1. a) Show that if the multiplication by u on an R-mod M

$u \cdot - : M \to M$

is injective for all u in a mult set U,

Then $\lambda : M \to U^{-1}M$

$m \mapsto \frac{m}{1}$

is also injective.

Ans: Suppose $\lambda(m_1) = \lambda(m_2)$

$\Rightarrow \frac{m_1}{1} = \frac{m_2}{1}$

$\Rightarrow \exists u : (m_1 \cdot 1 - m_2 \cdot 1) u = 0$

$\Rightarrow u \cdot m_1 = u \cdot m_2$

$\Rightarrow m_1 = m_2$ as $u \cdot -$ is inj

b) Show that if $u \cdot -$ is an isomorphism for any u, then λ is also an iso.

Ans: Let $\frac{a}{u} \in U^{-1}M$.

Since $u \cdot -$ is sur, $\exists a' : a = ua'$.

Now since $(a - ua') 1 = 0$

$\Rightarrow \frac{a}{u} = \frac{a'}{1} = \lambda(a')$

2. Construct a ring homo $\mathbb{Z}_{(p)} \to \mathbb{Z}/p\mathbb{Z}$.

$\mathbb{Z}_{(p)} / p\mathbb{Z}_{(p)}$

Ans: Consider $f : \mathbb{Z}_{(p)} \to \mathbb{Z}/p\mathbb{Z}$

$\frac{a}{b} : p \nmid b \mapsto ab^{-1} \mod p$

We have $f(0) = [0 \, b^{-1}] = [0]$

$f(1) = [1 \cdot 1^{-1}] = [1]$

$f(\frac{a}{b} \frac{c}{d}) = [ac(bd)^{-1}]$

$= [ab^{-1}][cd^{-1}]$

$= f(\frac{a}{b}) f(\frac{c}{d})$

3. a) Let M be a fin gen Ab group.
Compute the localisation $M_{(p)}$ of the $\mathbb{Z}$-mod M at (p) for a prime no. p.
Hint: ext of scalars.

Ans: $M_{(p)} \cong \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} M$
and we computed $\mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} M$ in the last tutorial.

b) Compute $\mathbb{Q} \otimes_{\mathbb{Z}} M$.

Ans: $\mathbb{Q} \otimes_{\mathbb{Z}} M \cong \mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}^r \oplus \mathbb{Z}/(n_1) \oplus \cdots \oplus \mathbb{Z}/(n_r))$
Note that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/(n) \cong 0$ because any element
$\frac{a}{b} \otimes x = \frac{a}{bn} \otimes nx = 0$.
So $\mathbb{Q} \otimes_{\mathbb{Z}} M \cong \mathbb{Q}^r$.