system of two ordinary differential equations. The value A_0 is the initial condition for this system. This determines the solution uniquelly. $\square$

de12.8 **12.8 Definition.** We say the tangent vector field A from the theorem 12.7 provides **parallel transport of the vector A along the motion γ on S.**

Assume the motion $\gamma(t)$ is the parametrized simple curve C on S. A reparametrization $t = \varphi(\tau)$ of C yields another motion $\gamma \circ \varphi$. Inserting this reparametrization into (3), derivation with respect to t are mutliplied by $\frac{d\varphi}{d\tau}$. Since expressions (3) are linear in $\frac{dU_i}{dt}$ a $\frac{du_i}{dt}$, $i = 1, 2$, differential equations $\frac{\nabla U_i}{dt} = 0$, $i = 1, 2$ for parallel transport are essentially the same as $\frac{\nabla(U_i \circ \varphi)}{d\tau} = 0$. Geometrically this means for different parametrizations of the curve C on S we get the same parallel transport of vectors. Thus we speak not only about paralle transport of vectors along a motion on S but also about **parallel transport of vectors along a curve on S.**

12.9 **12.9 Theorem.** If tangent vecros fields A and B along the motion γ provide parallel transport of vectors $A_0 \in T_{\gamma(t_0)}S$ and $B_0 \in T_{\gamma(t_0)}S$, respectively, then the field $k_1A + k_2B$ provides the parallel transport of the vector $k_1A_0 + k_2B_0$, k_1, $k_2 \in \mathbb{R}$.

Proof. The theorem 12.6 for constant $g = k$ yields $\frac{\nabla(kA)}{dt} = k\frac{\nabla A}{dt}$. Thus $\frac{\nabla(k_1A + k_2B)}{dt} = k_1\frac{\nabla A}{dt} + k_2\frac{\nabla B}{dt}$. Since the right hand side is zero, the same holds for the left hand side. $\square$

Geometrically this means the parallel transport preserves lienar combinations of vectors.

12.10 **12.10 Example.** Considering a plane ϱ as a surface in E_3 then for the tangent vector field $A(t) = \big(U_1(t), U_2(t)\big)$ along an arbitrary motion $\gamma(t)$ in ϱ has zero normal component of the vector $\frac{dA}{dt}$. Thus $A(t)$ is parallel transport along γ if and only if $\frac{dA}{dt} = 0$, i.e. $U_1(t)$ a $U_2(t)$ are constants. This is the classical parallel transport in the plane. This transport does not depend on the motion.

Now we shall show that, however, parallel transport on the sphere S does depend on the motion. Conside one eighth of the sphere according to the picture. The great circle in the plane $z = 0$ has the parametric expression $f(t) = (r\cos t, r\sin t)$. Its tangent vector is $v(t) = \frac{df}{dt} =$ JS: missing picture

$(-r\sin t, r\cos t)$. Thus $\frac{dv}{dt} = (-r\cos t, -r\sin t)$. The normal vector of the sphere at the point $f(t)$ is $(\cos t, \sin t)$ hence $\frac{dv}{dt} \in N_{f(t)}S$. Therefore tangent vectors to the great circle provide transport parallely along this circle. Denote by a the point with the parameter $t = 0$ and b the point with the parameter $t = \frac{\pi}{2}$.

Now let us transport this tangent vector $v = v(0)$ along the great circle in the plane perpendicular to v at the point c with the parameter $\frac{\pi}{2}$. The constant vector v is tangent along this curve, i.e. $\frac{dv}{dt} = o$. Now let us continue with the parallel transport along the small arc of the great circle from the point c to b. Here we again transport the tangent vector of the circle along this circle hence the result is again tangent to this circle. Thus the resulting transport of the vector v from the point a to b along two different motions I and II give rise to two different vectors which are actualy perpendicular. Although the second motion is not differentiable at one point, the "vertex" at the point c is not the reason of different parallel transports of the vector v along motions I and II.

12.11 **12.11.** Decompositions (12.1) can be used also to compute Christoffel symbols. This is particularly simple in the case when the parametric net is orthogonal, i.e. $g_{12} = (f_1, f_2) = 0$. Let us discuss the sphere $f(u_1, u_2) = r(\cos u_1 \cos u_2, \sin u_1 \cos u_2, \sin u_2)$ from 6.4. We found

$$\begin{aligned} & f_1 = r(-\sin u_1 \cos u_2, \cos u_1 \cos u_2, 0)\,, \\ (5) \qquad & f_2 = r(-\cos u_1 \sin u_2, -\sin u_1 \sin u_2, \cos u_2)\,, \end{aligned}$$

$$g_{11} = (f_1, f_1) = r^2 \cos^2 u_2\,, \quad g_{12} = (f_1, f_2) = 0\,, \quad g_{22} = (f_2, f_2) = r^2\,.$$

Further differentiation yields

e12.6
$$\begin{aligned} & f_{11} = r(-\cos u_1 \cos u_2, -\sin u_1 \cos u_2, 0)\,, \\ (6) \qquad & f_{12} = r(\sin u_1 \sin u_2, -\cos u_1 \sin u_2, 0)\,, \\ & f_{22} = r(-\cos u_1 \cos u_2, -\sin u_1 \cos u_2, -\sin u_2)\,. \end{aligned}$$

We consider scalar product of equations (1) with vectors f_1 and f_2 where we need to compute scalar products on the left hand side. We obtain

$$0 = \Gamma^1_{11} r^2 \cos^2 u_2, r^2 \sin u_2 \cos u_2 = \Gamma^2_{11} r^2, -r^2 \sin u_0 \cos u_0 = \Gamma^1_{12} r^2 \cos^2 u_2\,,$$

$$0 = \Gamma^2_{12} r^2\,, \quad 0 = \Gamma^1_{22} r^2 \cos u_2\,, \quad 0 = \Gamma^2_{22} r^2.$$

Thus

$$\Gamma^1_{11} = 0\,,\ \Gamma^2_{11} = \sin u_2 \cos u_2\,,\ \Gamma^1_{12} = -\tan u_2\,,\ \Gamma^2_{12} = 0\,,\ \Gamma^1_{22} = 0\,,\ \Gamma^2_{22} = 0\,.$$

12.12 **12.12.** The following theorem is an important tehoretical result. Equations (12.7) show that Christoffel symbols can be expressed using coefficirnts of the first fundamental form. THUS THE PARALLEL TRANSPORT OF VECTOS ON THE SURFACE BELONGS TO THE INNER GEOMETRY OF THE SURFACE.

Since $g_{11}g_{22}-g_{12}^2>0$, the square matrix (2×2)-matrix (g_{ij}) is reguklar. Denote by $(\widetilde{g}_{kl})$ its inverse matrix.

Theorem. We have

e12.7 (7)
$$\Gamma_{ij}^k=\frac{1}{2}\sum_{l=1}^{2}\widetilde{g}_{kl}\Big(\frac{\partial g_{jl}}{\partial u_i}+\frac{\partial g_{li}}{\partial u_j}-\frac{\partial g_{ij}}{\partial u_l}\Big)\,.$$

Proof. Differentiating $(f_1,f_j)=g_{ij}$ with respect to u_l, we obtain

e12.8 (8)
$$\frac{\partial g_{ij}}{\partial u_l}=(f_{il},f_j)+(f_i,f_{jl})\,.$$

It follows from (1) that $f_{ij}=\sum\limits_{m=1}^{2}\Gamma_{ij}^m f_m+h_{ij}n$. By substitution we obtain

$$(f_{il},f_j)=\sum_{m=1}^{2}\Gamma_{il}^m g_{mj}\,.$$

Thus (8) can be written as

e12.9 (9)
$$\frac{\partial g_{ij}}{\partial u_l}=\sum_{m=1}^{2}\big(\Gamma_{il}^m g_{mj}+\Gamma_{jl}^m g_{mi}\big)\,.$$

We obtain two more equations by a suitable change of indices, Dal? dv? rovnice z?sk me z m?nou index?

e12.10 (10)
$$\frac{\partial g_{il}}{\partial u_j}=\sum_{m=1}^{2}\big(\Gamma_{ij}^m g_{ml}+\Gamma_{lj}^m g_{mi}\big)\,,$$

e12.11 (11)
$$\frac{\partial g_{lj}}{\partial u_i}=\sum_{m=1}^{2}\big(\Gamma_{ji}^m g_{ml}+\Gamma_{li}^m g_{mj}\big)\,.$$

Summing (10) + (11) − (9) and using symmetris of g_{ij} and Γ_{ij}^k at lower indices, we obtain

e12.12 (12)
$$\frac{\partial g_{il}}{\partial u_j}+\frac{\partial g_{lj}}{\partial u_i}-\frac{\partial g_{ij}}{\partial u_l}=2\sum_{m=1}^{2}\Gamma_{ij}^m g_{ml}\,.$$

By division by 2 and for fixed i, j, we consider (Γ_{ij}^m) as rank two row vector. Then the matrix form of the right hand side of (12) is $(\Gamma_{ij}^m)(g_{ml})$. Now we compute the unkown Γ_{ij}^m by multiplication of the inverse matrix $(\widetilde{g}_{kl})$. This yields (7). □

12.13 **12.13.** Of course, (7) can be used also as a formula to compute Christoffel symbols. A simple example is the generalized cylinder from 11.16. where we found $g_{11} = 1$, $g_{12} = 0$, $g_{22} = 1$. All partial derivatives in (7) are thus zero since these are derivatives of a constant. Hence all Christoffel symbols of the generalized cylinder are zero.

12.14 **12.14.** Now we shall prove the Gauss' theorem egregium. We shall start with equations (1) which we shall wwrite in the form

e12.13
$$f_{ij} = \sum_{l=1}^{2} \Gamma_{ij}^l f_l + h_{ij} n \,. \tag{13}$$

We shall use the notation $f_{ijk} = \frac{\partial^3 f}{\partial u_i \partial u_j \partial u_k}$, $\Gamma_{ij,k}^l = \frac{\partial \Gamma_{ij}^l}{\partial u_k}$, $n_i = \frac{\partial n}{\partial u_i}$, $h_{ij,k} = \frac{\partial h_{ij}}{\partial u_k}$. Equations 7.(8) can be written in the form

e12.14
$$(n_i, f_j) = -h_{ij} \,. \tag{14}$$

Differentiating (13) with respect to u_k, we obtain

e12.15
$$f_{ijk} = \sum_{m=1}^{2} \Gamma_{ij,k}^m f_m + \sum_{n=1}^{2} \Gamma_{ij}^n f_{nk} + h_{ij,k} n + h_{ij} n_k \,. \tag{15}$$

Using (13), the second term on the right hand side has the form

e12.16
$$\sum_{n=1}^{2} \Gamma_{ij}^n f_{nk} = \sum_{m,n=1}^{2} \Gamma_{ij}^n (\Gamma_{nk}^m f_m + h_{nk} n) \,. \tag{16}$$

Taking the scalar product of (15) with the vector f_j and using (14), we obtain

e12.17
$$(f_{ijk}, f_l) = \sum_{m=1}^{2} \Gamma_{ij,k}^m g_{ml} + \sum_{m,n=1}^{2} \Gamma_{ij}^n \Gamma_{nk}^m g_{ml} - h_{ij} h_{kl} \,. \tag{17}$$

It follows from symmetry of third partial derivatives $f_{ikj} = f_{ijk}$ that (17) holds also after exchange of indices j and k. Subtracting these two equations, we obtain

e12.18
$$h_{ij} h_{kl} - h_{ik} h_{jl} = \sum_{m=1}^{2} g_{ml} \Big[\Gamma_{ij,k}^m - \Gamma_{ik,j}^m + \sum_{n=1}^{2} (\Gamma_{ij}^n \Gamma_{nk}^m - \Gamma_{ik}^n \Gamma_{nj}^m) \Big] \,. \tag{18}$$

Putting $i = 1$, $j = 1$, $k = 2$, $l = 2$ we obtain

Theorem (Gauss' equations). It holds

e12.19

$$h_{11}h_{22} - h_{12}^2 = \sum_{m=1}^{2} g_{m2}\Big[\Gamma^m_{11,2} - \Gamma^m_{12,1} + \sum_{n=1}^{2}(\Gamma^n_{11}\Gamma^m_{n2} - \Gamma^n_{12}\Gamma^m_{n1})\Big]. \qquad (19)$$

□

The right hand side depends only on coefficients g_{ij} of the first fundamental form and its partial derivatives of thr first and second order according to the theorem 12.12. We have found in 8.9 that $K = (h_{11}h_{22} - h_{12}^2)/(g_{11}g_{22} - g_{12}^2)$. Thus the Gaussian curvature K belongs to the inner geometry of the surface as the theorema egregium states.

We also remark that putting different indices i, j, k, l into (18), we obtain again the equation (19) or identity.

12.15

12.15. Information If we analogously decompose n_i, $i = 1, 2$ into the frame f_1, f_2, n and use the relation $\frac{\partial n_i}{\partial u_j} = \frac{\partial n_j}{\partial u_i}$ $\left(= \frac{\partial^2 n}{\partial u_i \partial u_j}\right)$, we obtain so called **Codazzi equations** (two of them are essentially)

e12.20

$$h_{ij,k} - h_{ik,j} + \sum_{l=1}^{2}(\Gamma^l_{ij}h_{lk} - \Gamma^l_{ik}h_{lj}) = 0. \qquad (20)$$

Using elementary techniques for systems of partial differential equations (e.g. the theorem 7.8 in the textbook [5]), one can prove following two statements about existence and uniqueness of solutions, which are sometimes called **The basic theorem of theory of surfaces**.

I. If S and $\bar{S}$ are two simple surfaces with parametrizations $f\colon D \to E_3$ and $\bar{f}\colon D \to E_3$, respectively on the same parameter space D, which have the same first and second fundamental form, then there exists an Euclidean motion $\varphi\colon E_3 \to E_3$ such that $\varphi \circ f = \bar{f}$.

Summarizing: surfaces, which have the same first and second fundamentzal form, are congruent.

II. Consider two quadratic forms Φ_1 and Φ_2 on $D \subset \mathbb{R}^2$ where Φ_1 is positive definite at all points. If Φ_1 and Φ_2 satisfy Gauss and Codazzi equations, then there locally exists a surface with a parametrization $f\colon D \to E_3$ such that Φ_1 and Φ_2 are its first and second fundamental form, respectively.

13
13 Geodetic curves

13.1 **13.1.** Given a surface S and a motion $\gamma\colon I \to S$, we can consider the field of tangent vectors of γ which we denote by $\dot\gamma$.

Definition. The motion $\gamma\colon U \to S$ is called **geodetic curve** if the field $\dot\gamma$ of its tanegnt vectors transports parallely along γ.

Consider a plane ϱ as a surface, this definition means that the vector $\dot\gamma = a$ is constant, i.e. γ is the motion $p + ta$ along a line, $p \in \varrho$.

13.2 **13.2.** Thus the condition for the motion γ to be geodetic means $\frac{\nabla\dot\gamma}{dt} = o$. Let $\gamma(t) = \big(u_1(t), u_2(t)\big)$. Hence we need to put $\gamma(t) = \big(u_1(t), u_2(t)\big)$ into
12 the relation .(12.3) and require the right hand side to be zero. I.e. geodetic motions are solutions of a system of two differential equations of the 2. order:

e13.1 (1)
$$\frac{d^2u_i}{dt^2} + \sum_{j,k=1}^{2} \Gamma^i_{jk}\big(u(t)\big)\frac{du_j}{dt}\,\frac{du_k}{dt} = 0\,, \quad i = 1, 2\,.$$

This system behaves more or less similarly as one differential equations of teh 2. order.

13.3 **13.3.** It is well known that a solution of a differential equation of the 2. order is fully determined by its initial value and initial velocity (i.e. the value of its derivative). Analogously, in teh case of the system (13.1) we have

Theorem. For every point $p \in S$ and every vector $A \in T_pS$ there exists an interval $0 \in I_A \subset \mathbb{R}$ and a unique geodetic motion $\gamma_A\colon I_A \to S$ such that $\gamma(0) = p$ and $\dot\gamma(0) = A$.

The interval I_A generally changes depending on A.

13.4 **13.4.** It is useful to observe the following property.

Lemma. If $\gamma(t)$ is a geodetic motion then also the motion $\gamma(at + b)$ is geodetic for each a, $b \in \mathbb{R}$, $a \neq 0$.

Proof. Put $\widetilde\gamma(t) = \gamma(at + b)$. We have $\frac{d\widetilde\gamma}{dt} = \frac{d\gamma}{dt}\,a$, $\frac{d^2\widetilde\gamma(t)}{dt^2} = \frac{d^2\gamma}{dt^2}\,a^2$. Multiply-
e13.1 ing equations by a^2 then also $\widetilde\gamma(t) = \big(\widetilde u_1(t), \widetilde u_2(t)\big)$ satisfies

$$\frac{d^2\widetilde u_i}{dt^2} + \sum_{j,k=1}^{2} \Gamma^i_{jk}\big(\widetilde u(t)\big)\frac{d\widetilde u_j}{dt}\,\frac{d\widetilde u_k}{dt} = 0\,, \quad i = 1, 2\,.$$

□

This lemma has a natural kinematic interpretation in the plane: If we change parametrization of a linear stadey motion, we obtain again a linear stady motion.

13.5 **13.5 Definition.** The curve $C \subset S$ is called **geodetic curve**, if there exists such a parametrization $\gamma(t)$ that γ is a geodetic motion.

We briefly talk about **geodetics**. Geodetic curves in the plane are lines. It follows from 13.3 and 13.4 that

Theorem. For each point $p \in S$ and each direction in T_pS there exists a unique geodetic on S which touches this direction at the point p.

13.6 **13.6.** It follows form properties of solutions of systems of differential equations of the 2. order (which we shall not discuss here in detail) that

Theorem. At each point $p \in S$ there exists a neighbourhood $Z \subset S$ such that for each two points $q_1 \neq q_2$ in Z there exists a unique geodetic in Z through points q_1 and q_2.

This property is analogous to the fact that each two points in the plane can be connected by a unique line. However, a simple examples shows that the locality assumption in the theorem is essential on surfaces. We shall show below that geodetic curves on the sphere S are great circles. Given any point $q_1 \in S$ and another point q_2 (different from the "opposite pole"), there is a unique great circe throught q_1 and q_2. However, if q_2 is the "opposite pole" to q_1 then there are infinitely many great circles throught points q_1 and q_2.

13.7 **13.7.** Recall the osculatung plane of a curve is not defined in inflection points. The following theorem provides "outer" characterization of geodesics.

Theorem. The curve $C \subset S$ is geodesic if and only of its osculating plane at each point contains the normal direction of the surface or is not defined.

Proof. The condition $\frac{\nabla\dot\gamma}{dt} = o$ means that the vector lies in the normal direction. If $\frac{d\dot\gamma}{dt} \neq o$ then vectors $\dot\gamma$ and $\frac{d\dot\gamma}{dt}$ determine the osculating plane which contains the normal direction. If $\frac{d\dot\gamma}{dt} = o$ then this is teh inflection point. In the opposite direction, consider a curve C parametrizad by the arc-length $\gamma(s)$. Then $\big(\dot\gamma(s), \dot\gamma(s)\big) = 1$. Differentiating the latter, we obtain $\big(\dot\gamma, \frac{d\dot\gamma}{ds}\big) = 0$. Je-li $\frac{d\dot\gamma}{ds} \neq o$ then vectors $\dot\gamma$ and $\frac{d\dot\gamma}{ds}$ determine the osculating plane and we assume this plane contains the normal line of the surface. Since the vector $\frac{d\dot\gamma}{ds}$ is perpendicular to the vector $\dot\gamma(s)$, it is parallel with the normal line. Thus $\frac{\nabla\dot\gamma}{ds} = o$. If $\frac{d\dot\gamma}{ds} = o$ then $\frac{\nabla\dot\gamma}{ds} = o$. □

Example. The great circle C on the sphere S is such circle whose center coincides with the center of the sphere. Its usculating plane coincides with the plane the circle lies in (at every point). Normal lines of the sphere along C lie in the same plane. Hence each great circle is a geodesic. On the other hand, at each point $p \in S$ and each direction in T_pS, there is a unique great circle. Using the theorem 13.5, other geodesics on S do not exist.

13.8 **13.8 Corollary.** If C is a geodesic curve and $\gamma(s)$ its arc-length parametrization then $\gamma(s)$ is a geodesic motion.

Proof. It follows from the theorem 13.7, the osculating plane of the curve contains the normal direction of the surface. The second part of the proof shows that $\frac{\nabla\dot\gamma}{ds} = o$. □

13.9 **13.9.** Let $Z \subset S$ is a neighbourhood of the point $p \in S$ with the property from the theorem 13.6

Theorem. For each $q \in Z$, $q \neq p$, the geodesic through points p and q is the shortes curve in Z which connects points p and q.

JS: missing picture

Proof. Let us denote by C the geodesic connecting points p and q. Choose a curve $\bar C$ through the point p perpendicular to C. Further, we have a geodetic curve through every point of $\bar C$ perpendicular to $\bar C$; these curves form a 1-parameter family of parametric curves. We choose its orthogonal trajectories as the second family of parametric curves. Choosing a parametr u_1 on C and a parameter u_2 on $\bar C$, we obtain a parametrization of the surface on some neighbourhood U of the point p. We can put $p = (0,0)$, $q = (a,0)$, $a > 0$.

Parametrizing curves $u_2 = c$ by the arc-length s, the parametrization $u_1 = s$, is a geodetic motion. This it satisfies equations (13). We have $\frac{du_2}{ds} = 0$, $\frac{d^2u_2}{ds^2} = 0$, since $u_2 = c$, and further $\frac{du_1}{ds} = 1$ (the parameter is arc-length) and $\frac{d^2u_1}{ds^2} = 0$. Putting this into (13.1), we obtain

$$\Gamma^1_{11} = 0\,, \quad \Gamma^2_{11} = 0\,.$$

Since the parametric net is orthogonal, we have $g_{12} = (f_1, f_2) = 0$. Using the decomposition 12.(1), we obtain

$$\frac{dg_{11}}{du_1} = \frac{\partial(f_1, f_1)}{\partial u_1} = 2(f_1, f_{11}) = 2\Gamma^1_{11}(f_1, f_1) = 0\,.$$

Differentiating $(f_1, f_2) = 0$, we find

$$0 = \frac{\partial(f_1, f_2)}{\partial u_1} = (f_{11}, f_2) + (f_1, f_{12}) = \Gamma^2_{11}(f_2, f_2) + (f_1, f_{12}) = 0\,,$$

hence $(f_1, f_{12}) = 0$. Now we obtain

$$\frac{\partial g_{11}}{\partial u_2} = \frac{\partial(f_1, f_1)}{\partial u_2} = 2(f_1, f_{12}) = 0\,.$$

Thus g_{11} is a constant $k > 0$.

Consider a curve from p to q given by the parametrization $u_1 = t$, $u_2 = f(t)$, $f(0) = 0$, $f(a) = 0$. Its length is equal to

e13.2 (2)
$$\int_0^a \sqrt{k + g_{22}\big(t, f(t)\big)\Big(\frac{df}{dt}\Big)^2}\, dt\,.$$

The length of the geodesic C is $\int_0^a \sqrt{k}\, dt = a\sqrt{k}$. Since $g_{22} > 0$, the integral (2) is greater or equal to $\sqrt{k}a$ and the equality holds obly for $\frac{df}{dt} = 0$. Using $f(0) = 0$, this means that $f(t) = 0$ for all t. □

13.10 **13.10 Remark.** The oldest approach to the notion of geodesics comes exactly from the property that these are shortest curves connecting two points on the surface. Thus differential equations were derived using the calculus of variation.

13.11 **13.11.** The curvature $\varkappa$ of a plane curve $f(s)$ satisfies $\varkappa = \big\|\frac{de_1}{ds}\big\|$, $e_1 = \frac{df}{ds}$. An analogy of this property is used to define geodetic curvature of the curve $C \subset S$. Assume that C is parametrized by the arclength $\gamma(s)$. Then $\dot\gamma = \frac{d\gamma}{ds}$ is a unit vector.

Definition. Geodetic curvature $\varkappa_g$ of **the curve** $\gamma(s)$ **on the surface** S is defined via the relation $\varkappa_g = \big\|\frac{\nabla\dot\gamma}{ds}\big\|$.

That is, the geodetic curvature belongs to the inner geometry of the surface.

Remark. Also a notion of **geodetic torsion of a curve on the surface** can be defined but this no more belongs to the inner geometry of the surface.

13.12 **13.12.** We shall show hot the usual curvature $\varkappa$ and the geodetic curvature $\varkappa_g$ of a curve C on the surface S are related.

Theorem. Let α be the angle between the normal direction of the surface and the osculating plane of the curve $C \subset S$. Then $\varkappa_g = \varkappa \sin \alpha$. Further, we have $\varkappa_g = 0$ in inflection points of the curve C.

Proof. In a non-inflection point, the vector $\frac{d\dot{\gamma}}{ds}$ lies in the osculating plane. It follows from the picture depictin the section by the plane perpendiculatr to the osculating plane ω of the curve C that $\left\|\frac{\nabla\dot{\gamma}}{ds}\right\| = \left\|\frac{d\dot{\gamma}}{ds}\right\| \sin \alpha$. In inflection points, we have $\frac{d\dot{\gamma}}{ds} = o$ thus also $\frac{\nabla\dot{\gamma}}{ds} = o$. □ JS: missing picture

13.13 **13.13 Corollary.** The curve C on the surface S is geodetic if and only if $\varkappa_g = 0$ at all points of C.

Proof. This follows directly from theorems 13.7 and 13.12. □

Lines in a plane are characterized by the property $\varkappa = 0$. This is one of analogies between lines on a plane and geodetic curves on a surface.

13.14 **13.14.** Using geodesics, we can extend certain constructions from the euclidean planes to surfaces. The simplest example is the notion of geodetic circles on the surface S.

It follows from properties of solutions of systems of 2nd order differential equations that for each point $p \in S$, there is a number $r_p > 0$ such that for each $0 < r < r_p$ and on every geodesic through p there are exactly two points q_1 and q_2 (one in each direction) such that the length of the arc between p and q_1 and of the arc between p and q_2 is equal to r. Moreover, it holds that the set $K(p, r)$ of all such points is a curve on S.

Definition. The curve $K(p, r)$ is called **geodetic circle of the radius r with the center p on the surface S.**

In the case of the sphere S with the raidus ϱ, we shall show that this constraction works generally onlu locally. If $r < \pi\varrho$ then $K(p, r)$ is a usual circle on S which is a curve. For $r = \pi\varrho$, one moves along geodesics in all directions to the point opposite to p on S. In a sense, putting $r = \pi\varrho$, the circle $K(p, r)$ "collapses" into a single point.

13.15 **13.15.** Geodetic circles have constant curvature in the plane and the same is true (as we have just shown) on spheres. This is not true in general, however. Already ellipsoid with axis of different lengths is an example of a surface where geodetic cicles do not have constant geodetic curvature. (We shall mention one more statement in this directions later in 14.7.)

14 Surfaces with constant Gaussian curvature

The main aim of this chapter is to show a relation between inner geometry of surfaces with noneuclidean geometries. These observation are not only geometrically nice but also played an important role in the history of mathematics. It was mainly understanding of inner geometry which (in the beginning of the 19. century) motiovated leading geometres to believe that noneuclidean geometry forms a mathematical theory in the same sense as euclidean geometry. Proofs in this chapter require either too extensive computations or tools going far beyond theory built so far. These proofs will be omitted — we are interested more in a general view than in a detailed technical matters.

14.1 **14.1.** We shall investigate local isometries of S with itself. Considering an arbitrary surface, the locallity assumption is so natural that the word "local" will be omitted. Consider a surface of revolution S from 8.15 with the parameter t changed to u. Hence the parametric expression of S is

e14.1 (1) $$f(u,v) = \big(g(u)\cos v, g(u)\sin v, h(u)\big)\,.$$

We found $g_{11} = g'^2 + h'^2$, $g_{12} = 0$, $g_{22} = g^2$ in 8.15. The surface of revolution obviously has a 1-parameter system of isometries given by rotations. One can see that also from the 1. fundamental form

e14.2 (2) $$\Phi_1 = g_{11}(u)\,du^2 + g_{22}(u)\,dv^2\,, \quad g_{11} = g'^2 + h'^2,\ g_{22} = g^2\,.$$

The mapping $u = \bar{u}$, $v = \bar{v} + c$ preserves this form since $du = d\bar{u}$, $dv = d\bar{v}$.

14.2 **14.2.** Assume in the opposite direction that the surface S has a 1-parameter system of isometries such that its trajectories form the system $\mathcal{L}$ of curves. Consider its orthogonal trajectories $\mathcal{L}'$. Curves form the system$\mathcal{L}'$ transform to each other by isometries since an isometry preserves angles. Choose $\mathcal{L}$ for coordinate curves $u =$ konst. and $\mathcal{L}'$ for coordinate curves $v =$ konst. Our isometries than have the form $\bar{u} = u$, $\bar{v} = v + c$. The form Φ_1 is preserved by isometries and also $g_{12} = 0$ by orthogonality of coordinate net. Hence

e14.3 (3) $$\Phi_1 = A_1(u)\,du^2 + A_2(u)\,dv^2\,, \quad A_1 > 0\,, A_2 > 0\,.$$

Change the parameter u to $\bar{u}$ such that $d\bar{u} = \sqrt{A_1}\,du$, i.e. $\bar{u} = \int \sqrt{A_1}\,du$ where we keep $\bar{v} = v$. Then we have

$$\Phi_1 = d\bar{u}^2 + B(\bar{u})\,d\bar{v}^2\,, \quad B > 0\,.$$

This is the 1. fundamental form of a surface of revolution given by graph of the function $x = \sqrt{B(z)}$ where we parametrize this curve by arc-length. Summarizing, we have proved

Theorem. If the surface S has a 1-parameter system of isometries then it is locally isometric to a surface of revolution.

14.3 **14.3.** We know that the Gaussian curvature is preserved by isometries. Hence if the surface S has more then one 1-parameter system of isometries then the Gaussian curvature must be constant. We know from 11.19 that a surface is locally isometric to a plane if it has zero Gaussian curvature. However, this holds more generally: two surfaces with the same constant Gaussian curvature are locally isometric, see 14.6 below.

The basic example of a surface with zero K is the plane. The basic example of a surface with a constant positive curvature $K = \frac{1}{r^2}$ is the sphere with radius r, see **??**. Now we shall present an example of a surface with a constant negative curvature $K = -\frac{1}{a^2}$.

14.4 **14.4.** We shall need to formula for the Gaussian curvature of the surface of revolution (1). We computed coefficients of 1. and 2. fundamental form in 8.15,

e14.4 (4)
$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2} = \frac{h'(g'h'' - h'g'')}{g(g'^2 + h'^2)^2} \,.$$

Since rotations are isometries, K is independent on the rotation parameter v.

14.5. A plane curve with the parametric expression

e14.5 (5)
$$x = a\sin u = g(u)\,,\ z = a\Big[\ln\big(\operatorname{tg}\frac{u}{2}\big) + \cos u\Big] = h(u)\,, \qquad u \in (0,\pi), u \neq \frac{\pi}{2}$$

is called **traktrix** with the parameter $a > 0$, see the picture. (Given $u = \frac{\pi}{2}$, we have $x = a$ and $\lim_{u \to \frac{\pi}{2}} h(u) = 0$. The point $(a, 0)$ does not lie on the curve, however. This is a sort of singularity.) Rotation around the z-axis yields the surface called **pseudosphere**. Its Gaussian curvature can be computed using (4). We have $g' = a\cos u$, $g'' = -a\sin u$, JS: Missing picture

$$h' = a\left(\frac{1}{\operatorname{tg}\frac{u}{2}\cos^2\frac{u}{2}}\cdot\frac{1}{2} - \sin u\right) = a\left(\frac{1}{\sin u} - \sin u\right), \quad h'' = a\left(-\frac{\cos u}{\sin^2 u} - \cos u\right).$$

Putting this to (4), we obtain $K = -\frac{1}{a^2}$.

Thus pseudosphere with the parameter a has negative constant Gauassian curvature $-\frac{1}{a^2}$. In a sense, this is "opposite" of the sphere with $K = \frac{1}{r^2}$ which also motivates the terminology

14.6 **14.6.** We know that isometries of the plane are Euclidean transformations, i.e. plane has 3-parameter system of isometries. Isometries of the sphere S are given by restriction of Euclidean motions in E_3 which preserve S. Geometrically, we can easily see that for each pair of points p, $\bar{p} \in S$ and each pair of unit perpendicular vectors e_1, $e_2 \in T_pS$ and $\bar{e}_1$, $\bar{e}_2 \in T_{\bar{p}}S$, there exists a unique isometry $g\colon S \to S$ such that $g(p) = \bar{p}$, $T_pg(e_1) = \bar{e}_1$, $T_pg(e_2) = \bar{e}_2$. A similar statement holds for an arbitrary suraface with a constant Gaussian curvature. We shall present this without proof.

Theorem (Minding)**.** Let S and $\bar{S}$ are surfaces with the same constant Gaussian curvature. Then for each pair of points $p \in S$ and $\bar{p} \in \bar{S}$ and each pair of unit perpendicular vectors e_1, $e_2 \in T_pS$ and $\bar{e}_1$, $\bar{e}_2 \in T_{\bar{p}}\bar{S}$, there exists a unique local isometry g from S to$\bar{S}$ such that $g(p) = \bar{p}$, $T_pg(e_1) = \bar{e}_1$, $T_pg(e_2) = \bar{e}_2$. □

Hence local isometries of every surface with constant Gaussian curvature form a 3-parameter system of isometries (as in the plane).

14.7 **14.7.** Consider the geodetic circle $K(p, r)$ on the surface S with a constant Gaussian curvature. It follows from Theorem 14.6 that there is a 1-parameter system of local isometries on S which preserve the point p (in a sense, these are rotations around the point p). Considering small r, this means that the geodetic curvature of the geodetic circle is the same at all points. The case of the sphere was already mentioned 13.15. One can show this holds also globally:

Theorem. Geodetic circles on surfaces with constant Gaussian curvature have constant geodetic curvature.

14.8 **14.8.** Our next tool will be the Gauss-Bonnet theorem. First we shall formulate necessary mathematical terminology.

Let C be a curveje with the parametrization $f(t)$, $t \in I$.

Definition. Segment U **of the curve** C corresponding to a closed interval $[a, b] \subset I$ is the set $f(t)$, $t \in [a, b]$.

14.9 **14.9.** Let $D \subset \mathbb{R}^2$ be an open set. **A simple region** $\Omega \subset D$ is an open, convex and bounded subset such that also its closure $\overline{\Omega}$ lies in D and its border $\partial\Omega$ is formed by a finite number of segments of curves. JS: missing picture

Definition. The subset $W \subset S$ is called **simple region on the surface** S if there exists such a parametrization $f\colon D \to E_3$ of the surface S such that $W = f(\Omega)$ where Ω is a simple region in D.

Note that one should carefully distinguish between a simple surface and a simple region on a surface.

14.10 **14.10.** Let h be a function defined on the curve C. Using the given parametrization $f\colon I \to E_3$ of the curve C, this is the function $h(t)\colon I \to \mathbb{R}$. Consider the segment U of the curve C which corresponds to the interval $[a,b] \subset I$. Then we define the **integral** $\int\limits_U h\,ds$ as

e14.6 (6)
$$\int\limits_U h\,ds = \int\limits_a^b h(t)\sqrt{(f_1')^2 + (f_2')^2 + (f_3')^2}\,dt\,.$$

This definition is independent on the choice of parametrization of the curve.

We define the **integral** $\int\limits_C h\,ds$ using deompocition of the curve C to segments.

14.11 **14.11.** Let h be a function defined on the surface S. Given a parametrization $f\colon D \to E_3$, this is the function $h(u_1,u_2)\colon D \to \mathbb{R}$. Let $\Omega \subset D$ be bounded region such that $\overline{\Omega} \subset D$. PWe write $W = f(\Omega)$. Recall that we introduced the volume element $dV = \sqrt{g_{11}g_{22} - g_{12}^2}\,du_1\,du_2$ of the surface in **??**. The **integral** $\iint\limits_W h\,dV$ is defined by

e14.7 (7)
$$\iint\limits_W h\,dV = \iint\limits_\Omega h(u_1,u_2)\sqrt{g_{11}g_{22} - g_{12}^2}\,du_1\,du_2\,.$$

Also this definition is independent on the choice of a parametrization of the surface.

14.12 **14.12.** Now we shall state (without a proof) one of most interesting results of the inner geometry of surfaces.

Theorem (Gauss-Bonnet theorem)**.** Let W be a simple region on the surface S whose border is the curve C of the class C^2. Let $\varkappa_g$ be the geodetic curvature of the curve C and K be the Gaussian curvature of the surface S. Then

e14.8 (8)
$$\iint\limits_W K\,dV = 2\pi - \int\limits_C \varkappa_g\,ds\,.$$

Example. To get a first experience with this theorem, we shall discuss the case of the simple region Ω in the plane whose border is the curve C of the class C^2. Its geodetic curvature is the usual curvature and we have $K = 0$ for the plane. Thus (8) yields

e14.9 (9)
$$\int_C \varkappa\, ds = 2\pi\,.$$

In the case of the circle $f(t) = (r\cos t, r\sin t)$, $t \in [0, 2\pi)$, this can be verified also by a direct computation. We have $\varkappa = \frac{1}{r}$, $ds = \sqrt{r^2\cos^2 t + r^2 \sin^2 t}\, dt = r dt$. Thus

$$\int_C \varkappa\, ds = \int_0^{2\pi} \frac{1}{r} r dt = 2\pi\,.$$

Observe moreover that we have $\int_{t_1}^{t_2} \varkappa\, ds = t_2 - t_1$ in this case.

14.13 **14.13.** One can extend the Gauss-Bonnet theorem also to some simple regions whose border is not differentiable because of "vertices". We shall discuss the case of the curvilinear triangle on the surface S. In the following definition, we understand a triangle in $\mathbb{R}^2$ as a closed set including its inner part.

Definition. The set $W \subset S$ is called **curvilinear triangle on the surface** S if there exists such local parametrization $f\colon D \to E_3$ of the surface S and such triangle $\Delta \subset D$ that $W = f(\Delta)$.

Thus a curvilinear triangle is a simple region on S. Its edges are segments U_1, U_2, U_3 of curves on S. Denote by β_1, β_2, β_3 inner angles of its tangent lines at vertices p_1, p_2, p_3 as on the picture. JS: missing picture

Theorem (Generalised Gauss-Bonnet)**.** It holds

e14.10 (10)
$$\iint_W K\, dV = 2\pi - \sum_{i=1}^{3} \beta_i - \sum_{i=1}^{3} \int_{U_i} \varkappa_g\, ds\,.$$

Idea of the proof. Let us "smooth" the border of W at the vertex p_i using a geodesic circle with $\widetilde{C}_i$ with a small radius r as on the picture. JS: missing picture Then one can show that $\lim\limits_{r\to 0} \int_{\widetilde{C}_i} \varkappa_g\, ds = \beta_i$. One can formulate this also as the idea that as a limit, we have the same situation as in the plane which we discussed at the end of example 14.12. It follows from (8) that be obtain the formulae (10) as a limit.

de14.14 **14.14 Definition.** A curvilinear triangle W is called **geodetic triangle on the surface** S if all its edges are segments of geodetic curves.

This notion is a direct a straightforward modification of the notion of a triangle for the inner geometry of surfaces.

We $\varkappa_g = 0$ in (10) in this case. This shows

Corollary. A geodetic triangle W on the surface S satisfies

e14.11 (11)
$$\iint\limits_{W} K\, dV = 2\pi - \beta_1 - \beta_2 - \beta_3\,.$$

14.15 **14.15.** The following geometrically very interseting statement follows directly from (11):

Theorem (Special Gauss-Bonnet)**.** If W is a geodetic triangle on a surface with constant Gauss curvature K of the area V and inner angles α_1, α_2, α_3 then the following holds:

e14.12 (12)
$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + KV\,.$$

Proof. PFor K constant, the integral in (11) is equal KV and inner angles are given by the relation $\beta_i = \pi - \alpha_i$, $i = 1, 2, 3$. □

14.16 **14.16. Examples.** a) Considering developable surfaces, we have $K = 0$ as in the plane. Then (12) yields the well known theorem about the sum of angles in a triangle. $\alpha_1 + \alpha_2 + \alpha_3 = \pi$.
b) Considering the sphere S with radius r, we have $K = \frac{1}{r^2}$. Thus the sum of angles in a geodetic triangle is greater then π. In fact, it follows from (12) that the sum af angles minus π is proportional to the area of the geodetic triangle. It is interesting to realiye that one can compute are of the sphere from (12). Since geodetic curves on S are great circles, one eighth of the sphere is a geodetic triangle with all three right angles. Denote by V its area. Since the Gaussian curvature of the sphere with radius r is $\frac{1}{r^2}$, it follows from (12) that $\frac{3\pi}{2} = \pi + \frac{1}{r^2}V$, i.e. $V = \frac{1}{2}\pi r^2$.

14.17 **14.17.** The 5th Euclid axiom states that for a given line p in the plane, there is a unique line nonintersecting p through every point $A \notin p$. Nowadays, this is known as the Euclid axiom about parallel lines. This statement is so essentially different from other Euclid axioms that many mathematicians for centuries tried to rpove that the 5th axiom is a consequence of remaining Euclid axioms. However, it was not possible to prove, despite of many (incorrect) attempts, to show that the opposite of the Euclid axiom about parallel lines leads to a contradiction.

Thus one of consequences of negation of the 5. axiom, i.e. an assumption of existence at least two lines through the point A nonitersecting p, is the statement that the sum $\Sigma$of angles in a triangle is smalles than π. Moreover, the difference $\pi - \Sigma$, so called **angle defect**, is propotional to the area of triangle. Such statement seemed absurd to many mathematicians. The special Gauss-Bonnet theorem shows this case happens in the inner geometry of a surface with negatiove constant Gaussian curvature. Also our theorem 14.6 about the three-parameter system of local isometries on a surface with constant Gaussian curvature corresponds to Euclidean transformations in the classical geometry of plane – with the Euklid axiom about parallel lines or with its negation. Precise constructions of noneuclidean geometries were found in the 2nd half of the 19. century (using essentially projective geometry tools).

14.18. Negation of the 5. axiom leads only to one type of noneuclidean geometries which are called **Lobachevsky geometries** or **hyperbolic geometries**. These correspond to surfaces with negative constant Gaussian curvature. However, D. Hilbert shown in the year 1901 that Lobachevsky plane cannot be globaly realized on a surface in E_3. (This is the essential explanation why the tractrix in **??**, whose rotation yiedls the pseudosphere, has a singular point.

An important analogy between properties of surfaces with a posotive and negative constant Gaussian curvature, stated in particular in theorems 14.3, 14.6 and 14.15, motivated to include also **elliptic geometries** (often related to the name B. Riemann) into noneuclidean geometries. These correspond to the inner geometry of surfaces with positive constant Gaussian curvature. Let us not the notion of **Riemannian geometry** is usually using in a meaning different from elliptic geometries. In fact, Riemannain geometry denotes far more important theory, which B. Riemann iniciated, and which is nowadays one of more important parts of differential geometry, see e.g. the textbook [5] for a basic information.

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