Lecture 2 - Grothendieck ∞-groupoids

- Just as cats & groupoids are graphs w/ structures, so ∞-cats / ∞-groupoids are ∞-graphs (aka globular sets) with structure.
- The globular cat $\mathbb{G}$ is the category gen by the graph

$$\{\ 0 \underset{\tau_1}{\overset{\sigma_1}{\rightrightarrows}} 1 \underset{\tau_2}{\overset{\sigma_2}{\rightrightarrows}} 2 \ \cdots\ n \underset{\tau_{n+1}}{\overset{\sigma_{n+1}}{\rightrightarrows}} n+1 \cdots \ \}$$

satisfying the coglobularity relations

$$\sigma_{n+1} \circ \sigma_n = \tau_{n+1} \circ \sigma_n \quad \&$$

$$\sigma_{n+1} \circ \tau_n = \tau_{n+1} \circ \tau_n .$$

- It follows that there are just 2 maps $n \underset{\tau_{n,m}}{\overset{\sigma_{n,m}}{\rightrightarrows}} m$ for $m > n$ & I write $n \underset{\tau}{\overset{\sigma}{\rightrightarrows}} m$ when context is clear.
- The category $[\mathbb{G}^{op}, Set]$ is the category of globular sets - a globular set $X : \mathbb{G}^{op} \longrightarrow Set$ consists of

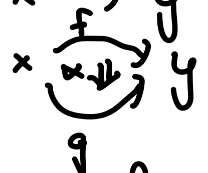
$$X(n+1) \underset{t_{n+1}}{\overset{s_{n+1}}{\rightrightarrows}} X(n) \ \cdots\ X(2) \underset{t_2}{\overset{s_2}{\rightrightarrows}} X(1) \underset{t_1}{\overset{s_1}{\rightrightarrows}} X(0)$$

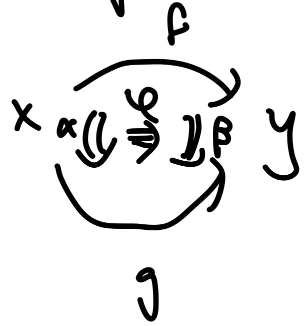
sat the globularity relations

$$s_n \circ s_{n+1} = s_n \circ t_{n+1} \quad \&$$

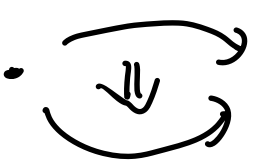
$$t_n \circ s_{n+1} = t_n \circ t_{n+1} .$$

- Here $X(n)$ is the set of <u>n-cells</u> of X

- In a globular set, we have
objects (or 0-cells): $x, y, z \ldots \in X(0)$
1-cells : $x \xrightarrow{f} y$ ~ i.e. $s_1 f \xrightarrow{f} t_1 f$
2-cells :  i.e.
$$\begin{array}{ccc} s_0 s_1 f & \xrightarrow{s_1 f} & t_0 s_1 f \\ \| & \Downarrow \alpha & \| \\ s_0 t_1 f & \xrightarrow[t_1 f]{} & t_0 t_1 f \end{array}$$

3-cells  etc . . .

- The Yoneda embedding
$$\begin{aligned} y : \mathbb{G} &\longrightarrow [\mathbb{G}^{op}, Set] \text{ sends} \\ n &\longmapsto y_n , \end{aligned}$$
the free n-cell (sometimes called n-globe).

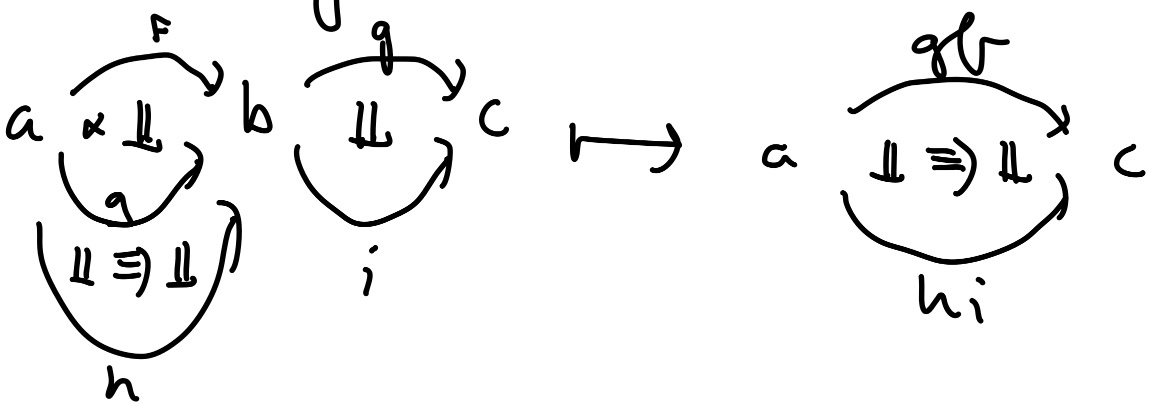
E.g. $y(2) = \{$  $\}$, where I have omitted labels of cells (all are distinct)

- Given a diagram as below left in an ∞-cat / ∞-gpd

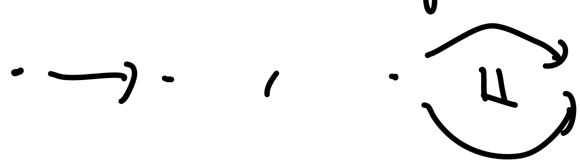
$$a \xrightarrow{f} b \xrightarrow{g} c \quad \longmapsto \quad a \xrightarrow{gf} c$$

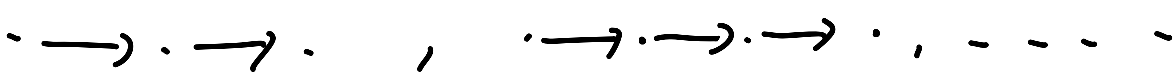
we want to compose it.

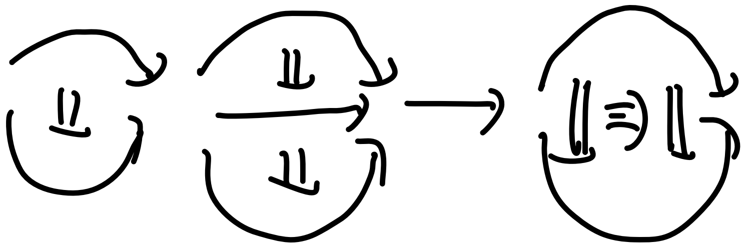
Similarly



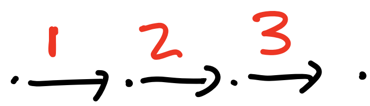
- The diagram shapes (arities) are the so-called globular pasting diagrams (gpd's) & include globular sets like:

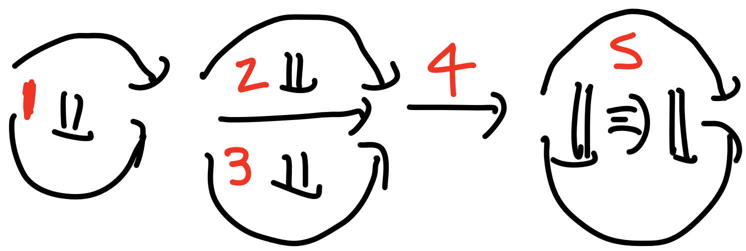
 ... all n-cells





How to parametrise such shapes?

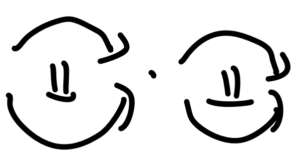
(1, 0, 1, 0, 1) $\sim$ 

(2, 0, 2, 1, 2, 0, 1, 0, 3) $\sim$ 

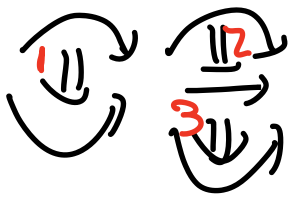
The sequence on the left is called a <u>table of dimensions</u> : it parametrises the associated globular pasting diag -

eg (2, 0, 2, 1, 2, 0, 1, 0, 3) says -

attach a 2-cell 1 to a 2-cell 2 along a 0-cell

 ;

then attach a 2-cell 3 to 2 along its 1-cell boundary



& so on ...

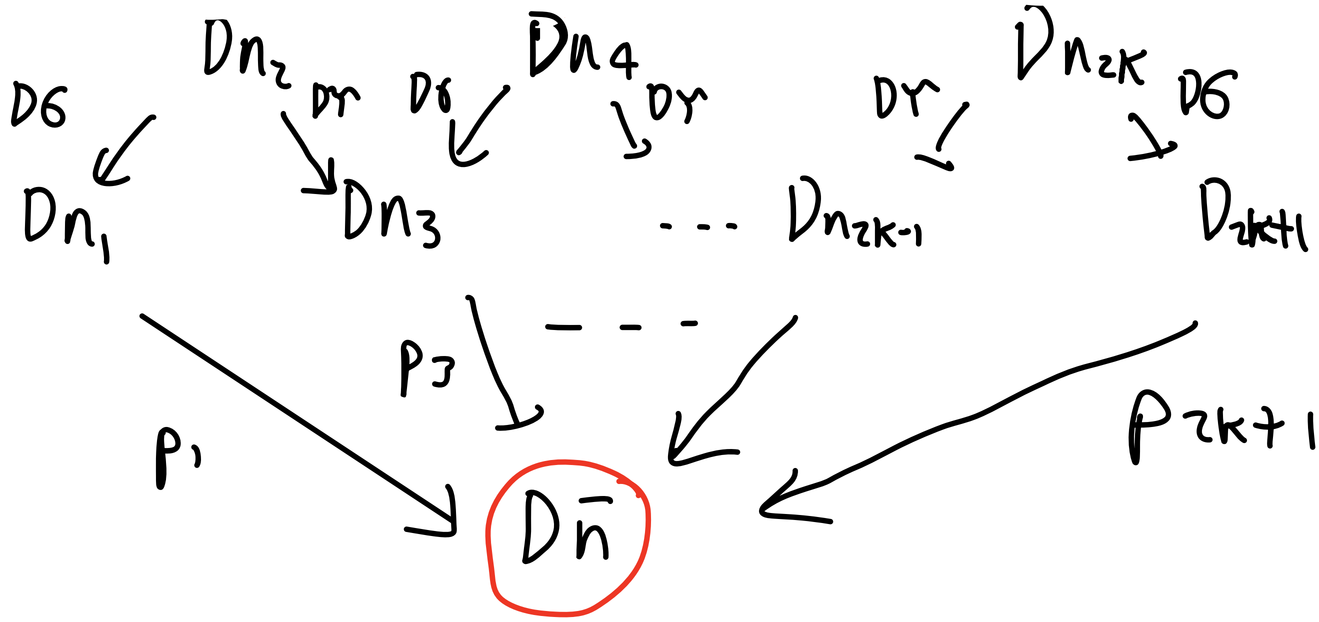
Def) A table of dimensions is a sequence $\bar{n} = (n_1, \dots, n_{2k+1})$ with

$$n_1 < n_2 > n_3 < n_4 > n_5 \cdots n_{2k-1} < n_{2k} > n_{2k+1}$$

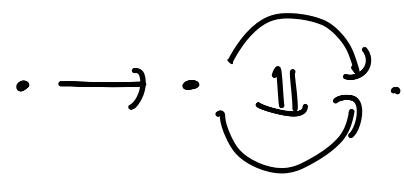
Given a <u>coglobular object</u> $\mathbb{G} \xrightarrow{D} \mathcal{C}$, we obtain a diagram in $\mathcal{C}$

$$\begin{array}{ccccccccc} & Dn_2 & & Dn_4 & & & & Dn_{2k} & \\ {\scriptstyle D\sigma}\swarrow & & \searrow{\scriptstyle D\tau} \quad {\scriptstyle D\sigma}\swarrow & & \searrow{\scriptstyle D\tau} & & {\scriptstyle D\tau}\swarrow & & \searrow{\scriptstyle D\sigma} \\ Dn_1 & & Dn_3 & & \cdots & Dn_{2k-1} & & & D_{2k+1} \end{array}$$

whose colimit, if it exists, we call a <u>globular sum</u> $D\bar{n}$



- In particular, for $\Upsilon : \mathbb{G} \longrightarrow [\mathbb{G}^{op}, Set]$, $\Upsilon\bar{n}$ is the corresponding g.p.d.

E.g. $\Upsilon(1,0,2) =$ 

- We write Θ_0 for the cat whose objects are the tables of dimensions & with
$$\Theta_0(\bar{n}, \bar{m}) = [\mathbb{G}^{op}, Set](\Upsilon\bar{n}, \Upsilon\bar{m}).$$
- Equiv, Θ_0 is skeletal full subcat of $[\mathbb{G}^{op}, Set]$ containing the gpds.

- Θ_0 is our category of arities.

- In partic., have

$$\begin{array}{ccccc} n & \longmapsto & (n) & \longmapsto & y\,n \\ \mathbb{G} & \xrightarrow{\ I\ } & \Theta_0 & \hookrightarrow & [\mathbb{G}^{op}, \mathrm{Set}] \end{array}$$

fully faithful

Remark

① Given $D : \mathbb{G} \longrightarrow \mathcal{C}$, globular sums $D(\bar{n})$ are equally the weighted colimits
$$y(\bar{n}) * D.$$

② - A t.o.d. has dimension d where d is the maximum nat. no. appearing in the sequence.

Eg. $\dim(1,0,2) = 2$.

- There is just one t.o.d. of dimension 0,
$$(0) = \bullet$$

The 1-d t.o.ds are those of the form $(1,0,1,0,1,\ldots)$ ie.
$$\bullet \longrightarrow \longrightarrow \longrightarrow \ldots$$
capturing the finite ordinals.

In particular $\Delta_0 \subseteq \Theta_0$ is the full subcategory containing the tods of dimension ≤ 1.

Universal property of $\mathcal{O}_0$

Def^n) A globular theory is an id. on obs functor $J : \Theta_0 \to \mathbb{T}$ preserving globular sums.

Given a globular theory $\mathbb{T}$, the category of $\mathbb{T}$-models in $\mathcal{C}$ is the full subcategory $\mathrm{Mod}(\mathbb{T}, \mathcal{C}) \hookrightarrow [\mathbb{T}^{op}, \mathcal{C}]$ whose obs are those functors $\mathbb{T}^{op} \xrightarrow{X} \mathcal{C}$ sending globular sums to globular products - ie. sending the specified colimits to limits.

- We write $\mathrm{Mod}(\mathbb{T})$ for the category of $\mathbb{T}$-models in $\mathbf{Set}$.
- Of course, each representable $\mathbb{T}(-, \bar{n})$ is a $\mathbb{T}$-model, since reps send all colimits to limits.

- Restricting along $\mathbb{G} \xrightarrow{I} \Theta_0 \xrightarrow{J} \mathbb{T}$, with composite $D : \mathbb{G} \to \mathbb{T}$,

  induces a forgetful functor

  $$\mathrm{Mod}(\mathbb{T}) \xrightarrow{U} [\mathbb{G}^{op}, \mathrm{Set}]$$

  so each $\mathbb{T}$-model has und. glob. set

- The category <u>$\mathbb{G}$-Th of globular theories</u> has morphisms given by commutative triangles

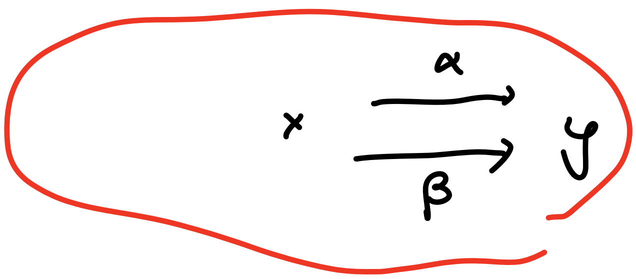
$$\begin{array}{ccc} & \Theta_0 & \\ {}^{J_{\mathbb{T}}}\swarrow & \cong & \searrow^{J_{\mathbb{S}}} \\ \mathbb{T} & \xrightarrow[K]{} & \mathbb{S} \end{array}$$

- Such automatically preserve globular sums.

Contractible globular sets

- In a globular set X, n-cells α, β are said to be parallel if
- $n = 0$ or $s_n \alpha = s_n \beta$ & $t_n \alpha = t_n \beta$.

picture



A glob. set X is <u>contractible</u>

- if $X(0)$ is non-empty &
- given parallel $\alpha, \beta \in X(n)$ $\exists\, \theta \in X(n+1)$ st $s_{n+1}(\theta) = \alpha$ & $t_{n+1}(\theta)$

ie. given 0-cells x, y $\exists\, x \to y$.

Given 1-cells $x \overset{f}{\underset{g}{\rightrightarrows}} y$

$\exists\; x \overset{f}{\underset{g}{\Downarrow}} y$ etc.

Contractible Theories

Consider a glob theory $J : \Theta_0 \longrightarrow \mathbb{T}$,
with $D = J \circ I : \mathbb{G} \longrightarrow \mathbb{T}$
$n \longmapsto (n)$

Def^n) A globular theory $\mathbb{T}$ is contractible if each globular set
$U\mathbb{T}(-, \bar{n}) = \mathbb{T}(D-, \bar{m})$
is contractible.

What does this mean in elementary terms?

- Well $\Theta_0((0), \bar{m}) \subseteq \mathbb{T}((0), \bar{m})$ so $\mathbb{T}(D0, \bar{m})$ always non-empty.
- Let us call elements of $\mathbb{T}(Dn, \bar{m})$ n-cells in $\bar{m}$.
- Two such $Dn \overset{f}{\underset{g}{\rightrightarrows}} \bar{m}$ are parallel in $\mathbb{T}(D-, \bar{m})$

$\iff$ $n = 0$ or $f \circ D\sigma_n = g \circ D\sigma_n$ &
$f \circ D\tau_n = g \circ D\tau_n$.

- Contractibility of Π says that given $\bar{m}$ a glob. sum &

$$D_n \underset{g}{\overset{f}{\rightrightarrows}} \bar{m} \text{ parallel in } \Pi$$

$$D_0 \perp \perp D_n \downarrow \qquad D(n+1) \dashrightarrow \exists h$$

Defⁿ) A Grothendieck ∞-groupoid is a model for some contractible globular theory Π.

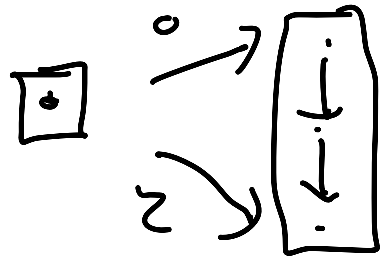
Remark) First outlined by Grothendieck in his letter to Daniel Quillen 1983 at start of Pursuing Stacks.
Exposed & made fully precise by Maltsiniotis in 2010.

So the idea is that models of such Π should have structure of a weak ∞-groupoid.

Let's investigate why?

Well in Θ_0, have maps

$(0) \overset{0}{\underset{2}{\rightrightarrows}} (1,0,1)$



so in Π

$(0) \overset{D0}{\underset{D2}{\rightrightarrows}} (1,0,1)$

$D\sigma \perp \perp D\tau$

$(1) \dashrightarrow \exists c_0^1$

which in a Π-model X gets sent to

$$\hat{G}(\cdot \to \cdot \to \cdot, X) \cong X(1,0,1) \xrightarrow{Xc_0^1} X1 \cong \hat{G}(\cdot \to \cdot, X)$$

giving composition.

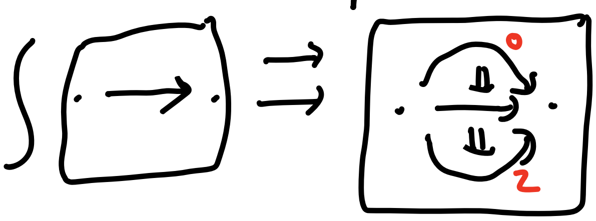
- Have $(0) \overset{D\tau}{\underset{D\sigma}{\rightrightarrows}} (1)$

  $D\sigma \perp D\tau$

  $(1) \to \exists \, in$

  corresponding to inverses map

$(1) \overset{0}{\underset{2}{\rightrightarrows}} (2,1,2) \in \Theta_0$



parallel, so

$(1) \overset{D0}{\underset{D2}{\rightrightarrows}} (2,1,2)$

$D\sigma \perp \perp D\tau$

$(2) \dashrightarrow \exists c_1^2$

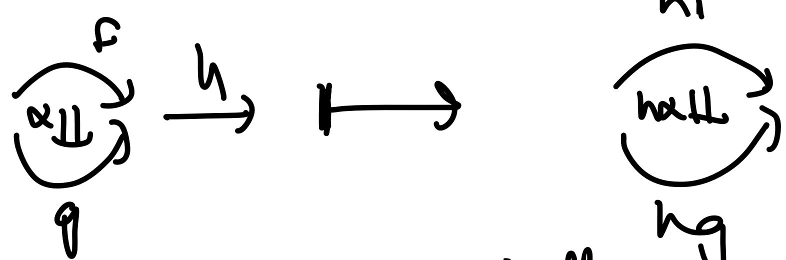
giving comp. of 2-cells along 1-cells.

Similarly get composition

$(n) \xrightarrow{c_i^n} (n,i,n)$ of n-cells along i-cell boundaries

all $i<n$.

- I call all these complexity 1 operations
- At level 2, we get whiskering ops



via $(1) \xrightarrow{c_0^1} (1,0,1) \rightrightarrows (1,0,2)$ (source 1-cell, target 1-cells)

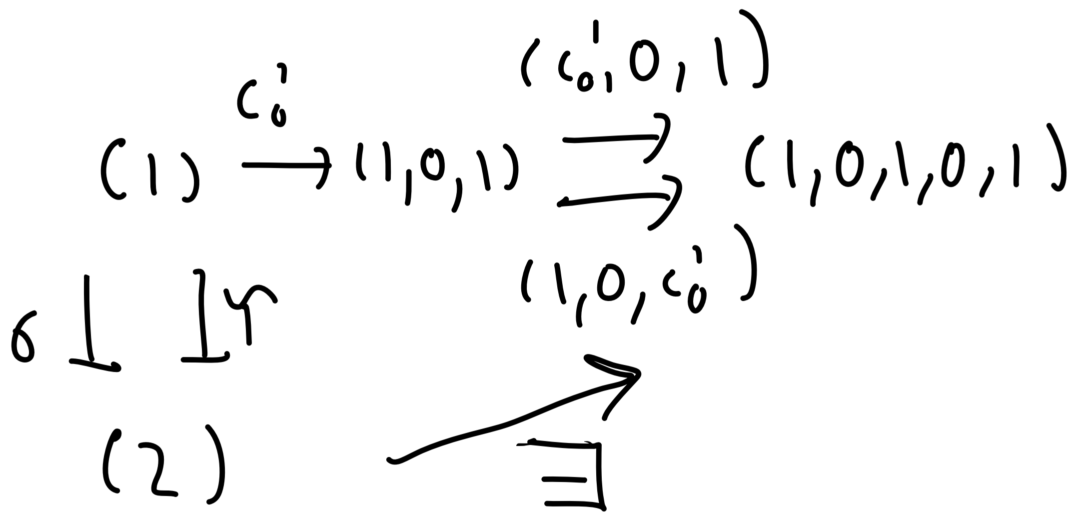
$D_0 \downarrow \downarrow D_1$ $\quad \exists$ $(2) \to (1,0,2)$

- For $\xrightarrow{f}\xrightarrow{g}\xrightarrow{h} \in X$,

we get the associator

$(hg)f \Rrightarrow h(gf)$ via

the lifting



I call them <u>complexity 2 operations</u> because they are dependent upon complexity 1 operations

Exercise Convince yourself that any contractible globular theory Π encodes the structure you would like in a weak ω-groupoid.

Examples

- We can construct the globular theories of strict ω-categories Θ, by factoring

$$\begin{array}{ccccc} \Theta_0 & \longrightarrow & \hat{G} & \xrightarrow{F} & \text{Strict } \infty\text{-cat} \\ & \searrow & & \nearrow & \\ & & \Theta & & \end{array}$$

called Joyal's cat Θ. Will re-appear later in defs of weak ∞-cat & (∞, n)-cat

- It admits a simple description, in fact, using "wreath products".

- The globular theory Θ_{gr} for strict ∞-groupoids does not admit a known simple description.

  Eg. - free ∞-groupoid on $(2) = 0 \overset{\Downarrow}{\rightrightarrows} 1$ has infinitely many 1-cells, so

  $\Theta_{gr}((1),(2))$ is infinite . . .

- Takes some work to show it is contractible (Ara - Strict ∞-groupoids are Grothendieck..)

Next time

- Fundamental ∞-groupoid of a top. space
- The homotopy hypothesis