5.3 (iv) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} x^n$

$$\limsup_{n\to\infty} \underbrace{\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}}}_{\left(\left(1+\frac{1}{n}\right)^{n^2}\right)^{\frac{1}{n}} = \left(1+\frac{1}{n}\right)^{n}} =$$

$$= \limsup \left(1+\frac{1}{n}\right)^{n} = e$$

$\Rightarrow$ pol. konvergence je $\frac{1}{e}$

- $\left(-\frac{1}{e}, \frac{1}{e}\right)$ řada konverguje
- $\left(-\infty, -\frac{1}{e}\right) \cup \left(\frac{1}{e}, \infty\right)$ řada nekonverguje

$x = \frac{1}{e}$: $\sum_{n=1}^{\infty} \underbrace{\left(1+\frac{1}{n}\right)^{n^2} \left(\frac{1}{e}\right)^{n}}_{c_n}$ = d~~iverguje~~

neboť

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^n\right)^2 \left(\frac{1}{e}\right)^n =$$

$$= \lim_{n\to\infty} \cancel{\left[\left(1+\frac{1}{n}\right)^n \left(\frac{1}{e}\right)^n\right]^2}$$

$$= e^2 \cdot \underbrace{\lim_{n\to\infty} \left(\frac{1}{e}\right)^n}_{=0} = 0$$

platí nutná podmínka konvergence

- $\underline{x = -\frac{1}{e}}$: $\sum (-1)^n \underbrace{\left(1+\frac{1}{n}\right)^{n^2} \left(\frac{1}{e}\right)^n}_{\to 0}$

konverguje pro $n \to \infty$

dle Leibnizova kritéria

- $x = \frac{1}{e}$ nevíme

3.3 (v) $\sum_{n=1}^{\infty} \underbrace{\frac{n^5}{(2+(-1)^n)^n}}_{c_n} x^n$

- $\lim_{n\to\infty} \sqrt[n]{\frac{n^5}{(2+(-1)^n)^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^5}}{2+(-1)^n}$

neexistuje

nebot́ posloupnost $\sqrt[n]{\frac{n^5}{(2+(-1)^n)^n}}$

má dva hromadné body

$\frac{1}{3}$ a 1

↳ n sudé ↳ n liché

- $\limsup_{n\to\infty} \sqrt[n]{\frac{n^5}{(2+(-1)^n)^n}} = 1$

$\Rightarrow$ poloměr konvergence 1

Krajní body: $x = 1$

$$\sum_{n=1}^{\infty} \frac{n^5}{(2+(-1)^n)^n}$$ diverguje

↳ lichá členy n^5

$x = -1$

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^5}{(2+(-1)^n)^n}$$

• n liché $\quad \sum_{m=1}^{\infty} -(2m-1)^5 = -\infty$

• n sudé $\quad \sum_{m=1}^{\infty} \underbrace{\frac{(2m)^5}{3^{2m}}}_{\to 0 \text{ pro } m \to \infty} = ?$

(nevíme, jestli řada sudých členů konv)

Celkově $\sum_{n=1}^{\infty} (-1)^n \frac{n^5}{(2+(-1)^n)^n} = -\infty$

<u>Závěr</u>: řada konverguje na intervalu $(-1, 1)$

<u>6.1</u> (i) $f(x) = \dfrac{e^x}{x}$

$$f(x) \simeq f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) + \frac{1}{3!}(x - x_0)^3 f'''(x_0) + \dots$$

v bodě x_0

$f(1) = e$, $f'(1) = 0$

$f''(1) = e$

$x_0 = 1$

$$f'(x) = \frac{e^x \cdot x - e^x}{x^2} = \frac{e^x}{x} - \frac{e^x}{x^2}$$

$$f''(x) = \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right) - \frac{e^x \cdot x^2 - 2e^x x}{x^4}$$

$$= \frac{e^x}{x} - \frac{2e^x}{x^2} + \frac{2e^x}{x^3}$$

$$f'''(x) = \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right) - 2\,\frac{e^x \cdot x^2 - 2 \cdot e^x \cdot x}{x^4}$$

$$+ 2\,\frac{e^x \cdot x^3 - 3 \cdot e^x \cdot x^2}{x^6}$$

$$= \left(\frac{e^x}{x} - \frac{e^x}{x^2}\right) - 2\left(\frac{e^x}{x^2} - \frac{2e^x}{x^3}\right)$$

$$+ 2\left(\frac{e^x}{x^3} - \frac{3e^x}{x^4}\right)$$

$$= \frac{e^x}{x} - 3\frac{e^x}{x^2} + 6\frac{e^x}{x^3} - 6\frac{e^x}{x^4}$$

$$f'''(1) = e - 3e + 6e - 6e = -2e$$

Záver: $T_1^3(x) = e + \frac{1}{2}e(x-1)^2$

$+ \frac{1}{6}\cdot(-2e)(x-1)^3$

(ii) Taylorov rozvoj funkcie

$f(x) = \ln(x+1)$ v bode $x_0 = 0$

$$f'(x) = \frac{1}{x+1}$$

$$f''(x) = -\frac{1}{(x+1)^2} = -(x+1)^{-2}$$

$$f'''(x) = 2\,\frac{1}{(x+1)^3}$$

$$f^{IV}(x) = -6\,\frac{1}{(x+1)^4}$$

$$f^{(V)}(x) = 24\,\frac{1}{(x+1)^5}$$

$$\vdots$$

$$f^{(\ell)}(x) = (-1)^{\ell+1}(\ell-1)!\,\frac{1}{(x+1)^{\ell}}, \quad \ell \geq 1$$

$$f^{(\ell)}(0) = (-1)^{\ell+1}(\ell-1)!$$

$$\sum_{n=1}^{\infty} \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n =$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} (-1)^{n+1}(n-1)!\, x^n$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} x^n$$

Taylorův rozvoj

Určíme por. konvergence řady $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} x^n$

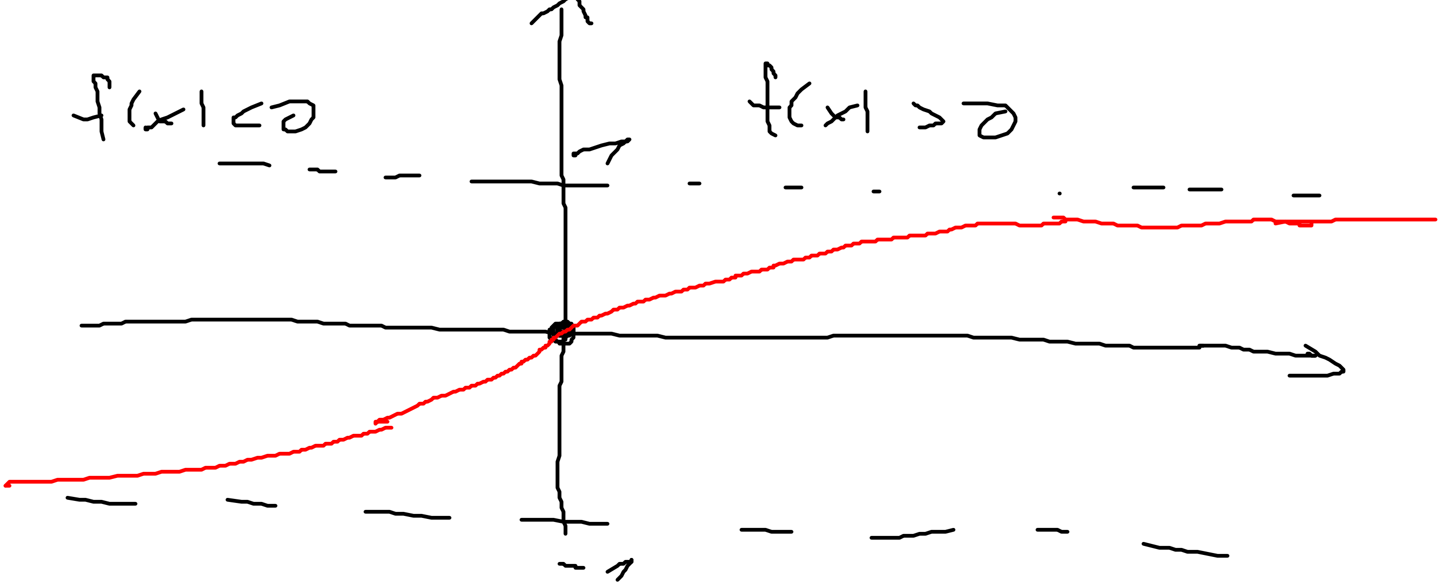
$$\lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \frac{1}{n}\right|} = 1 \Rightarrow \text{por. konv.} = 1$$

krajní body

- $x = 1$: $\sum (-1)^{n+1} \frac{1}{n}$ konv.
- $x = -1$: $\sum \left(-\frac{1}{n}\right)$ diverguje

Závěr: řada konv. pro $x \in (-1, 1]$

Př. 6.2 (i) Obor hodnot funkce $f(x) = \frac{e^x - 1}{e^x + 1}$



- $$\lim_{x \to \infty} \frac{e^x - 1}{e^x + 1} = 1$$

- $$\lim_{x \to -\infty} \frac{e^x - 1}{e^x + 1} = -1$$

$$\frac{e^x - 1}{e^x + 1} = \frac{1 - \frac{1}{e^x}}{1 + \frac{1}{e^x}}$$

$$f'(x) = \left(\frac{e^x - 1}{e^x + 1}\right)' = \frac{e^x(\cancel{e^x} + 1) - (\cancel{e^x} - 1)e^x}{(e^x + 1)^2}$$

$$= \frac{2e^x}{(e^x + 1)^2} > 0 \Rightarrow \text{funkce } f(x)$$

je rostoucí na $\mathbb{R}$

Závěr: obor hodnot je $(-1;1)$

6.2 (ii) Průběh funkce

$f(x) = \sqrt[3]{|x|^3+1}$

$D(f) = \mathbb{R}$

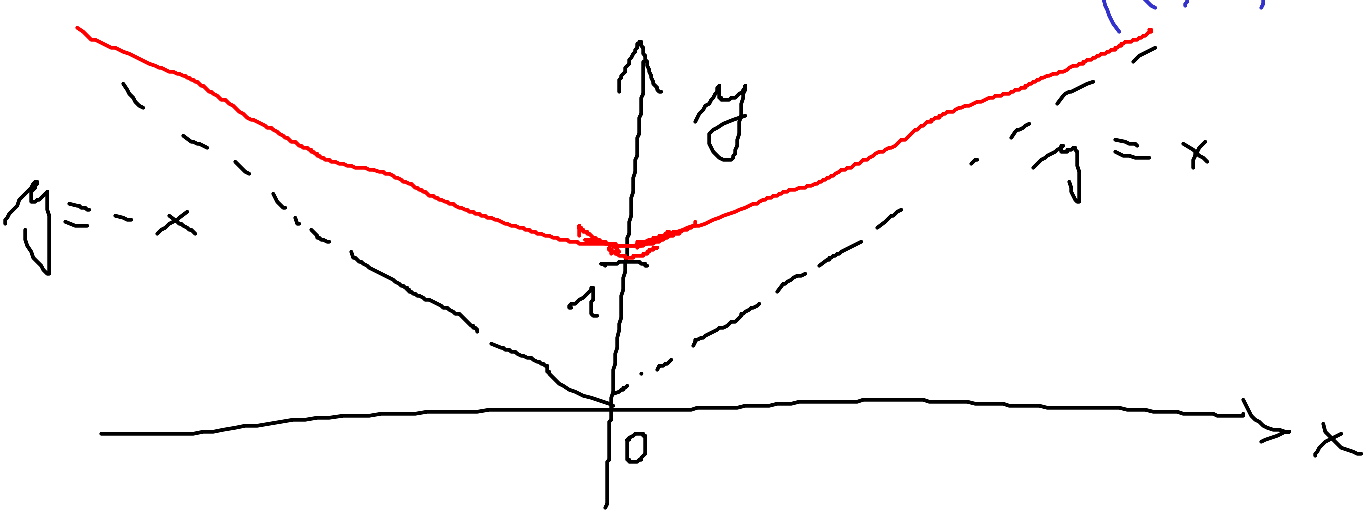
$H(f) = ?$

- sudá, spojitá

- $f'(x) = \left| x>0 \right| = \left(\sqrt[3]{x^3+1}\right)' = \left((x^3+1)^{\frac{1}{3}}\right)' = \frac{1}{3}(x^3+1)^{-\frac{2}{3}} \cdot 3x^2$

$= \dfrac{x^2}{\sqrt[3]{(x+1)^2}} > 0$ pro $x>0$

$f(x)$ rostoucí



- $f(x) = \sqrt[3]{x^3+1}$ pro $x \geq 0$

$\Rightarrow f'_+(0) = 0$ $\qquad$ $f'(0) = 0$

- symetricky pro $x \leq 0$

$f'_-(0) = 0$

$$f'(x) = |x<0| = \left(\sqrt[3]{(-x)^3+1}\right)' = \left((-x^3+1)^{\frac{1}{3}}\right)'$$

$$= \frac{1}{3}(-x^3+1)^{-\frac{2}{3}} \cdot (-3x^2)$$

$$= \frac{-x^2}{\sqrt[3]{(-x^3+1)^2}} < 0$$

- jediný lokální extrém v bodě $x = 0$
- Určíme 2. derivaci

$\underline{x > 0}$: $f''(x) = \left((x^3+1)^{-\frac{2}{3}} \cdot x^2 \right)' =$

$= -\frac{2}{3} (x^3+1)^{-5/3} \cdot 3x^2 \cdot x^2$

$+ (x^3+1)^{-\frac{2}{3}} \cdot 2x$

$= (x^3+1)^{-5/3} \left[-2x^4 + (x^3+1) \cdot 2x \right]$

$= \dfrac{2x}{(x^3+1)^{5/3}}$ pro $x > 0$

• podobně zleva

$\Rightarrow f''(x) = 0$ pouze při

$x = 0$ (lok. minimum)

• $\lim\limits_{x \to \infty} \sqrt[3]{|x|^3 + 1} = \infty$

$$\lim_{x\to\infty} \frac{\sqrt[3]{|x|^3+1}}{x} = 1 = a$$

asymptota $ax+b$

Ultimo b:

$$b = \lim_{x\to\infty} \left(\sqrt[3]{|x|^3+1} - x\right) =$$

$$= \lim_{x\to\infty} \left(\sqrt[3]{x^3-1} - x\right) \frac{\left(\sqrt[3]{x^3+1}\right)^2 + x\sqrt[3]{x^3+1} + x^2}{\text{totale}}$$

$$= \lim_{x\to\infty} \frac{1}{\left(\sqrt[3]{x^3+1}\right)^2 + x\sqrt[3]{x^3+1} + x^2} = 0$$

$$\left(\sqrt[3]{x^3+1} - x\right)\left(\left(\sqrt[3]{x^3+1}\right)^2 + x\sqrt[3]{x^3+1} + x^2\right)$$

$$= (x^3+1) - x^3 = 1$$

$$(A-B)(A^2+AB+B^2) = A^3 - B^3$$