Def. An ideal $Q \subseteq R$ is primary $\Leftrightarrow$
$Q \neq R$ & $\forall x, y \in R$, $xy \in Q \Rightarrow x \in Q$ or $y^n \in Q$ for some $n > 0$.

Rk. Q is primary $\Leftrightarrow$ $R/Q \neq 0$ and all zero div in R/Q are nilpotent.

Def. Let $I \subseteq R$ be an ideal. The expression
$$I = \bigcap_{i=1}^{n} Q_i$$
is called a primary decomposition where Q_i's are primary.
A minimal primary decomposition is that $\mathrm{rad}(Q_i)$ are distinct and $Q_i \not\supseteq \bigcap_{i \neq j} Q_j$ for each i

$q^n \in Q_i$

1. a) Express an ideal in $\mathbb{Z}$ in the form of primary decomposition.

Ans: Write $n = p_1^{k_1} \cdots p_r^{k_r}$.
It is clear that $(n) = \bigcap_{i=1}^{r} (p_i^{k_i})$. It suffices to check that $(p_i^{k_i})$ is primary.
Indeed, if $xy \in (p_i^{k_i})$, then $xy = a\, p_i^{k}$, where $(a, p_i) = 1$ and $k_i < k$. If $x = a_x\, p_i^{k_x}$ where $(a_x, p_i) = 1$ and $k_x < k_i$, then $y = a_y\, p_i^{k_y}$ where $(a_y, p_i) = 1$, $a_x a_y = a$, and $k_x + k_y = k$. Take $\mathrm{lcm}(k_y, k_i) =: l$, then $k_i \mid k_y l$, so $y^l \in (p_i^{k_i})$.

b) Is this decomposition minimal?

Ans: Yes. $p_i \in \mathrm{rad}(p_i^{k_i})$ but $p_i \notin \mathrm{rad}(p_j^{k_j})$ for $i \neq j$.
It is also clear that $(p_i^{k_i}) \not\supseteq \bigcap_{i \neq j} (p_j^{k_j})$.

2. Express concretely the exactness of

a) $0 \to A \to 0$

b) $0 \to A \to B$

c) $A \to B \to 0$

d) $0 \to A \to B \to 0$

e) $0 \to A \xrightarrow{f_2} B \xrightarrow{f_3} C$

f) $A \xrightarrow{f_1} B \xrightarrow{f_2} C \to 0$

g) $0 \xrightarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} C \xrightarrow{f_4} 0$

Ans: a) $\operatorname{im} f_1 = \ker f_2$

$0 = A$

b) $0 = \ker f_2 \Rightarrow f_2$ is mono

c) $\ker f_2 = B = \operatorname{im} f_1 \Rightarrow f_1$ is epi

d) f_2 is iso by (b) & (c)

e) By (b), f_2 is mono, and $(A, f_2) = \ker f_3$:

Consider

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! h"'] \arrow[dr, "x"] & & \\ A \arrow[r, tail, "f_2"'] & B \arrow[r, shift left, "f_3"] \arrow[r, shift right, "0"'] & C \end{tikzcd}$$

If $f_3 x = 0$, then $\operatorname{im} x \subseteq \ker f_3 = \operatorname{im} f_2$, so x factors through A uniquely as f_2 is mono.

f) By (c), f_2 is epi, and $(C, f_2) = \operatorname{coker} f_1$:

Consider

$$\begin{tikzcd} A \arrow[r, shift left, "f_1"] \arrow[r, shift right, "0"'] & B \arrow[r, two heads, "f_2"] \arrow[dr, "y"'] & C \arrow[d, dashed, "\exists! k"] \\ & & Y \end{tikzcd}$$

If $y f_1 = 0$, then $\ker f_2 = \operatorname{im} f_1 \subseteq \ker y$, so y factors through C uniquely as f_2 is epi.

g) By (e) & (f), f_2 is mono, f_3 is epi, and $\ker f_3 = \operatorname{im} f_2$.

5. Prove that if g is the pullback of g'

$$\begin{tikzcd} \{(b,c) : g'(b) = f'(c)\} =: B \times_A C \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f'"] \\ B \arrow[r, "g'"'] & A \end{tikzcd}$$

in an Abelian cat

then the induced map $k : \ker g \xrightarrow[\cong]{f|_{\ker g}} \ker g'$ is an isomorphism

Pf. Claim: $\operatorname{im} f|_{\ker g} \subseteq \ker g'$:

$$g' f|_{\ker g} = f' g|_{\ker g} \equiv 0 .$$

It is clear that we have a map $\ker g' \overset{i}{\hookrightarrow} B$ and also a map $\ker g' \xrightarrow{0} C$, so we have a cone

$$\begin{tikzcd} \ker g' \arrow[r, "0"] \arrow[d, "i"'] & C \arrow[d, "f'"] \\ B \arrow[r, "g'"'] & A \end{tikzcd}$$

$$\begin{tikzcd} \ker g' \arrow[dr, "h"] \arrow[rr, bend left, "0"] \arrow[ddr, "i"'] & & \\ & B \times_A C \arrow[r, "g"] \arrow[d, "f"] & \\ & B & \end{tikzcd}$$

and by the universal property, $\exists!\ h : \ker g' \to B \times_A C$.

Claim: $\operatorname{im} h \subseteq \ker g$:

Since $gh = 0$, $\operatorname{im} h \subseteq \ker g$. So $h : \ker g' \to \ker g$.

Now, we have a cone from $\ker g$ as shown below

$$\begin{tikzcd} \ker g \arrow[r, "f|_{\ker g}"] \arrow[drrr, bend left=40, "g|_{\ker g}"] \arrow[ddrr, bend right=40, "f|_{\ker g}"'] & \ker g' \arrow[dr, "h"] \arrow[drr, bend left, "0"] \arrow[ddr, "i"'] & & \\ & & B \times_A C \arrow[r, "g"] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f'"] \\ & & B \arrow[r, "g'"'] & A \end{tikzcd}$$

By uni prop, the inclusion $\iota : \ker g \hookrightarrow B \times_A C$ is the unique map making the diag commutes. Yet, we have $f h f|_{\ker g} = i f|_{\ker g} = f|_{\ker g}$ & $f' \circ f|_{\ker g} = g|_{\ker g} \equiv 0$, so $h f|_{\ker g} = \iota$. Restricting onto $\ker g$, $hk = 1$.

Next, consider

$$\begin{tikzcd}
\ker g' \arrow[r, "h"] \arrow[rrrr, bend left=40, "0"] \arrow[rrrd, bend right=40, "i"'] & \ker g \arrow[r, "f|_{\ker g}"] & \ker g' \arrow[dr, "h"] \arrow[rr, bend left, "0"] \arrow[rd, bend right, "i"'] & & \\
& & & B \times_A C \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "f'"] \\
& & & B \arrow[r, "g'"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
\ker g' \arrow[d, dashed, "\exists!"'] \arrow[dr] & & \\
\ker g' \arrow[r, hook, "i"'] & B \arrow[r, shift left, "g'"] \arrow[r, shift right, "0"'] & A
\end{tikzcd}$$

where we have $f h f|_{\ker g} h = f(1) h = fh = i$.

So we have by the uni prop of $\ker g'$,

$$\begin{tikzcd}
\ker g' \arrow[d, dashed, "f|_{\ker g} h"', "\exists!"] \arrow[drr, "fhf|_{\ker g}h = i"] & & & \\
\ker g' \arrow[rr, hook, "i"'] & & B \arrow[r, shift left, "g'"] \arrow[r, shift right, "0"'] & A
\end{tikzcd}$$

where the unique map is 1, so $f|_{\ker g} h = 1$. $\square$