

Def. An ideal $Q \subseteq R$ is primary \Leftrightarrow
 $Q \neq R$ & $\forall x, y \in R, xy \in Q \Rightarrow x \in Q$ or $y^n \in Q$ for some $n > 0$.

Rk. Q is primary $\Leftrightarrow R/Q \neq 0$ and all zero div in R/Q are nilpotent.

Def. Let $I \subseteq R$ be an ideal. The expression

$$I = \bigcap_{i=1}^n Q_i$$
 is called a primary decomposition where Q_i 's are primary.
 A minimal primary decomposition is that $\text{rad}(Q_i)$
 are distinct and $Q_i \not\subseteq \bigcap_{j \neq i} Q_j$ for each i \searrow
 $Q_i \in \mathcal{Q}_I$

1. a) Express an ideal in \mathbb{Z} in the form of primary decomposition.

Ans: Write $n = p_1^{k_1} \dots p_r^{k_r}$.

It is clear that $(n) = \bigcap_{i=1}^r (p_i^{k_i})$. It suffices to check that $(p_i^{k_i})$ is primary.

Indeed, if $xy \in (p_i^{k_i})$, then $xy = a p_i^k$, where $(a, p_i) = 1$ and $k_i < k$. If $x = a_x p_i^{k_x}$ where $(a_x, p_i) = 1$ and $k_x < k_i$, then $y = a_y p_i^{k_y}$ where $(a_y, p_i) = 1$, $a_x a_y = a$, and $k_x + k_y = k$. Take $\text{lcm}(k_y, k_i) =: l$, then $k_i \mid k_y l$, so $y^l \in (p_i^{k_i})$.

b) Is this decomposition minimal?

Ans: Yes. $p_i \in \text{rad}(p_i^{k_i})$ but $p_i \notin \text{rad}(p_j^{k_j})$ for $i \neq j$.
 It is also clear that $(p_i^{k_i}) \not\subseteq \bigcap_{j \neq i} (p_j^{k_j})$.

2. Express concretely the exactness of

a) $0 \rightarrow A \rightarrow 0$

b) $0 \rightarrow A \rightarrow B$

c) $A \rightarrow B \rightarrow 0$

d) $0 \rightarrow A \rightarrow B \rightarrow 0$

e) $0 \rightarrow A \xrightarrow{f_1} B \xrightarrow{f_2} C$

f) $A \xrightarrow{f_1} B \xrightarrow{f_2} C \rightarrow 0$

g) $0 \xrightarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} C \xrightarrow{f_4} 0$

ans: a) $\text{im } f_1 = \ker f_1$

$0 = A$

b) $0 = \ker f_2 \Rightarrow f_2$ is mono

c) $\ker f_2 = B = \text{im } f_1 \Rightarrow f_1$ is epi

d) f_2 is iso by (b) & (c)

e) By (b), f_2 is mono, and $(A, f_2) = \ker f_3$:

Consider X

$$\begin{array}{ccc} \exists! h \downarrow & \searrow x & \\ A & \xrightarrow{f_2} B & \xrightarrow{f_3} C \\ & \searrow & \searrow \end{array}$$

If $f_3 x = 0$, then $\text{im } x \subseteq \ker f_3 = \text{im } f_2$, so x factors through A uniquely as f_2 is mono.

f) By (c), f_2 is epi, and $(C, f_2) = \text{coker } f_1$:

$$\begin{array}{ccc} \text{Consider } A & \xrightarrow{f_1} B & \xrightarrow{f_2} C \\ & \searrow y & \downarrow \exists! k \\ & & Y \end{array}$$

If $y f_1 = 0$, then $\ker f_2 = \text{im } f_1 \subseteq \ker y$, so y factors through C uniquely as f_2 is epi.

g) By (e) & (f), f_2 is mono, f_3 is epi, and $\ker f_3 = \text{im } f_2$.

5. Prove that if g is the pullback of g'

$$\{(b, c) : g'(b) = f'(c)\} =: B \times_A C \xrightarrow{g} C$$

$$\begin{array}{ccc} & \downarrow f & \\ & B & \xrightarrow{g'} A \end{array}$$

in an Abelian cat

then the induced map $k: \ker g \xrightarrow{\cong} \ker g'$ is an isomorphism

Pf. Claim: $\text{im } f|_{\ker g} \subseteq \ker g'$:
 $g' f|_{\ker g} = f' g|_{\ker g} = 0$.

It is clear that we have a map $\ker g' \xrightarrow{i} B$ and also a map $\ker g' \xrightarrow{0} C$, so we have a cone

$$\begin{array}{ccc} \ker g' & \xrightarrow{0} & C \\ i \downarrow & & \downarrow f' \\ B & \xrightarrow{g'} & A \end{array}$$

$$\begin{array}{ccc} \ker g' & \xrightarrow{0} & C \\ \downarrow h & & \downarrow g \\ B \times_A C & \xrightarrow{g} & C \\ i \downarrow & & \downarrow f \\ B & \xrightarrow{g'} & A \end{array}$$

and by the universal property, $\exists! h: \ker g' \rightarrow B \times_A C$.

Claim: $\text{im } h \subseteq \ker g$:
 since $gh = 0$, $\text{im } h \subseteq \ker g$. So $h: \ker g' \rightarrow \ker g$.

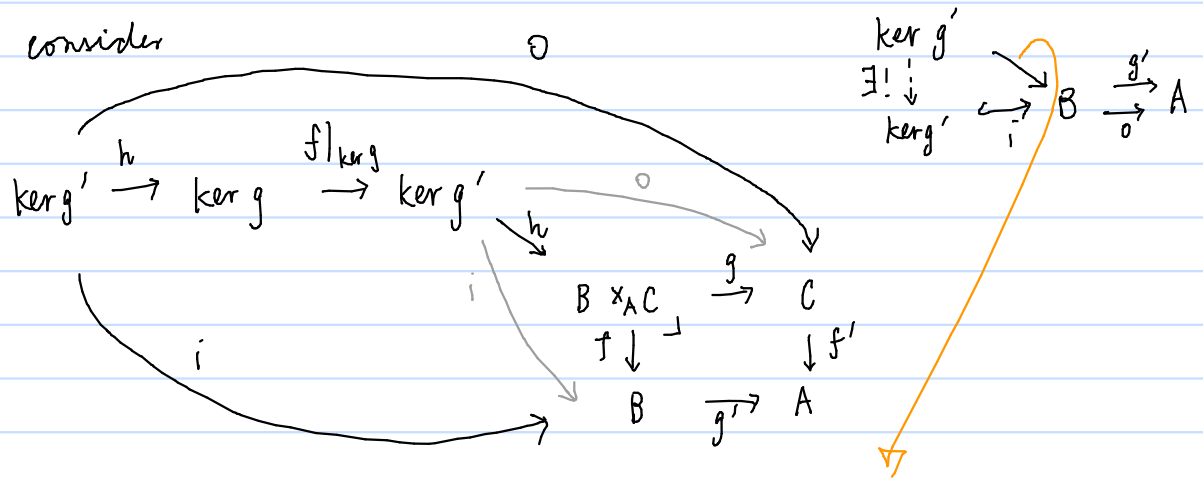
Now, we have a cone from $\ker g$ as shown below

$$\begin{array}{ccc} \ker g & \xrightarrow{f|_{\ker g}} & \ker g' \\ & \searrow f|_{\ker g} & \downarrow h \\ & & B \times_A C \\ & \searrow f|_{\ker g} & \downarrow f \\ & & B \end{array}$$

$$\begin{array}{ccc} \ker g & \xrightarrow{g|_{\ker g}} & C \\ & \searrow f|_{\ker g} & \downarrow f' \\ & & A \end{array}$$

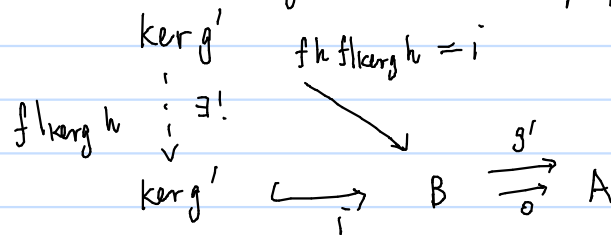
By Uni prop, the inclusion $\iota: \ker g \hookrightarrow B \times_A C$ is the unique map making the diag commutes. Yet, we have $f \circ h \circ f|_{\ker g} = \iota \circ f|_{\ker g}$ & $f' \circ f|_{\ker g} = g|_{\ker g} = 0$, so $h \circ f|_{\ker g} = \iota$.
 Restricting onto $\ker g$, $h \circ k = 1$.

Next, consider



where we have $f h f|_{\ker g} h = f(1) h = f h = i$.

So we have by the univ prop of $\ker g'$,



where the unique map is 1, so $f|_{\ker g} h = 1$. \square