Def. Let A be a ring. An A-module M is Noetherian $\Leftrightarrow$ M satisfies the ascending chain condition (ACC) for submod, i.e., every increasing sequence of submod of M

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M$$

terminates, so $\exists\, N > 0 : M_N = M_{N+1} = M_{N+2} = \cdots$.

If A is a Noetherian A-mod, then A is called a Noetherian ring. since an ideal is an A-submod of A

Def. An A-mod M is generated by a set $S \Leftrightarrow \forall\, m \in M$,
$m = \sum_{i=1}^{k} a_i s_i$ where $a_i \in A$, $s_i \in S$.

M is finitely generated $\Leftrightarrow$ S is finite.
any mod M is gen by itself

1. Prove that an A-mod M is Noetherian $\Leftrightarrow$ every submod of M is fin gen.

Pf. ($\Rightarrow$): Suppose $N \subseteq M$ is not a fin gen submod of M.
Then there exists $x_1 \in N$, $N \neq (x_1)$, so $\exists\, x_2 \in N$: $x_2 \in N \setminus (x_1)$.
Inductively, we have a chain of submod of M

$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots$$

that does not terminate.

($\Leftarrow$): Let $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ be a chain of submod of M.
Let $J = \bigcup_{i=1}^{\infty} M_i$, which is a submod of M.
Since M_i's are fin gen, so is J.
Let $S = \{s_1, \ldots, s_k\}$ be the generating set for J.
Then $\forall\, j = 1, \ldots, k$, $s_j \in M_{i_j}$ for some i_j.
Set $N = \max_j (i_j)$.
So $s_1, \ldots, s_n \in M_N$, which means $M_N = J$.
Hence $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_N = M_{N+1} = \cdots$ terminates at M_N. $\square$

2. Prove that an A-mod M is Noetherian $\Leftrightarrow$ any non-empty collection P of submod of M has a maximal element, i.e., $\exists N \in P$: if $Q \in P$ and $N \subseteq Q$, then $Q \subseteq N$.

Pf. ($\Rightarrow$): Suppose P is a collection of submod of M without a maximal element. Axiom of Dependent Choice

Thus for $M_1 \in P$, there exists $M_2 \in P$ s.t. $M_1 \subsetneq M_2$.

Inductively, we form a strictly increasing chain of submod of M

$$M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \cdots$$

so M is not Noetherian.

($\Leftarrow$): Let $M_0 \subseteq M_1 \subseteq \cdots$ be an increasing seq of submod of M.

Let P be the collection $\{M_0 \subseteq M_1 \subseteq \cdots\}$, by assumption, P has a maximal element, so $\exists N \in P$ s.t.

$$M_0 \subseteq M_1 \subseteq \cdots = M_n = M_{n+1} = \cdots$$

where $M_n = N$. $\square$

3. Determine if A is a Noetherian ring:

i) $A = \Bbbk[x_1, \cdots]$, the poly ring of infin variables.

ii) $A = C(\mathbb{R})$, the ring of cont functions.

Ans: i): No. We have an increasing chain

$$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots$$

ii): No. Consider the ideal for $k \geq 1$,

$$I_k = \{ f \in C(\mathbb{R}) : f(x) = 0 \quad \forall x \geq k \}$$

Then we have an increasing chain

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

4. Let $0 \xrightarrow{f_0} N \xrightarrow{f_1} M \xrightarrow{f_2} M/N \xrightarrow{f_3} 0$ be a SES of A-mod, i.e., $\operatorname{im} f_i = \ker f_{i+1}$.

Show that M is Noetherian $\Leftrightarrow$ both N and M/N are Noetherian.

Hints: If we use the fin gen property:

For $(\Rightarrow)$:

- f_1 is injective so we may identify a submod of N with that of M.
- for a submod L of M/N, $f_2^{-1}(L)$ is a submod of M.

For $(\Leftarrow)$:

- Let K be a submod of M and consider $f_1^{-1}(K)$ and $f_2(K)$.

Alternatively, one can also prove using the ACC property.

Pf. $(\Rightarrow)$: Let M be Noetherian.

Any submod of N is iso to one of M via the injective f_0, which is fin gen, so N is Noeth.

Let L be a submod of M/N. Then $f_2^{-1}(L)$ is a submod of M, which is fin gen.

Since f_2 is surjective, $f_2|_{f_2^{-1}(L)} : f_2^{-1}(L) \to L$ is also surj, so any generator of L comes from one of $f_2^{-1}(L)$, this implies L is fin gen, so M/N is Noeth.

$(\Leftarrow)$: Let N, M/N be Noeth.

Let K be a submod of M.

The the submods $f_1^{-1}(K)$ and $f_2(K)$ are fin gen, by $x_1', \dots, x_r'$ and $y_1', \dots, y_s'$ resp..

Let $x_i := f_1(x_i') \in K$ and since f_2 is surjective, we can find $y_j \in K$ s.t. $f_2(y_j) = y_j'$.

Now let $m \in K$.

Then $f_2(m) = \sum_{j=1}^{s} b_j y_j' = f_2\left(\sum_{j=1}^{s} b_j y_j\right)$ where $b_j \in A$.

$\Rightarrow \quad m - \sum_{j=1}^{s} b_j y_j \in \ker f_2 \cap K = \operatorname{im} f_1 \cap K$

$\Rightarrow \quad m - \sum_{j=1}^{s} b_j y_j = f_1(n)$ for some $n \in f_1^{-1}(K)$.

Now $n = \sum_{i=1}^{r} a_i x_i'$, so $f_1(n) = \sum_{i=1}^{r} a_i f_1(x_i') = \sum_{i=1}^{r} a_i x_i$.

$\therefore \quad m = \sum_{i=1}^{r} a_i x_i + \sum_{j=1}^{s} b_j y_j$

Thus K is fin gen by $\{x_1, \dots, x_r, y_1, \dots, y_s\}$. $\square$

Rk. $(\Rightarrow)$: Let $K_1 \subseteq K_2 \subseteq \cdots$ be in M, then $f_1^{-1}(K_n)$ is a chain in N and $f_2(K_n)$ is a chain in M/N.

$(\Leftarrow)$: Any ascending chain in N or M/N gives rise to a chain in M.

5. Show that if A is a Noeth ring, then any fin gen A-mod M is a Noeth module.

Hint: Use Q4.

Pf. There is a SES

$$0 \to A \to A \oplus A \to A \to 0$$

So by Q4, $A \oplus A$ is Noeth.

Now there is a SES

$$0 \to A \oplus A \to A \oplus A \oplus A \to A \to 0$$

So by Q4, $A \oplus A \oplus A$ is Noeth.

Inductively, $A^{\oplus n}$ is Noeth.

Now since any fin gen A-mod M is a quotient of $A^{\oplus n}$, so there is a SES

$$0 \to \ker q \to A^{\oplus n} \to M \to 0,$$

by Q4, M is a Noeth mod. $\square$