sometimes called rank

7. Let $\dim : \text{FreeFGMod}_R \to \mathbb{Z}$ be a function which outputs the dimension (rank) of a free f.g. R-module, such that if there is a SES $A \rightarrowtail B \twoheadrightarrow C$, then
$$\dim B = \dim A + \dim C.$$

a) Describe $\dim$ as a familiar concept in
i) $R = \mathbb{R}$, i.e., $\text{FinVect}_{\mathbb{R}}$
ii) $R = \mathbb{Z}$, i.e., FGAb
Hint: $\dim G := \log |G|$ and consider a subgroup $H \leq G$

Ans: i) Let $f : V \to W$ be a linear map.
Then we have a SES
$$\ker f \rightarrowtail V \overset{f}{\twoheadrightarrow} \operatorname{im} f$$
so $\dim V = \dim(\ker f) + \dim(\operatorname{im} f)$, which is indeed the Rank-Nullity Theorem
nullity   rank
$$\dim V = \text{null } f + \text{rank } f.$$

ii) We have a SES
$$H \rightarrowtail G \twoheadrightarrow G/H$$
and so
$$\log |G| = \log |H| + \log |G/H|$$
$$\Leftrightarrow \quad |G| = |H| \cdot |G/H|$$
where $|G/H|$ is the no. of cosets of H in G, which is equal to the index $[G:H]$.
Thus, this is indeed Lagrange's Theorem.

b) Define the Euler characteristic of a finite chain complex C as
$$\chi(C) := \sum_n (-1)^n \dim C_n.$$

i) Recall that the Euler char of the homology of C is defined as
$$\chi(H_*(C)) := \sum_n (-1)^n \dim (H_n(C))$$
Show that $\chi(C) = \chi(H_*(C))$
Hint: $Z_n := \ker d$, $B_{n-1} := \operatorname{im} d$, $H_n := H_n(C)$, consider the SES's
$$Z_n \rightarrowtail C_n \twoheadrightarrow B_{n-1},$$
$$B_n \rightarrowtail Z_n \twoheadrightarrow H_n.$$

ii) From this, recover the classical formula in Algebraic Topology:
$$|V| - |E| + |F| = 1$$
for any topological spaces with homology same as $H_*(P^2)$, and
$$|V| - |E| + |F| = 2$$
for those with homology same as $H_*(S^2)$.

Ans: i) Set $Z_n := \ker d$ and $B_{n-1} := \operatorname{im} d$.

Then we have a SES

$$Z_n \rightarrowtail C_n \twoheadrightarrow B_{n-1}.$$

Write $H_n := H_n(C)$, we have another SES

$$B_n \rightarrowtail Z_n \twoheadrightarrow H_n.$$

Now
$$\dim C_n = \dim Z_n + \dim B_{n-1}$$
$$= \dim B_n + \dim H_n + \dim B_{n-1}$$

So $(-1)^n \dim C_n = (-1)^n \dim B_n + (-1) \dim H_n + (-1)^n \dim B_{n-1}$

$\Rightarrow \sum_n (-1)^n \dim C_n = \sum_n [(-1)^n \dim B_n + (-1)^n \dim B_{n-1}] + \sum_n (-1)^n \dim H_n.$

Note that

$$\sum_n [(-1)^n \dim B_n + (-1)^n \dim B_{n-1}] = \sum_n [(-1)^n \dim B_n - (-1)^{n-1} \dim B_{n-1}]$$
$$= \sum_n (-1)^n \dim B_n - \sum_{n-1} (-1)^{n-1} \dim B_{n-1} = 0$$

$\therefore \quad \chi(C) = \chi(H_*(C))$.

ii) In Algebraic Topology, we consider a chain complex C where C_n is an Abe group generated by n-cells in a topological space.

It is clear that

$|V| = \dim C_0, \quad |E| = \dim C_1, \quad |F| = \dim C_2.$

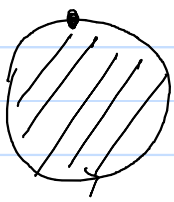
Now $\chi(D^2) = 1 - 1 + 1 = 1$

and if $H_*(C) \cong H_*(D^2)$, then

$\chi(C) = \chi(H_*(C)) = \chi(H_*(D^2)) = \chi(D^2) = 1.$

Similarly, $\chi(S^2) = 1 - 0 + 1 = 2$, if $H_*(C) \cong H_*(S^2)$, then $\chi(C) = \chi(S^2) = 2$.

D^2



S^2