Lecture 4 - Cellularity & the small object argument

- Consider $J \subseteq \mathrm{Mor}(\mathcal{C})$.
- The small object argument will factor

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \\ & C \arrow[ur, "h"'] & \end{tikzcd}$$

$^{\square}(J^{\square}) \supseteq \mathrm{Cell}(J) \ni g$, $h \in J^{\square}$

- I just want to explain this in the case $B = 1$: then for each $A \in \mathcal{C}$ we form

$$A \xrightarrow{f \in \mathrm{Cell}(J)} A^* \in \mathrm{Inj}(J).$$

- This is weakly universal in the sense that given

$$\begin{tikzcd} A \arrow[rr, "g"] \arrow[dr, "f"'] & & B \in \mathrm{Inj}(J) \\ & A^* \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Indeed

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "!"] \\ A^* \arrow[r, "!"] \arrow[ur, dashed, "\exists"] & 1 \end{tikzcd}$$

$^{\square}(J^{\square}) \ni f$, $! \in J^{\square}$

but we will also explain how to

make it really universal.

- I will also assume that each $j : A \longrightarrow B \in J$ has A a finitely presentable object, which implies each map f to a colim of a chain

$$\begin{tikzcd}
 & & & A \arrow[d, "\exists"] \arrow[rrr, "f"] & & & B_\omega \\
B_0 \arrow[r] & B_1 \arrow[r] & \cdots & B_n \arrow[r] & B_{n+1} \arrow[r] & \cdots \arrow[r] & B_\omega
\end{tikzcd}$$

factors through some earlier stage.

- Also that $\mathcal{C}$ is locally small & cocomplete.

The classical small object argument

(detailed explanation)

- Consider $X \in \mathcal{C}$.
- Need to find $X \longrightarrow X^*$ w' X^+ J-inj.

- Consider the solid part of

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "J \ni j"'] & X \arrow[d, "\eta_X"] \\ B \arrow[r, dashed] & X^* \end{tikzcd}.$$

Certainly we need a dotted filler, so might define X^* as universal ob. equipped with arrow $X \xrightarrow{\eta_X} X^*$ & filling function φ as below

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "J \ni j"'] & X \arrow[d, "\eta_X"] \\ B \arrow[r, "\varphi(j,f)"] & X^* \end{tikzcd}.$$

Its universal property is that given a second pair $(X \xrightarrow{k} Y, \theta)$ filling function

$\exists!$ $X^* \xrightarrow{k'} Y$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "J \ni j"'] & X \arrow[d, "\eta_X"] \arrow[ddr, "k"] & \\ B \arrow[r, "\varphi(j,f)"] & X^* \arrow[dr, "k'"'] & \\ & & \, \end{tikzcd}$$

$$\Theta(j,f) \longrightarrow Y$$

This can be captured as the pushout on right below

$$\begin{tikzcd}
A \arrow[r, "u(j,f)"] \arrow[d, "j"'] \arrow[rr, bend left, "f"] \arrow[dr, phantom, "\text{coprod inclusions}"] & \sum\limits_{\substack{(j:A\to B\in J,\\ f:A\to X)}} A \arrow[r, "\varepsilon"] \arrow[d, "\sum \varepsilon_j"] & X \arrow[d, "\eta_X"] \\
B \arrow[r, "v(j,f)"'] \arrow[rr, bend right, "\Theta(j,f)"'] & \sum\limits_{\substack{(j:A\to B\in J,\\ f:A\to X)}} B \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & X^*
\end{tikzcd}$$

so η_X is J-cellular.

Problem : given

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & X^* \\
B &
\end{tikzcd}$$

if

f factors as $A \xrightarrow{f'} X \xrightarrow{\eta_X} X^*$,

get filler

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "j"'] \arrow[d, phantom, "\perp"] & X \arrow[d, phantom, "\perp"] \arrow[d, "\eta_X"] \\ B \arrow[r, "\theta(j,f')"'] \arrow[ur, phantom, "="] & X^* \end{tikzcd} \quad A \xrightarrow{f''} X^*$$

but if $f : A \longrightarrow X^*$ does not factor through η_X, perhaps no filler - X^* not J-injective

So we repeat :

setting $X_0 = X$;

$X_{n+1} = (X_n)^*$

$$\begin{tikzcd} X = X_0 \arrow[r, "\eta_{X_0}"] \arrow[rrrr, bend right, "\eta_{0,\omega}"'] & X_1 \arrow[r, "\eta_{X_1}"] & X_2 \arrow[r, dotted] & X_n \arrow[r, dotted] & X_\omega \end{tikzcd}$$

& X_ω the colimit of the chain.

- Then η is J-cellular by construction.

- Consider

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & X_\omega \\ B & \end{tikzcd}.$$

Then

$$\begin{tikzcd} & X_n \arrow[dr, "\eta_{n,\omega}"] & \\ A \arrow[ur, "\exists f'"] \arrow[rr, "f"] & & X_\omega \end{tikzcd}$$

as A finitely pres.,

so

$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[d, "j"'] \arrow[rrd, bend left, "f"] \arrow[dr, phantom, "="] & X_n \arrow[d, "\eta_{n,n+1}"] \arrow[dr, "\eta_{n,\omega}"] \arrow[d, phantom, shift left=6, "="] & \\ B \arrow[r, "\varphi^*(j,f')"'] & X_{n+1} \arrow[r, "\eta_{n+1,\omega}"'] & X_\omega \end{tikzcd}$$

gives filler.

Thus $X_\omega \in \mathrm{Inj}(J)$ & this completes the usual small object argument.

- The small object argument has some odd features.
- At stage 1, we add a canonical lifting

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[r, "n_{0,1}"] & X_1 \\ B \arrow[urr, "\varphi^1(j,f)"'] & & \end{tikzcd} \qquad J \ni j$$

& then at stage 2 a lifting

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[r, "n_{0,1}"] & X_1 \arrow[r, "n_{1,2}"] & X_2 \\ B \arrow[urrr, "\varphi^2(j, n_{0,1} \circ f)"'] & & & \end{tikzcd} \qquad J \ni j$$

So now two liftings

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "J \ni j"'] & X \arrow[r, "\eta_{0,1}"] & X_1 \arrow[r, "\eta_{1,2}"] & X_2 \\ B \arrow[urr, "\varphi^1(j,f)", "?"'] \arrow[urrr, "\varphi^2(j,\eta_{0,1}\circ f)"'] & & & \end{tikzcd}$$

For the same problem which need not be the same !!

In particular this means that in X_ω we have added many fillers for the same lifting problem - this prevents X_ω from having canonical liftings / a universal property.

There are two solutions

① This involves forming coequalisers

identifying the liftings →
the algebraic small object argument.

(2) A simpler solution is what I'll call the efficient small object argument:
it is simpler than (1), but in the cases we are interested in, they coincide.

The efficient small object argument

This starts exactly as before:

$X_0 = X$,

$X_0 \xrightarrow{\eta_{0,1}} X_1$ is $X \xrightarrow{\eta_X} X^*$.

Suppose we have $X_n \xrightarrow{\eta_{n,n+1}} X_{n+1}$.

Call $A \xrightarrow{f} X_{n+1}$ irredundant if it does not factor through $X_n \xrightarrow{\pi_{n,n+1}} X_{n+1}$.

- We define X_{n+2} as the universal object equipped with a filler

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "J \ni j"'] & X_{n+1} \arrow[d, "\pi_{n+1,n+2}"] \\ B \arrow[r, "\varphi(j,f)"] & X_{n+2} \end{tikzcd}$$

for each pair (j,f) with f irredundant.

- Then $X_{n+1} \longrightarrow X_{n+2}$ is again a pushout of a coproduct of maps in J just as before, only we only consider irredundant f.
- Now take the colim of the chain

$$X \to X_1 \to \cdots \to X_n \longrightarrow X_e$$

as before -

an easy adaptation of the prev. proof shows X_e is J-injective

& $X \longrightarrow X_e$ is J-cellular by construction.

In fact, under further assumptions X_e is the free algebraic injective.

Algebraic injectivity

A J-algebraic injective (X, φ) is an object $X \in \mathcal{C}$ + a lifting function

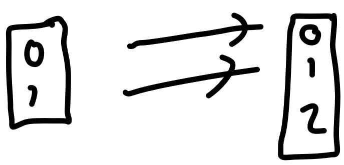
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j \in J"'] & X \\ B \arrow[ur, "\varphi(j,f)"'] & \end{tikzcd}$$

A morphism $g : (X, \varphi) \longrightarrow (Y, \theta)$ of algebraic injectives is $g : X \to Y$ such that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j \in J"'] & X \arrow[r, "g"] & Y \\ B \arrow[ur, "\varphi(j,f)"'] \arrow[urr, bend right, "\theta(j,gf)"'] & & \end{tikzcd}$$

These form a cat J-Alg, which comes with a forgetful functor
$U : \text{J-Alg} \longrightarrow \mathcal{C}$.

Example

In Set, consider
$$j : 2 \hookrightarrow 3 \quad \& \quad J = \{j\}.$$


A J-alg. injective (X, φ) gives
$$\begin{tikzcd} 2 \arrow[r, "{(a,b)}"] \arrow[d, "j"'] & X \\ 3 \arrow[ur, "{(a,b,m(a,b))}"'] & \end{tikzcd}$$
ie. a function $X^2 \xrightarrow{m} X$.

Thus J-Alg is the category of magmas.

More generally, any category

Ω-Alg for Ω a signature in universal algebra is of form
J-Alg for J a set of monos between finite sets

& let us compare the efficient & classical small object arguments

Let $X \in \mathbf{Set}$. In both cases, X_1 is universally equipped with fillers

$$\begin{tikzcd} 2 \arrow[r, "{(a,b)}"] \arrow[d, hook] & X \arrow[d, hook] \\ 3 \arrow[r, "{(a,b,m(a,b))}"'] & X_1 \end{tikzcd} \quad \text{so} \quad X_1 = X \cup \{ m(a,b) : a,b \in X \}$$

- At stage 2, $2 \xrightarrow{(u,v)} X_1$ is irredundant just when at least one of u,v does not belong to X.
  i.e. one is of form $m(a,b)$.

- So in efficient soa, we have fillers like

$$\begin{tikzcd}
2 \arrow[r, "{(a,m(b,c))}"] \arrow[d] & X_1 \arrow[d] \\
3 \arrow[r, "{(a,m(b,c),m(a,m(b,c)))}"'] & X_2
\end{tikzcd}$$

X_0 - a,b,c

X_1 - $a,b,m(a,b),\ldots$

X_2 - $m(a,m(b,c)),\ m(m(a,b),m(c,d))\ldots$

$X_e = \bigcup X_n$ is free magma on X!

- Classical soa produces $m(a,b)$, but also $m'(a,b)$ at stage 2 - useless..

This suggest efficient soa produces free algebraic injectives & , under some assumptions, it does.

Remark: One possible advantage of classical soa is due to its simplicity - simply iterating a functor $X \mapsto X^*$.

I have not checked functoriality of the efficient soa. Of course, under the assumptions below, it will be functorial - even a monad.

$\boxed{\text{Theorem}}$ Let J be a set of monos with f.p. domain & $\mathcal{C}$ cocomplete. Suppose J-cellular maps are mono.

Then $X \xrightarrow{\eta_{0,\omega}} X_e$ is the free J-algebraic injective on X.

Remark) In Set or $[\mathcal{C}, \mathrm{Set}]$ this holds.

Main point is that pushouts of mono are mono. In Set, each mono of form $X \overset{i}{\rightarrowtail} X+Y$ &

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i"'] & Z \arrow[d] \\ X+Y \arrow[r] & Z+Y \end{tikzcd}$$

mono - mono

.

~~Proof~~ First, we give X_e structure of object of J-Alg.

Given $f : A \longrightarrow X_e$, let's define the complexity of f as the least natural number st

f factors as
$$\begin{tikzcd} A \arrow[r, "f'"] \arrow[dr, "f"'] & X_n \arrow[d, tail, "\eta_{n,\omega}"] \\ & X_e \end{tikzcd}.$$

(Such an n exists as A fin. pres.)

By assumption, $\eta_{n,\omega}$ is mono – hence the factorisation f' is unique.

Given
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "J \ni j"'] & X_e \\ B \end{tikzcd}$$
where f has complexity n, we define

$$\begin{tikzcd}[column sep=large, row sep=large] A \arrow[r, "f'"] \arrow[d, "j"'] \arrow[dr, phantom, "="] \arrow[rr, bend left=40, "f"] & X_n \arrow[d, "\eta_{n,n+1}"] \arrow[dr, "\eta_{n,\omega}"] \arrow[dr, phantom, "=", near start, shift left=3] \arrow[dr, phantom, "=", shift right=3] & \\ B \arrow[r, "\varphi^{n+1}(j,f')"'] \arrow[rr, bend right=50, "\varphi(j,f)"'] & X_{n+1} \arrow[r, "\eta_{n+1,\omega}"'] \arrow[r, phantom, ":=", shift right=6] & X_e \end{tikzcd}.$$

Since n & f' are uniquely determined by f, $\varphi(j,f)$ is well defined.

Consider $(Y,\theta) \in J\text{-Alg}$ & $g : X \to Y \in \mathcal{C}$.

We must show $\exists!\ (X_e, \varphi) \xrightarrow{\bar{g}} (Y,\theta)$ such that

$$\begin{tikzcd}
 & X_e \arrow[dr, "\bar{g}"] & \\
X \arrow[ur, "n_{0,\omega}"] \arrow[rr, "g"'] & {=} & Y
\end{tikzcd}$$

To give such an extension is to give a system

$$\begin{tikzcd}
X_0 \arrow[d, "n_{0,1}"'] \arrow[drr, "g_0"] & & \\
X_1 \arrow[d] \arrow[rr, "g_1"] & & Y \\
\vdots & & \\
X_n \arrow[d, "n_{n,n+1}"'] \arrow[uurr, "g_n"] & & \\
X_{n+1} \arrow[uuurr, "g_{n+1}"'] & & \\
\vdots & &
\end{tikzcd}$$

with $g_0 = g$

& to say that g preserves fillers of complexity n is to say that

$\forall\, A \xrightarrow{f} X_n$ non-redundant

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "="] & X_n \arrow[d, "\eta_{n,n+1}"] \arrow[dr, "\eta_{n,\omega}"] \arrow[d, phantom, shift left=6ex, "="] & & \\
B \arrow[r, "\varphi^{n+1}(j,f)"'] \arrow[rrr, bend right=40, "\theta(j,\ \bar{g}.\eta_{n,\omega}.f)"'] & X_{n+1} \arrow[r, "\eta_{n+1,\omega}"'] \arrow[rr, phantom, shift right=8ex, "="] & X_\omega \arrow[r, "\bar{g}"'] & Y
\end{tikzcd}$$

ie

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "="] & X_n \arrow[d, "\eta_{n,n+1}"] \arrow[dr, "g_n"] \arrow[d, phantom, shift left=6ex, "="] & \\
B \arrow[r, "\varphi^{n+1}(j,f)"'] \arrow[rr, bend right=40, "\theta(j,\ g_n f)"'] & X_{n+1} \arrow[r, "g_{n+1}"] \arrow[r, phantom, shift right=8ex, "="] & Y
\end{tikzcd}$$

- But by the universal property of X_{n+1}, given g_n $\exists!$ g_{n+1} extending g_n & having this property.
- Since $g_0 = g$, we hence obtain a unique extension $\bar{g}$ preserving fillers for morphisms of complexity n $\forall n$ – that is, preserving all fillers. $\square$

Closest thing to a ref for this stuff:

- JB – Iterated algebraic injectivity & the Faithfulness conjecture.

Builds on

- Nikolaus – Algebraic models for higher cats.

- For more on algebraic small obs. arg,

Garner – Understanding the small obs. argument