Lecture 3

Last time I mentioned

Proposition

Consider $D : \mathbb{G} \longrightarrow \mathcal{C}$ where $\mathcal{C}$ has D-glob. sums.

Then

$$\begin{tikzcd} \Theta_0 \arrow[dr, "\mathrm{lan}_I D"] & \\ \mathbb{G} \arrow[u, "I"] \arrow[r, "D"'] \arrow[ur, phantom, "\cong"] & \mathcal{C} \end{tikzcd}$$

exists and sends $\bar{n} \longmapsto D(\bar{n})$.

It preserves I-globular sums, & is the unique up to iso extension w' this property.

$\boxed{\text{Proof}}$ The following proof is probably too formal!

- From enriched cat. theory, the pwise left Kan ext. will exist $\iff$ the weighted colims $\Theta_0(I-, \bar{m}) * D$ exist $\forall \bar{m}$, where $\Theta_0(I-, \bar{m}) : \mathbb{G}^{op} \to \mathrm{Set}$.
- It is then defined as the ess. unique functor sending the canonical cocone
$$\Theta_0(I-, \bar{m}) \xrightarrow{1} \Theta_0(I-, \bar{m})$$
to a weighted colimit in $\mathcal{C}$.
But $\Theta_0(I-, \bar{m}) \cong$ (applying $J : \Theta_0 \to [\mathbb{G}^{op}, \mathrm{Set}]$)
$[\mathbb{G}^{op}, \mathrm{Set}](y-, y(\bar{m})) \cong$ (by Yoneda)
$y(\bar{m})$ & then
$$y(\bar{m}) \cong \Theta_0(I-, \bar{m}) \text{ is the cocone exhib.}$$
$$\bar{m} \text{ as } y(\bar{m}) * I.$$
- But since weighted cols of the $y(\bar{m})$ are precisely the globular sums, the result follows. $\square$

The globular theory of topological spaces

- Firstly, we will construct the globular theory of spaces & show it is contractible.
- Firstly, we construct

$$D : \mathbb{G} \longrightarrow \mathrm{Top}$$
$$n \longmapsto D^n = \{ x \in \mathbb{R}^n \mid |x| \leq 1 \}$$

$\shortparallel$

$\bullet$, -1 —— 1 , 

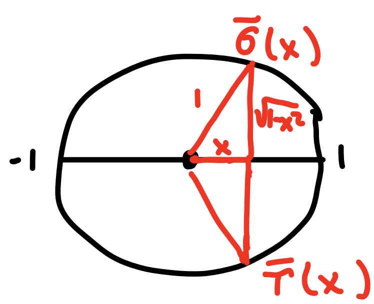
Then

$$n \underset{\tau}{\overset{\sigma}{\rightrightarrows}} n+1 \longmapsto D^n \underset{\bar\tau}{\overset{\bar\sigma}{\rightrightarrows}} D^{n+1}$$

are the north/south hemisphere maps

$$\bar\sigma(x) = (x, \sqrt{1-x^2}), \quad \bar\tau(x) = (x, -\sqrt{1-x^2})$$

eg.



- Then we obtain the Kan extension

$$\begin{aligned} D : \Theta_0 &\longrightarrow Top \text{ sending} \\ \bar{n} &\longmapsto D(\bar{n}). \end{aligned}$$

We obtain the <u>globular theory Π_{Top}</u> of spaces by factoring

$$\begin{tikzcd} \Theta_0 \arrow[r, "I"] \arrow[rr, bend right, "D"'] & \Pi_{Top} \arrow[r, "J"] & Top \end{tikzcd}$$

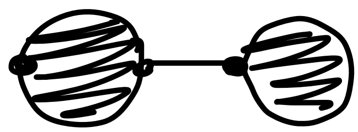
as id. on obs / FF.

<u>Proposition</u>

Π_{Top} is contractible.

<u>Proof</u>

- The spaces $D(\bar{n})$ are things like

 etc.

$D(\bar{n})$ is always <u>contractible as a space</u>, ie $D(\bar{n}) \to 1$ a homotopy equivalence.

We must show that given

$$
\begin{tikzcd}
D^n \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, phantom, "\bar{\sigma}\ \perp\ \perp\ \bar{\tau}"] & D(\bar{m}) \\
D^{n+1} \arrow[ur, "\exists h"'] &
\end{tikzcd}
$$

parallel

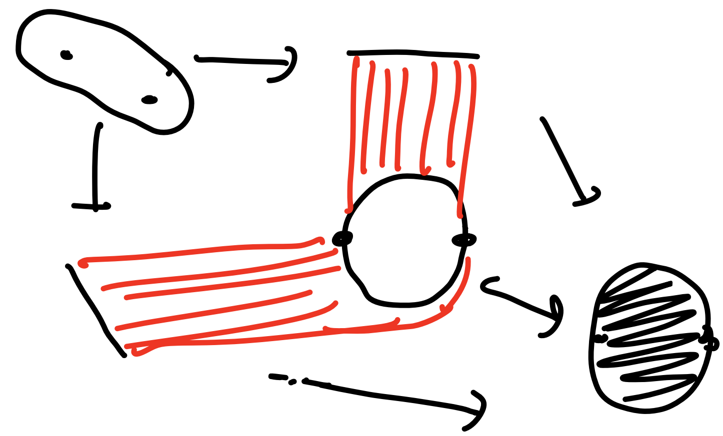
Now to say that f & g are parallel is, by definition, to say that

$$
\begin{tikzcd}
D^{n-1} + D^{n-1} \arrow[r, "\bar{\sigma}+\bar{\tau}"] \arrow[d, "\bar{\sigma}+\bar{\tau}"'] & D^n \arrow[d, "f"] \arrow[dl, phantom, "="] \\
D^n \arrow[r, "g"'] & D(\bar{m})
\end{tikzcd}
$$

Note that $S^n = \{ x : |x| = 1 \}$ is the pushout

$$\begin{tikzcd}
D^{n-1}+D^{n-1} \arrow[r, "\bar{\sigma}+\bar{\tau}"] \arrow[d, "\bar{\sigma}+\bar{\tau}"'] & D^n \arrow[d, hook, "i"] \arrow[ddr, "\bar{\sigma}"] & \\
D^n \arrow[r, hook, "j"'] \arrow[drr, "\bar{\tau}"'] & S^n \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, hook, "k"] & \\
 & & D^{n+1}
\end{tikzcd}$$

eg.



So using <u>pushout property</u>

obtain

$$\begin{tikzcd}
D^n \arrow[d, shift right, hook, "i"'] \arrow[d, shift left, hook, "j"] \arrow[dr, bend left, "f"] \arrow[dr, "g"'] & \\
S^n \arrow[r, "\langle f, g \rangle"] \arrow[d, hook, "k"'] & D(\bar{n}) \arrow[d, "!"] \\
D^{n+1} \arrow[r, "!"'] & 1
\end{tikzcd}$$

But now k is a <u>cofibration</u> in the Quillen model structure on Top, & $D(\bar{n}) \to 1$ a <u>triv. fibration</u>

So

$$\begin{tikzcd}
D^n \arrow[d, shift right, "i"'] \arrow[d, shift left, "j"] \arrow[drr, bend left, "f"] \arrow[drr, bend right=10, "g"'] & & \\
S^n \arrow[rr, "\langle f, g \rangle"] \arrow[d, hook, "\text{cof}\quad k"'] & & D(\bar{n}) \arrow[d, "!\quad \text{triv fib}"] \\
D^{n+1} \arrow[rr, "!"'] \arrow[urr, "\exists"] & & 1
\end{tikzcd}$$

proving contractibility. $\square$

Fundamental ∞-groupoid of a space

Now $\Theta_0 \xrightarrow{I} \Pi_{Top} \xrightarrow{J} Top$ induces

$$Top \xrightarrow{N_J} [\Pi_{Top}^{op}, Set]$$

& $N_J X = Top(J-, X) = Top(-, X) \circ J$

sends glob. sums to globular products

since J pres. glob. sums, $Top(-,X)$ sends colims to lims.

Hence each $N_J X$ is a model of Π_{Top} – we obtain a factorisation

$$Top \xrightarrow{N_J} Mod(\Pi_{Top}) \hookrightarrow [\Pi_{Top}^{op}, Set]$$

$N_J X$ is the fundamental ∞-groupoid of X.

- What does it look like?

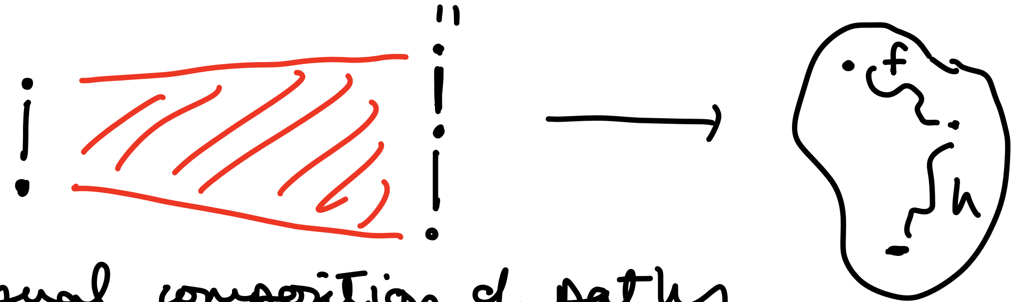
$N_J X(\bar{n}) = Top(D(\bar{n}), X)$.

- In partic, underlying globular set has

$$N_J X(n) = \mathrm{Top}(D^n, X).$$

- composition

$$D(1) \xrightarrow{c_0^1} D(1,0,1) \xrightarrow{\langle f, g \rangle} X$$



is usual composition of paths (up to htpy).

So it really does look like the fundamental ∞-groupoid should look.

Weak equivalences of ∞-groupoids

What is a weak equiv. of top spaces?
Usually defined as
$f: X \longrightarrow Y$ inducing
- $\pi_0 X \longrightarrow \pi_0 Y$ bij^n on path comps
- bij $\pi_n(X,x) \longrightarrow \pi_n(Y, f(x))$ on homotopy groups $\forall n, x \in X$.

Remark: group structure not actually relevant to this def...

- Similarly, can define homotopy groups of ∞-groupoids.
- If X is model of contractible theory Π, let $\Pi_0 X$ = set of connected components of underlying graph of X.

Also, for all n
can form groupoid $n\text{-}Gr(X)$ whose
- objects are n-cells α, β
- morphisms: equiv. classes of $(n+1)$-cells

e-rel $\quad \alpha \xrightarrow{\varphi} \beta$ where .

where $\varphi \sim \varphi'$ if $\exists$ $(n+2)$-cell $\varphi \to \varphi'$.

Exercise

- Each $n\text{-}Gr(X)$ is a groupoid, using the composition operations in Π

eg $\quad (n) \xrightarrow{c^n_{n-1}} (n, n-1, n) \in \Pi$

& this is independent of the choice of composition operation.

- Now given $x \in X_0$, obtain $k_n(x) \in X_n$, where $k_n(x)$ is the "identity n-cell" on x, arising from the contractibility of Π

Then $\Pi_{n+1}(X, x) := n\text{-}Gr(X)(x, x)$

$k_1 x$

Eg $\Pi(X,x)$ has $\quad x$

e.g. $\pi_1(X,x)$ has $\wedge$ $[\alpha] \Vert$ X

e-class $\qquad$ $k_1 X$

This gives a functor

$$\pi_0 : \mathrm{Mod}(\Pi) \longrightarrow \mathrm{Set} \quad \&$$

$$\pi_n : \mathrm{Mod}(\Pi)_* \longrightarrow \mathrm{Grp}$$

pointed models
& point preserving maps

taking the homotopy groups of a Π-model.

- In fact, we have

$$\begin{tikzcd}
(\mathrm{Top}, *) \arrow[rr, "N_J"] \arrow[dr, "\pi_n"'] \arrow[rr, phantom, shift right=6, "="] & & (\mathrm{Mod}(\Pi_{\mathrm{Top}}), *) \arrow[dl, "\pi_n"] \\
& \mathrm{Grp} &
\end{tikzcd}$$

Def^n) A morphism $f : X \to Y$ of Π-models is a weak equiv.

if - $\Pi_0 X \longrightarrow \Pi_0 Y$ bij^n on path comps

- bij $\Pi_n(X, x) \longrightarrow \Pi_n(Y, f(x))$

$\forall n, x \in X_0$.

Corollary

$$\mathrm{Top} \xrightarrow{N_J} \mathrm{Mod}(\Pi_{\mathrm{Top}})$$

preserves & reflects weak equivalences.

Proof

Since the triangle commutes.

The homotopy hypothesis?

- A good guess for what the homotopy hypothesis should say is that the above functor induces an equiv. of cats when we invert the weak equivs on either side.
- Close, but not quite Grothendieck's formulation anyway ...

Weakness (aka cellularity)

So far, we talked about contractibility which leads to ∞-groupoids.

But suppose we want to single out the theories of weak ∞-groupoids?

How to do it?

Grothendieck's formulation was (essentially):

Π is a coherator if it is contractible & Π is the colimit of a chain

$$\Theta_0 = \Pi_0 \to \Pi_1 \to \cdots \to \Pi_n \xrightarrow{J_n} \Pi_{n+1} \to \cdots \to \Pi \in \mathfrak{G}\text{-Th}$$

where · there is a set P_n of parallel pairs $(\ell) \overset{u}{\underset{v}{\rightrightarrows}} \overline{m}$ in Π_n

such that Π_{n+1} is obtained by freely adding a lifting

$$\begin{tikzcd} (\ell) \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] \arrow[d, shift left] \arrow[d, shift right] & \overline{m} \\ (\ell+1) \arrow[ur, "\varphi_{u,v} \in \Pi_{n+1}"'] & \end{tikzcd}$$

for each $(u,v) \in P_n$.

What is the idea?

E.g. if we take all parallel pairs at each stage, then:

in Π_1, get liftings such as

$$\begin{tikzcd}
(0) \arrow[d, shift right] \arrow[d, shift left] \arrow[rr, shift left, "0"] \arrow[rr, shift right, "2"'] & & \\
(1) \arrow[r] & \arrow[r] & (1,0,1)
\end{tikzcd}$$

& then

$0_0^1 \in \Pi_1$

$$\begin{tikzcd}
(1) \arrow[r, "0_0^1"] & (1,0,1) \arrow[r, shift left, "{(0_0^1, 1)}"] \arrow[r, shift right, "{(1, 0_0^1)}"'] & (1,0,1,0,1)
\end{tikzcd}$$

is a parallel pair in Π_1 but not equal since we added the liftings freely.

Thus we don't force any equations like associativity.

More conceptually, it corresponds to cellularity, which is just another part of the same story as contractibility.

Cellularity & contractibility

- let J be a class of morphisms in a cat $\mathcal{C}$.

Write $j \perp f$ if
$$\begin{tikzcd} a \arrow[r, "r"] \arrow[d, "j"'] & c \arrow[d, "f"] \\ b \arrow[r, "s"'] \arrow[ur, "\exists"] & d \end{tikzcd}$$

& $J^{\square} = \{ f : j \perp f \ \forall j \in J \}$

& ${}^{\square}J = \{ f : f \perp j \ \forall j \in J \}$

Then $J \subseteq {}^{\square}(J^{\square})$.

- We say $({}^{\square}(J^{\square}), J^{\square})$ is a weak factorisation system if each arrow factors as a

J-cofibration (in ${}^{\square}(J^{\square})$)

followed by a

J-contraction (in $J^{\square}$)

- These can be generated using Quillen's **small object argument.**

if. eg J is a set & $\mathcal{C}$ is nice,

eg. locally presentable,

This produces factorisations

$$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "g"'] & & b \\ & c \arrow[ur, "h"'] & \end{tikzcd}$$

$${}^{\square}(J^{\square}) \supseteq \underline{J\text{-cell}} \ni g \qquad h \in J^{\square}$$

J-cell is the closure of $J \subseteq \mathrm{Mor}(\mathcal{C})$

under
- pushouts of maps in J
- coproducts
- transfinite composition

- Then the J-cofibrations are the retracts of the J-cellular maps.

- We say an object X is
  - J-cellular if $\emptyset \to X$ is a J-cellular map
  - J-contractible (or J-injective) if
$$X \to 1 \in J^{\square}.$$

i.e.
$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "J \ni j"'] & X \\ b \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

Next time, I will describe a set of maps Θ in the category Θ_0-Th such that:

- The (J-cellular, J-contractible)-objects are exactly Grothendieck's coherators, i.e. the globular theories for weak ∞-groupoids.