Kan complexes & quasicategories

- Kan complexes $\equiv$ simplicial ∞-groupoids
- Quasicats $\equiv$ simplicial $(\infty,1)$-categories,
  meaning all n-morphisms above dimension 1 are invertible.
  Also called <u>∞-categories</u>.
- So Kan complexes $\equiv$ simplicial $(\infty,0)$-cats
  quasicats $\equiv$ - - - - $(\infty,1)$-cats
  ? $\equiv$ - - -

- Both are defined as <u>simplicial sets with properties</u>.

- Here Δ = simplicial cat of non-empty finite ordinals
  $[n] = \{0 < \ldots < n\}$ for $n \geq 0$
  & order preserving maps.

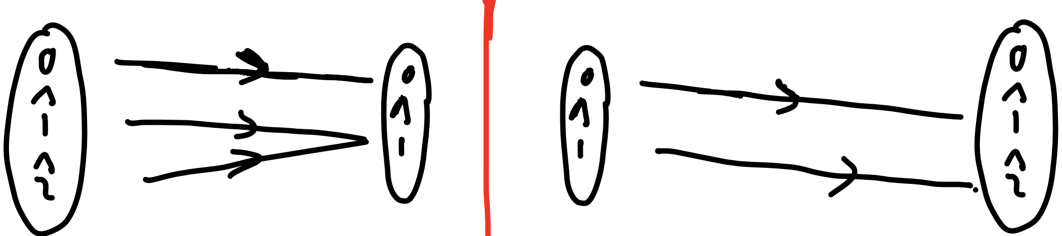
The factorisation system

Δ has a strict fact. system (Surj / Inj).

- The <u>surjections</u> are generated by the maps $\sigma_i^n : [n+1] \twoheadrightarrow [n]$ for $0 \leq i \leq n$ taking value @ i twice;
- The <u>injections</u> are generated by the maps $\delta_i^n : [n-1] \rightarrowtail [n]$ for $0 \leq i \leq n$ which omit i.

$\sigma_1^1 : [2] \to [1]$	$\delta_0^2 : [1] \longrightarrow [2]$

E.g



- Δ is freely generated by these maps subject to the <u>simplicial identities</u>:
  - comp. of two δ's
  - comp. of two σ's
  - Rewrite $\sigma . \delta$

  which I won't use.

Simplicial sets

- Write sSet for $[\Delta^{op}, \mathrm{Set}]$, the cat. of simplicial sets.
- A simplicial set X comes equipped with

$$X_n \xrightarrow{d_n^i} X_{n-1}, \quad X_n \xrightarrow{s_n^i} X_{n+1}$$

for $0 \le i \le n$.

- Elements of X_n called n-simplices.

- In low dimensions, have

$$\begin{tikzcd}
\cdots X_2 \arrow[r, shift left=4] \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] \arrow[r, shift right=4] & X_1 \arrow[r, shift left=3, "d_1^0"] \arrow[r, leftarrow, "s_1^0"] \arrow[r, shift right=3, "d_1^1"'] & X_0
\end{tikzcd}$$

$$\begin{array}{ccc}
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\
\begin{tikzcd}[sep=small] & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "\alpha"] & & c \end{tikzcd} & a \xrightarrow[F]{} b & a, b
\end{array}$$

No joy drawing beyond X_3!

Yoneda embedding

$$\begin{array}{rcl} y : \Delta & \longrightarrow & sSet \\ n & \longmapsto & \Delta^n, \text{ the } n\text{-simplex.} \end{array}$$

Δ^0 $\quad$ | $\quad$ $\bullet$

Δ^1 $\quad$ | $\quad$ $0 \longrightarrow 1$

Δ^2 $\quad$ |

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & \arrow[u, phantom, "\Uparrow"] & 2 \end{tikzcd}$$

+ degenerate higher cells

Just write

$$n \xrightarrow{f} m \quad \longmapsto \quad \Delta^n \xrightarrow{\ f\ } \Delta^m$$

Two embeddings

- We have the full inclusion

$$J : \Delta \longrightarrow \mathrm{Cat} .$$

indeed, we saw it earlier as the inclusion of the graphical theory of categories.

- Also $\Delta \xrightarrow{K} \mathrm{Top}$

$$[n] \longmapsto |\Delta^n|$$

$$= \{(x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \mid \Sigma x_i \leq 1\}$$

views $[n]$ as standard n-simplex $|\Delta^n|$,

$$\& \; f : [n] \to [m] \longmapsto |\Delta^n| \xrightarrow{|f|} |\Delta^m|$$

$$|f|(x_1, \dots, x_n) = (y_0, \dots, y_m)$$

where $y_j = \sum_{i \in f^{-1}(j)} x_i$

These induce

$$(*) \qquad \begin{tikzcd} & \mathrm{SSet} & \\ \mathrm{Top} \arrow[ur, "N_K"] & \Delta \arrow[l, "K"'] \arrow[u, "y"] \arrow[r, "J"'] & \mathrm{Cat} \arrow[ul, "N_J"'] \end{tikzcd}$$

- Here $N_K X := \mathrm{Sing}(X)$ is the <u>singular complex of X</u>, with value $\mathrm{Sing}X_n = \mathrm{Top}(|\Delta^n|, X)$ the set of <u>n-simplices in X.</u>

- $N_J C := NC$ is the <u>nerve of C</u>:
  $NC_n = \mathrm{Cat}([n], C)$
  $= \{$composable sequences of length n $\}$

⊛

$$
\begin{tikzcd}
 & \mathrm{SSet} \arrow[dl, shift right, "|-|"'] \arrow[dr, shift left, "h"] & \\
\mathrm{Top} \arrow[ur, shift right, "N_K"'] \arrow[ur, phantom, "\perp"] & \Delta \arrow[l, "K"'] \arrow[u, "y"] \arrow[r, "J"] & \mathrm{Cat} \arrow[ul, shift left, "N_J"] \arrow[ul, phantom, "\perp"]
\end{tikzcd}
$$

- Since SSet is the free cocompletion of Δ both of these have left adjoints, as depicted.

- $|X|$ is the geometric realisation of X, $|X| = \int^{[n] \in \Delta} X_n . |\Delta^n|$

- hX is category w/ obs X_0, arrows generated by 1-simplices $x \xrightarrow{f} y \in X_1$ subject to relations:
  - $x \xrightarrow{s_i(x)} x = id_x$
  - $$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\alpha"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & {} & z \end{tikzcd} \in X_2 \implies g \circ f = h \in hX.$$

$\ast$

$$
\begin{tikzcd}
 & \mathrm{SSet} \arrow[dl, shift right, "|-|"'] \arrow[dl, phantom, "\perp"] \arrow[dr, shift left, "h"] \arrow[dr, phantom, "\perp"] & \\
\mathrm{Top} \arrow[ur, shift right, "\mathrm{Sing}"'] & \Delta \arrow[l, "K"] \arrow[u, "y"] \arrow[r, "J"'] & \mathrm{Cat} \arrow[ul, shift left, "N"]
\end{tikzcd}
$$

- Certainly topological spaces give rise to ∞-groupoids (as we know) so Sing$X$ should be a simplicial ∞-groupoid.
- Likewise NC should be an ∞-category.
- As such,

∞-cats should be a common generalisation of Sing$X$ & NC – so lets explore some of their common properties.

Images, boundaries & horns

- Consider a set of maps $\{f_i : A_i \to B : i \in I\}$ in SSet.
- Factoring

$$\begin{tikzcd}
\sum\limits_{i\in I} A_i \arrow[rr, "\Sigma f_i"] \arrow[dr, two heads] & & B \\
& \mathrm{im}(f_I) \arrow[ur, hook] &
\end{tikzcd}$$

as (pointwise surj / pointwise mono) produces the image of the f_i –

- explicitly $\mathrm{im}(f_I)[n]$ consists of those $x \in B[n]$ which are in the image of some $(f_i)_n : A_i(n) \to B(n)$

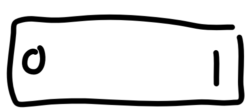
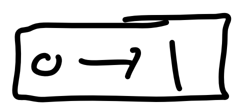
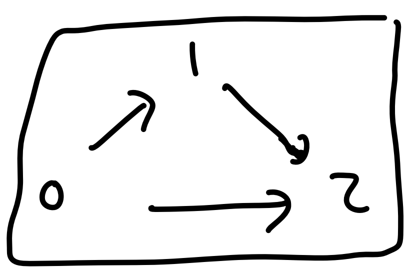
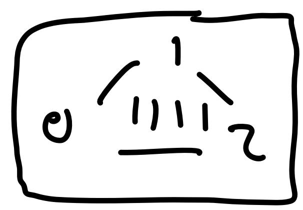
Examples

(1) The joint image of

$$\{\sigma_i^n : \Delta^{n-1} \longrightarrow \Delta^n : 0 \le i \le n\}$$

produces $\partial\Delta^n \hookrightarrow \Delta^n$,

the boundary of the n-simplex.

$n=0$	$\emptyset \hookrightarrow \{\bullet\}$
$n=1$	 $\hookrightarrow$ 
$n=2$	 $\hookrightarrow$ 

② The joint image of the maps $\sigma_i^n : \Delta^{n-1} \to \Delta^n$ for $i \neq k$, $n \geq 1$ produces the <u>k'th horn inclusion</u>

$$\Lambda^n_k \hookrightarrow \Delta^n$$

It is called an

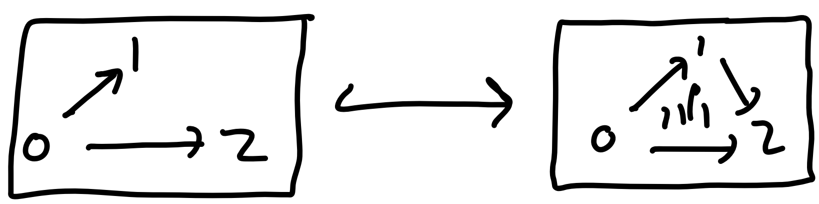
- <u>outer horn</u> if <u>$k = 0$ or n</u>
- <u>inner horn</u> if <u>$0 < k < n$</u>.

$n = 1$ $[0] \hookrightarrow [0 \to 1]$, $[1] \hookrightarrow [0 \to 1]$

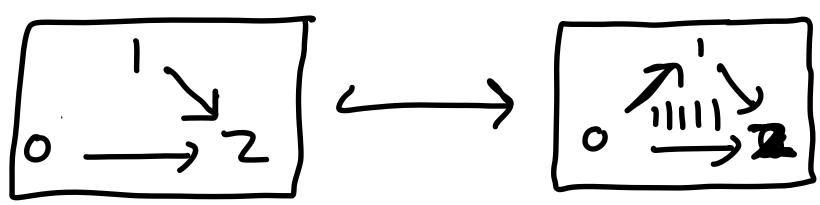
both <u>outer</u>.

$n = 2$

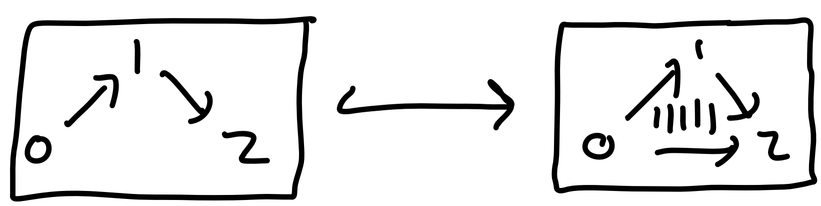
<u>Outer</u>



<u>Outer</u>



<u>Inner</u>



③ For $[1] \xrightarrow{\theta_{i,i+1}} [n]$ & $n \geq 1$

$0 < 1 \longmapsto i < i+1$

the joint image of the

$$\theta_{i,i+1} \quad \Delta^1 \longrightarrow \Delta^n$$

produces the spine of Δ^n:

$$Sp\,\Delta^n \hookrightarrow \Delta^n,$$

which just looks like

$\boxed{0 \to 1 \to 2 \to 3 \to \cdots \to n}$ with no

1-simplices $n \to m$ unless $m = n+1$.

Note: The inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ & $\Lambda^n_k \hookrightarrow \Delta^n$ completely determine the classical model structure on sSet: they are the gen. cofibrations & trivial cofibrations.

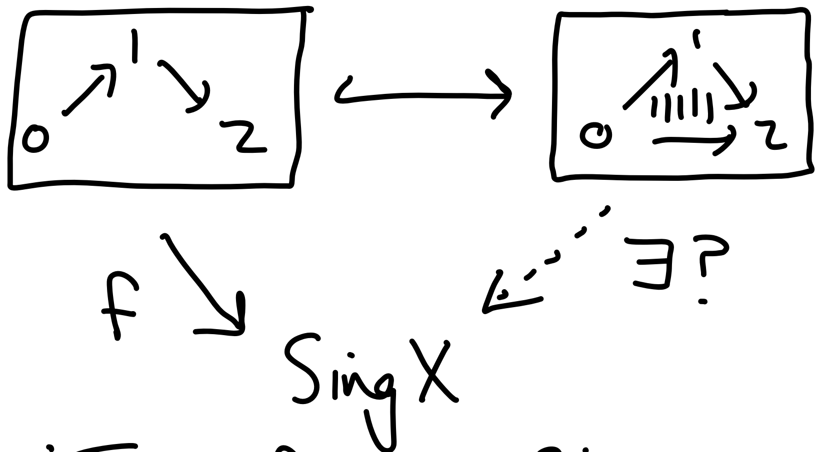
Two embeddings

- We have the full inclusion
  $J : \Delta \longrightarrow \mathrm{Cat}$ -
  indeed, we saw it earlier as the inclusion of the graphical theory of categories.

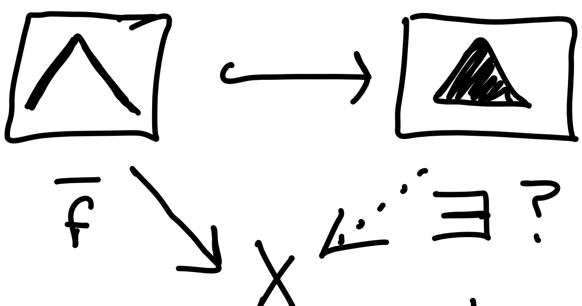
- Also $\Delta \xrightarrow{K} \mathrm{Top}$
  $$[n] \longmapsto |\Delta^n| = \{(x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \mid \Sigma x_i \leq 1\}$$
  views $[n]$ as standard n-simplex $|\Delta^n|$,
  & $f : [n] \to [m] \longmapsto |\Delta^n| \xrightarrow{|f|} |\Delta^m|$
  $$|f|(x_1, \dots, x_n) = (y_0, \dots, y_m)$$
  where $y_j = \sum_{i \in f^{-1}(j)} x_i$

Properties of Sing X

- Consider the inner horn



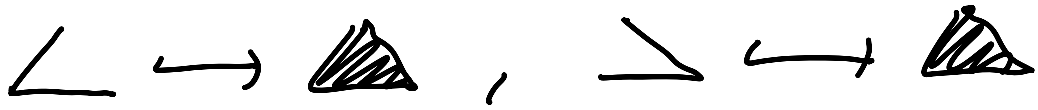
- By adjointness for $|\cdot| \dashv \mathrm{Sing}$, this is to give



& we can find a filler by composing paths.

- In fact, the horizontal map is a deformation retraction (in particular a split mono) & has section s – then $\bar{f} \circ s$ is filler.

- Similarly all horn inclusions
$$\Lambda^n_k \hookrightarrow \Delta^n \in SSet$$
are sent by $|-| : SSet \longrightarrow Top$ to deformation retractions.

- E.g 

  - Hence $\mathrm{Sing}X$ is injective to each horn inclusion.

$\boxed{\text{Def}^n}$ A simplicial set X is a Kan-complex (aka ∞-groupoid) if $X \in \mathrm{Inj}$ (Horn Inclusions).

Properties of NC

- A map $\Lambda^2_1 = \boxed{\begin{tikzcd}[column sep=small] & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}} \longrightarrow NC$

  picks out a pair $a \xrightarrow{f} b \xrightarrow{g} c \in C$.

- <u>Unique extension</u> $\Delta^2 \longrightarrow NC$ picking out 2-simplex
$$\begin{tikzcd}[column sep=small] & b \arrow[dr, "g"] \arrow[d, phantom, "="] & \\ a \arrow[ur, "f"] \arrow[rr, "gf"'] & & c \end{tikzcd}$$
in NC.

- <u>Not true for outer horns</u>

$$\boxed{\begin{tikzcd}[column sep=small] & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}} \quad \& \quad \boxed{\begin{tikzcd}[column sep=small] & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}}$$

unless X a <u>groupoid</u>.

unique lifts

Proposition

Each NC is orthogonal to each inner horn inclusion.

~~Proof~~ - We have checked the case Λ^1_2 above.

- It suffices to check each $\Lambda^n_k \longrightarrow \Delta^n$ is inverted by $h : SSet \longrightarrow Cat$.
- Now h determined by 2-truncation & for $n \geq 4$, these have same 2-truncation - only omit some $(n-1)$-cells.
- Remains to consider Λ^3_1, Λ^3_2.

Λ^1_3 looks like

- So we see $h\Lambda^3_1$ = free cat gen by
  $0 \to 1 \to 2 \to 3$,
  ie. $n = h\Delta^n$, as required.
- Case Λ^3_1 similar. $\square$

Definition) A simplicial set X is a quasicategory (aka ∞-cat)
$X \in \mathrm{Inj}(\text{Inner Horns})$.

Corollary) Both SX & NC are ∞-cats.

Quasicategories - basic properties

- let X be an ∞-cat.
- Its $0/1$-simplices we call objects & morphisms.
- Given $a \xrightarrow{f} b \xrightarrow{g} c \in X$, we say $h : a \to c$ is a composite of g & f

if $\exists$ 2-simplex
$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & & c \end{tikzcd}$$
in X.

- We then say that the triangle
$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & & c \end{tikzcd}$$
commutes & write $gf \sim h$.

- By filling Λ^2_1, each composable pair has a composite.

- Composites are not unique, but they are unique up to homotopy.

- Also write $a \xrightarrow{1} a$ for $s_0^1(a)$ - this will be identity.

Def^n) $a \overset{f}{\underset{g}{\rightrightarrows}} b \in X$ are htpic ($f \simeq g$) if any of

① $f \circ 1_a \sim g$ ② $1_b \circ f \sim g$

③ $g \circ 1_a \sim f$ ④ $1_b \circ g \sim f$.

Prop^n) The above 4 relations are equivalent in an ∞-cat X & an equivalence relation.

<u>Proof</u>

- First above that in an ∞-cat X, given a diagram

$$\boxed{\begin{tikzcd} & b \arrow[r, "j"] \arrow[d, "g"] & d \\ a \arrow[ur, "f"] \arrow[r, "h"'] & c \arrow[ur, "i"'] & \end{tikzcd} \qquad \begin{tikzcd} & b \arrow[r, "j"] & d \\ a \arrow[ur, "f"] \arrow[urr, "k"] \arrow[r, "h"'] & c \arrow[ur, "i"'] & \end{tikzcd}}$$

then A commutes $\Longleftrightarrow$ B commutes.

Follows from filling Λ^3_1, Λ^3_2.

Or can draw as:

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[rr, bend left, "h"] \arrow[drr, bend right, "k"'] \arrow[dr, phantom, "A"] & b \arrow[r, "g"] \arrow[dr, "j"] \arrow[u, phantom, "\circ"] \arrow[r, phantom, "\circ"'] & c \arrow[d, "i"] \\
& & d
\end{tikzcd}$$

& outside as B.

- Then

$$\begin{tikzcd}
a \arrow[r, "1"] \arrow[rr, bend left, "f"] \arrow[drr, bend right, "g"'] & a \arrow[r, "f"] \arrow[dr, "f"] \arrow[u, phantom, "\circ"] \arrow[r, phantom, "\circ"'] & b \arrow[d, "1"] \\
& & b
\end{tikzcd}$$

shows 1 $f \circ 1 \sim g$ $\iff$ 2 $1 \circ f \sim g$.

$1 \circ f \sim g$

- Likewise 3 $\iff$ 4 (swap f & g)

-

$$\begin{tikzcd}
a \arrow[r, "f"'] \arrow[rr, bend left, "f"] \arrow[drr, bend right, "g"'] & b \arrow[r, "1"] \arrow[dr, "1"] \arrow[u, phantom, "\circ"] \arrow[r, phantom, "\circ"'] & b \arrow[d, "1"] \\
& & b
\end{tikzcd}$$

$1 \circ g \sim f \implies$

$1 \circ f \sim g$

shows 2 $\iff$ 4.

Sim 1 $\iff$ 3.

For the equiv. relation,

- $f \simeq f$ since
$$\begin{tikzcd} & a \arrow[dr, "f"] \arrow[d, phantom, "\circ"] & \\ a \arrow[ur, "1"] \arrow[rr, "f"'] & {} & b \end{tikzcd}$$

- If $f \simeq g$ then $f \circ 1 \sim g$ so $g \circ 1 \sim f$ so $g \simeq f$.
- If $f \simeq g$ & $g \simeq h$ then

$$\begin{tikzcd} & & b \arrow[dd, "1"] \\ a \arrow[r, "1"'] \arrow[urr, bend left, "f"] \arrow[drr, bend right, "h"'] & a \arrow[ur, "g"'] \arrow[dr, "g"] & \\ & & b \end{tikzcd}$$

gives $f \simeq h$.

$\square$

$\boxed{\text{Prop}}$ Composites are unique up to htpy.

$\boxed{\text{Proof}}$ Suppose
$$\begin{tikzcd} & b \arrow[dr, "g"] \arrow[d, phantom, "\circ"] & \\ a \arrow[ur, "f"] \arrow[rr, "h"'] & {} & c \end{tikzcd} \quad \& \quad \begin{tikzcd} & b \arrow[dr, "g"] \arrow[d, phantom, "\circ"] & \\ a \arrow[ur, "f"] \arrow[rr, "i"'] & {} & c \end{tikzcd}.$$

Then

$$\begin{tikzcd} & & c \arrow[dd, "1"] \\ a \arrow[r, "f"] \arrow[urr, bend left, "h"] \arrow[drr, bend right, "h'"'] & b \arrow[ur, "g"] \arrow[dr, "g"] & \\ & & c \end{tikzcd}$$

shows $h \simeq h'$.

Now we define a cat <u>$ho(X)$</u> :

- obs as in X_0
- arrows are htpy classes of arrows.

Composition

- Given $a \xrightarrow{[f]} b \xrightarrow{[g]} c$ define $[g]\circ[f] : a \longrightarrow c$ to be $[h] : a \longrightarrow c$ where $h \sim g \circ f$. Check well-defined.

- Identity $a \xrightarrow{[1_a]} a$.

Associativity

$$
\begin{tikzcd}
a \arrow[r, "f"] \arrow[rr, bend left, "j"] \arrow[drr, bend right, "k"'] & b \arrow[r, "g"] \arrow[dr, "i"] & c \arrow[d, "h"] \\
& & d
\end{tikzcd}
$$

suppose $[g]\circ[f] = [j]$

$[h]\circ[g] = [i]$

$[i]\circ[f] = [k]$

Then $([h]\circ[g])\circ[f] = [i]\circ[f] = [k]$.

& $[h]\circ([g]\circ[f]) = [h]\circ[j]$ so must show $[h]\circ[j] = [k]$. See diagram.

Clearly unital & so a category.

<u>Exercise</u>

$ho(X)$ as above equals $h(X)$, as earlier constructed.