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Existence and uniqueness of Cournot equilibrium: a contraction mapping approach

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Abstract

We provide a new proof of existence and uniqueness of a Cournot equilibrium. The contraction mapping approach is used. Equilibrium is characterized in terms of marginal costs. The result is useful for applications to two-stage games.

Keywords: Cournot equilibrium; Existence; Uniqueness

JEL classification: L13; L23

1. Introduction

This note provides a new proof of existence and uniqueness of Cournot equilibrium, for any arbitrary number of firms. Previous proofs of existence and/or uniqueness include Friedman (1977), Szidarovsky and Yakowitz (1977), Nishimura and Friedman (1981), Novshek (1985), Kolstad and Mathieson (1986), and Gaudet and Salant (1991).

Our approach is different. We prove existence and uniqueness by applying the contraction mapping theorem to a function involving marginal costs. The advantage of our approach is that the equilibrium is characterized in terms of marginal costs, which can be manipulated by firms in an earlier stage, before the Cournot game takes place. This facilitates the study of a class of two-stage Cournot games. (See Long and Soubeyran (1999) for examples of these games.)

2. Sufficient conditions

There are m firms. Let $M = \{1, \ldots, m\}$. Let $q_i \geq 0$ denote firm i's output. Its cost function is

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$C_i(q_i, \alpha_i)$ where $\alpha_i \geq 0$ is a parameter. (We have in mind applications to two-stage Cournot games, where, in the second stage, firms are Cournot rivals, and in the first stage, firms incur costs to change the parameter α_i. For example, α_i may be firm i's capital stock, which is chosen in stage one.) The inverse demand function is $P(Q)$, and $Q = \Sigma_{i \in M} q_i$. We make the following assumptions.

A1: There exists some $\bar{Q} > 0$ such that $P(Q) > 0$ for $Q \in [0, \bar{Q})$, and $P(Q) = 0$ for $Q \geq \bar{Q}$.

A2: $P''(Q)$ is continuous, $P(0) = \bar{P} > 0$, $P'(Q) < 0$ for $Q \in [0, \bar{Q})$.

A3: $C_i(q_i, \alpha_i)$ is twice continuously differentiable in q_i, and $\partial C_i / \partial q_i > 0$ for all $q_i \in (0, \bar{Q}]$.

A4: $C_i(0, \alpha_i) = 0$, $\partial C_i(0, \alpha_i)/\partial q_i = 0$, and $\partial^2 C_i / \partial q_i^2 > 0$ for all $q_i \in (0, \bar{Q}]$.

A5: $q_i P''(Q) + 2P'(Q) < 0$ for all $Q \in [0, \bar{Q})$, and $q_i \in (0, \bar{Q}]$.

A5(b): $q_i P''(Q) + P'(Q) < 0$ for all $Q \in [0, \bar{Q})$, and $q_i \in (0, \bar{Q}]$.

A6: $-P'(Q) > \delta > 0$ for $Q \in [0, \bar{Q})$, and, for all $q_i \in (0, \bar{Q}]$, $\partial^2 C_i / \partial q_i^2 < b$ for some $b > 0$.

(Note that A5(b) implies A5, it ensures that reaction functions have a negative slope)

Define the marginal cost of firm i as

$$\theta_i = \frac{\partial C_i(q_i, \alpha_i)}{\partial q_i} \tag{1}$$

(Eq. (1)) yields

$$q_i = \rho_i(\theta_i, \alpha_i) \tag{2}$$

with $\partial \rho_i / \partial \theta_i > 0$ for all $\theta_i \geq 0$. Furthermore, because of A4, $\rho_i(0, \alpha_i) = 0$.

Consider the first order condition for an interior equilibrium

$$\hat{q}_i P'(\hat{Q}) + P(\hat{Q}) = \theta_i \tag{3}$$

where the hat denotes equilibrium values. Summing (3) over all $i \in M$, we obtain

$$\hat{Q} P'(\hat{Q}) + mP(\hat{Q}) = m\theta_M \tag{4}$$

where $\theta_M = (1/m)\ \Sigma_{i \in M} \theta_i$.

Define $\psi(Q) = QP'(Q) + mP(Q)$. By A2 and A5, $\psi'(Q) < 0$ for all $Q \in [0, \bar{Q})$. Note that $\psi(0) = m\bar{P} > 0$ and $\psi(\bar{Q}) < 0$. It follows that for all $\theta_M \in [0, \bar{P}]$, there exists a unique $\hat{Q}(\theta_M) \geq 0$ that satisfies (4), and

$$\frac{\partial \hat{Q}}{\partial \theta_M} = \frac{m}{\psi'} < 0$$

From (3),

$$\hat{q}_i = \frac{P(\hat{Q}) - \theta_i}{[-P'(\hat{Q})]} \tag{5}$$

Using (5) and (2), we get the equilibrium condition for the marginal cost of firm i, given θ_M and α_i,

$$\rho_i(\theta_i, \alpha_i) = \frac{P(\hat{Q}(\theta_M)) - \theta_i}{[-P'(\hat{Q}(\theta_M))]} \tag{6}$$

This equation has a unique solution

$$\hat{\theta}_i = \gamma_i(\theta_M, \alpha_i) \tag{7}$$

The uniqueness of $\hat{\theta}_i$, given θ_M and α_i, follows from the following facts: (i) the left-hand side of (6) is increasing in θ_i, and takes the value 0 at $\theta_i = 0$, and (ii) the right-hand side of (6) is strictly decreasing in θ_i, given θ_M and α_i, and takes the positive value $P(\hat{Q}(\theta_M))/[-P'(\hat{Q}(\theta_M))]$ at $\theta_i = 0$.

The function $\gamma_i(\theta_M, \alpha_i)$ is continuous for all $\theta_M \in [0, P]$. Define the function

$$\Gamma(\theta_M, \boldsymbol{\alpha}) = (1/m) \sum_{i \in M} \gamma_i(\theta_M, \alpha_i) \tag{8}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$. For given $\boldsymbol{\alpha}$, the function $\Gamma(\theta_M, \boldsymbol{\alpha})$ is continuous in θ_M for all $\theta_M \in [0, \bar{P}]$ and maps the set $[0, P]$ into itself. Thus, from Kakutani's fixed point theorem (Kakutani, 1941), there exists a fixed point $\hat{\theta}_M$ that satisfies the equation $\Gamma(\theta_M, \boldsymbol{\alpha}) = \theta_M$. We now show the uniqueness of $\hat{\theta}_M$ by showing that $\Gamma(\theta_M, \boldsymbol{\alpha})$ is a contraction mapping.

Lemma. *$\Gamma(\theta_M, \boldsymbol{\alpha})$ is a contraction mapping.*

Proof. Let us find $\partial\gamma_i/\partial\theta_M$. From (6),

$$\frac{\partial \gamma_i}{\partial \theta_M} = \frac{\Lambda_i}{D_i}\left[\frac{-\partial \hat{Q}}{\partial \theta_M}\right] \tag{9}$$

where

$$D_i = [-P'(\hat{Q}(\theta_M))]\frac{\partial \rho_i}{\partial \theta_i} + 1 > \frac{\delta}{b} + 1 = D > 1$$

and

$$\Lambda_i = -\hat{q}_i P''(\hat{Q}(\theta_M)) - P'(\hat{Q}(\theta_M)) > 0$$

Hence

$$0 < \frac{\partial \gamma_i}{\partial \theta_M} < \frac{1}{D}\left[\frac{m\Lambda_i}{-P''(\hat{Q}(\theta_M))\hat{Q}(\theta_M) - (m+1)P'(\hat{Q}(\theta_M))}\right]$$

by A5(b) and A6. Thus

$$0 < \frac{\partial \Gamma}{\partial \theta_M} = \frac{1}{D}\left[\frac{-P''(\hat{Q}(\theta_M))\hat{Q}(\theta_M) - mP'(\hat{Q}(\theta_M))}{-P''(\hat{Q}(\theta_M))\hat{Q}(\theta_M) - (m+1)P'(\hat{Q}(\theta_M))}\right] < \frac{1}{D} < 1$$

This shows that $\Gamma(\theta_M, \boldsymbol{\alpha})$ is a contraction mapping.

From the Lemma and the preceding argument, we can now state:

Proposition. *Under assumptions* $A1$ *to* $A6$, *there exists a unique Cournot equilibrium*.

3. Concluding remarks

Our approach has two distinct advantages. First, the contraction mapping approach facilitates numerical computation of the equilibrium. Second, the Cournot equilibrium is characterized in terms of marginal costs, and this facilitates the study of two-stage Cournot games of cost manipulations, where in the first stage firms manipulate their marginal costs by choosing the parameters α_i $(i = 1, \ldots, m)$ and incurring a cost of manipulation, $\phi_i(\alpha_i)$.

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