1. Let C_∞ be the infinite cyclic gp C_∞ with generator t.
Find a proj resol of $\mathbb{Z}$ in $\mathrm{Mod}_{\mathbb{Z}[C_\infty]}$.

Ans : Note that $\mathbb{Z}[C_\infty] \cong \mathbb{Z}[t, t^{-1}]$, the ring of Laurent poly.

Again, $\mathbb{Z}[C_\infty] \overset{ev_1}{\twoheadrightarrow} \mathbb{Z}$ is clearly an epi.

Since $\sum_{i=-\infty}^{+\infty} a_i t^i \, (t-1) = 0 \iff \sum_{i=-\infty}^{+\infty} a_i t^i = 0$,

we have a SES

$$0 \xrightarrow{0} \mathbb{Z}[C_\infty] \xrightarrow{t-1} \mathbb{Z}[C_\infty]$$

$\therefore$ The proj resol is given by

$$0 \xrightarrow{0} \mathbb{Z}[C_\infty] \xrightarrow{t-1} \mathbb{Z}[C_\infty] \overset{ev_1}{\twoheadrightarrow} \mathbb{Z} .$$

2. Find an injective coresolution of $\mathbb{Z}$ in $\mathrm{Mod}_{\mathbb{Z}}$.

Ans : $\mathbb{Z} \overset{inc}{\hookrightarrow} \mathbb{Q}$ is clearly a mono.

Indeed, we have a SES

$$0 \to \mathbb{Z} \overset{inc}{\hookrightarrow} \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \to 0$$

this gives an inj cores of $\mathbb{Z}$.

Recall that we define $\mathcal{P}$ as a collection of $\mathcal{P}$-proj obj, such that

- every obj admits an epi from a $\mathcal{P}$-proj obj ;
- $\mathcal{P}$ is closed under kernels of epis,

  i.e., for a SES $A \rightarrowtail B \twoheadrightarrow C$, if B & C are both in $\mathcal{P}$, then so is A.

Def. Let $\mathcal{P}$ be adopted to F, i.e., exact on SES of $\mathcal{P}$-proj. The left derived functor $L_* F$ is defined by

$$L_* F(A) := H_*(FP)$$

where $P \to A[0]$ is a $\mathcal{P}$-proj resolution.

3. Show that $L_* F$ is independent of the class $\mathcal{P}$, i.e., if $\mathcal{Q}$ is another class of proj obj and there is a $\mathcal{Q}$-proj resol $Q' \to A[0]$, show that $H_*(FP) \cong H_*(FQ')$.

Hint: Consider

$$\begin{tikzcd}
P' \arrow[r] \arrow[d] & Q' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & Q \arrow[d] \\
A[0] \arrow[r, equal] & A[0] \arrow[r, equal] & A[0] \arrow[r, equal] & A[0]
\end{tikzcd}$$

where $P^{(\prime)} \to A[0]$ are $\mathcal{P}$-proj resol & $Q' \to A[0]$ is a $\mathcal{Q}$-proj resol.

Pf. Recall from previous tutorial & lecture, we can complete the diagram

$$\begin{tikzcd}
P' \arrow[r, two heads, "\sim"] \arrow[d] & Q' \arrow[r, two heads, "\sim"] \arrow[d] & P \arrow[r, two heads, "\sim"] \arrow[d] & Q \arrow[d] \\
A[0] \arrow[r, equal] & A[0] \arrow[r, equal] & A[0] \arrow[r, equal] & A[0]
\end{tikzcd}$$

mono's too, as usual pf of chain htpy equiv is quasi-iso

so the maps on the top $P' \to P$ and $Q' \to Q$ between $\mathcal{P}$-proj & $\mathcal{Q}$-proj resp. are epi quasi-iso.

From the lecture, we know that if f is an epi between $\mathcal{P}$-proj res, then $\ker f$ is acyclic, hence $F \ker f \cong \ker Ff$ is also acyclic.

Recall that for $A \rightarrowtail B \twoheadrightarrow C$, A is acyclic $\Leftrightarrow$ $B \twoheadrightarrow C$ is quasi-iso.

$A = \ker Ff$

So $F(P' \to P)$ and $F(Q' \to Q)$ are quasi-iso's.

By 2-out-of-6 property, we have $F(Q' \to P)$ being a quasi-iso, so $H_*(FP) \cong H_*(FQ')$, i.e., $L_*^{\mathcal{P}} F \cong L_*^{\mathcal{Q}} F$. $\square$