Energy generation in stars

So far, we have only considered the dynamical properties of the star, and the state of the stellar material. But what is the source of the stellar energy?

Let’s consider the origin of the energy i.e. the conversion of energy from some form in which it is not immediately available into some form that it can radiate.

How much energy does the Sun need to generate?

$$L(Sun) = 4\times10^{26}\ \text{W} = 4\times10^{26}\ \text{Js}^{-1}$$

Sun has not changed flux in 10^9 yr ($3\text{x}10^6$s)

$\Rightarrow$ Sun has radiated $1.2\ \text{x}10^{43}$ J

$$E = mc^2$$

$$\Rightarrow m_{lost} = 10^{26}\ \text{kg} = 10^{-4} M(Sun)$$

Source of energy generation

What is the source of this energy? We can think of four possibilities:

- Cooling
- Contraction
- Chemical Reactions
- Nuclear Reactions

Cooling and contraction

These are closely related, so we consider them together. Cooling is simplest idea of all. Suppose the radiative energy of the Sun is due to it being much hotter when it was formed, and has since been cooling down.

Or is the Sun slowly contracting with consequent release of gravitational potential energy, which is converted to radiation?

Source of energy generation

In an ideal gas, the thermal energy of a particle; where n_f … *number of degrees of freedom (= 3)*

$$= \frac{kT}{2} n_f = \frac{3kT}{2}$$

Total thermal energy per unit volume $= \frac{3knT}{2}$

n … *number of particles per unit volume*

Now Virial Theorem: $$3\int_0^{V_s} P dV + \Omega = 0$$

Assume that stellar material is ideal gas (negligible P_r)

$$\Rightarrow P = nkT$$

$$3\int_0^{V_s} nkT dV + \Omega = 0$$

Source of energy generation

Now lets define U ... *integral over volume of the thermal energy per unit volume*

Thermal energy per unit volume $= \frac{3knT}{2} \Rightarrow 2U + \Omega = 0$

The negative gravitational energy of a star is equal to twice its thermal energy. This means that the time for which the present thermal energy of the Sun can supply its radiation and the time for which the past release of gravitational potential energy could have supplied its present rate of radiation differ by only a factor two.

Negative gravitational potential energy of a star is related by the inequality

$$-\Omega > \frac{GM_s^2}{2r_s} \quad \text{as an approximation assume} \quad -\Omega \sim \frac{GM_s^2}{2r_s}$$

Source of energy generation

Total release of gravitational potential energy would have been sufficient to provide radiant energy at a rate given by the luminosity of the star L_s, for a time

$$t_{th} \sim \frac{GM_s^2}{L_s r_s}$$

For the values of the Sun: $t_{th}(Sun) = 3 \times 10^7$ years.

t_{th}… *thermal timescale (or Kelvin-Helmholtz timescale)*

Hence if the Sun is powered by either contraction or cooling, it would have changed substantially in the last 10 million years. A factor of ~100 too short to account for the constraints on age of the Sun imposed by fossil and geological records.

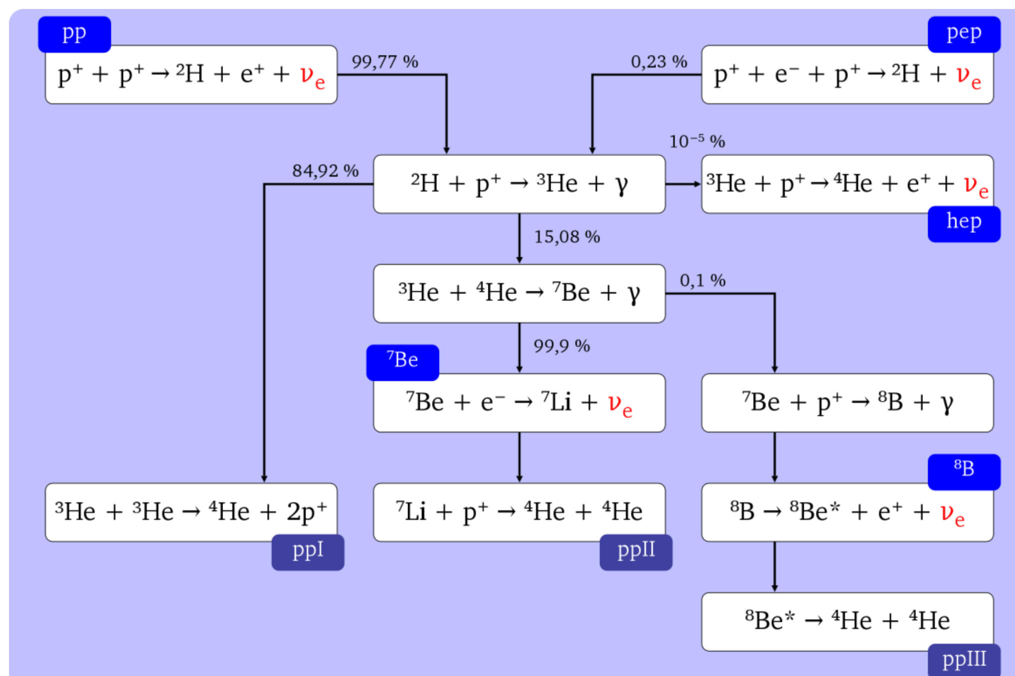
Chemical Reactions

We calculated above that we need to find a process that can produce at least 10^{-4} of the rest mass energy of the Sun. Chemical reactions such as the combustion of fossil fuels release ~ 5×10^{-10} of the rest mass energy of the fuel.

Source of energy generation

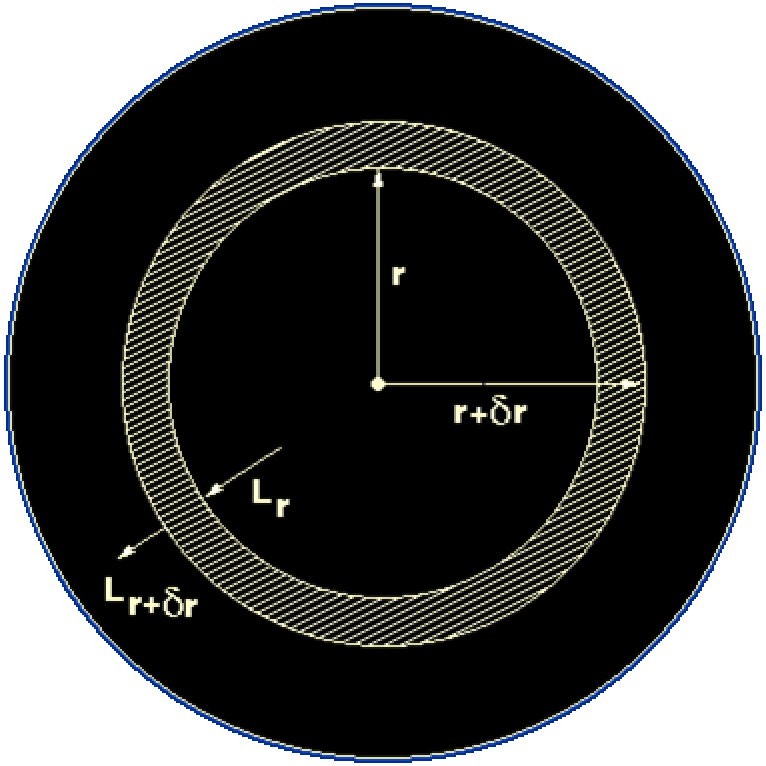
Nuclear Reactions

Hence the only known way of producing sufficiently large amounts of energy is through nuclear reactions. There are two types of nuclear reactions, fission and fusion. Fission reactions, such as those that occur in nuclear reactors, or atomic weapons can release ~ 5×10^{-4} of rest mass energy through fission of heavy nuclei (uranium or plutonium).



Equation of energy production

The third equation of stellar structure: relation between energy release and the rate of energy transport



Consider a spherically symmetric star in which energy transport is radial and in which time variations are unimportant.

$L(r)$ … rate of energy flow across sphere of radius r

$L(r+\delta r)$ … rate of energy flow across sphere of radius $r+\delta r$

Because the shell is thin:

$$\delta V(r) = 4\pi r^2 \delta r$$

and $\delta m(r) = 4\pi r^2 \rho(r) \delta r$

We define ε ... *energy release per unit mass per unit volume* (Wkg^{-1})

Hence energy release in a shell is written as:

$$4\pi r^2 \rho(r)\delta r\varepsilon$$

Conservation of energy leads us to

$$L(r+\delta r) = L(r) + 4\pi r^2 \rho(r)\delta r\varepsilon$$

$$\Rightarrow$$

$$\frac{L(r+\delta r)}{\delta r} - L(r) = 4\pi r^2 \rho(r)\delta r\varepsilon$$

and for $\delta r \to 0$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon$$

This is the ***equation of energy production***.

We now have three of the equations of stellar structure. However we have five unknowns $P(r), M(r), L(r), \rho(r), \varepsilon(r)$. The next step is to investigate the energy transport inside stars.

Method of energy transport

There are three ways energy can be transported in stars

- Convection – energy transport by mass motions of the gas
- Conduction – by exchange of energy during collisions of gas particles (usually e^-)
- Radiation – energy transport by the emission and absorption of photons

Conduction and radiation are similar processes – they both involve transfer of energy by direct interaction, either between particles or between photons and particles.

Which is the more dominant in stars?

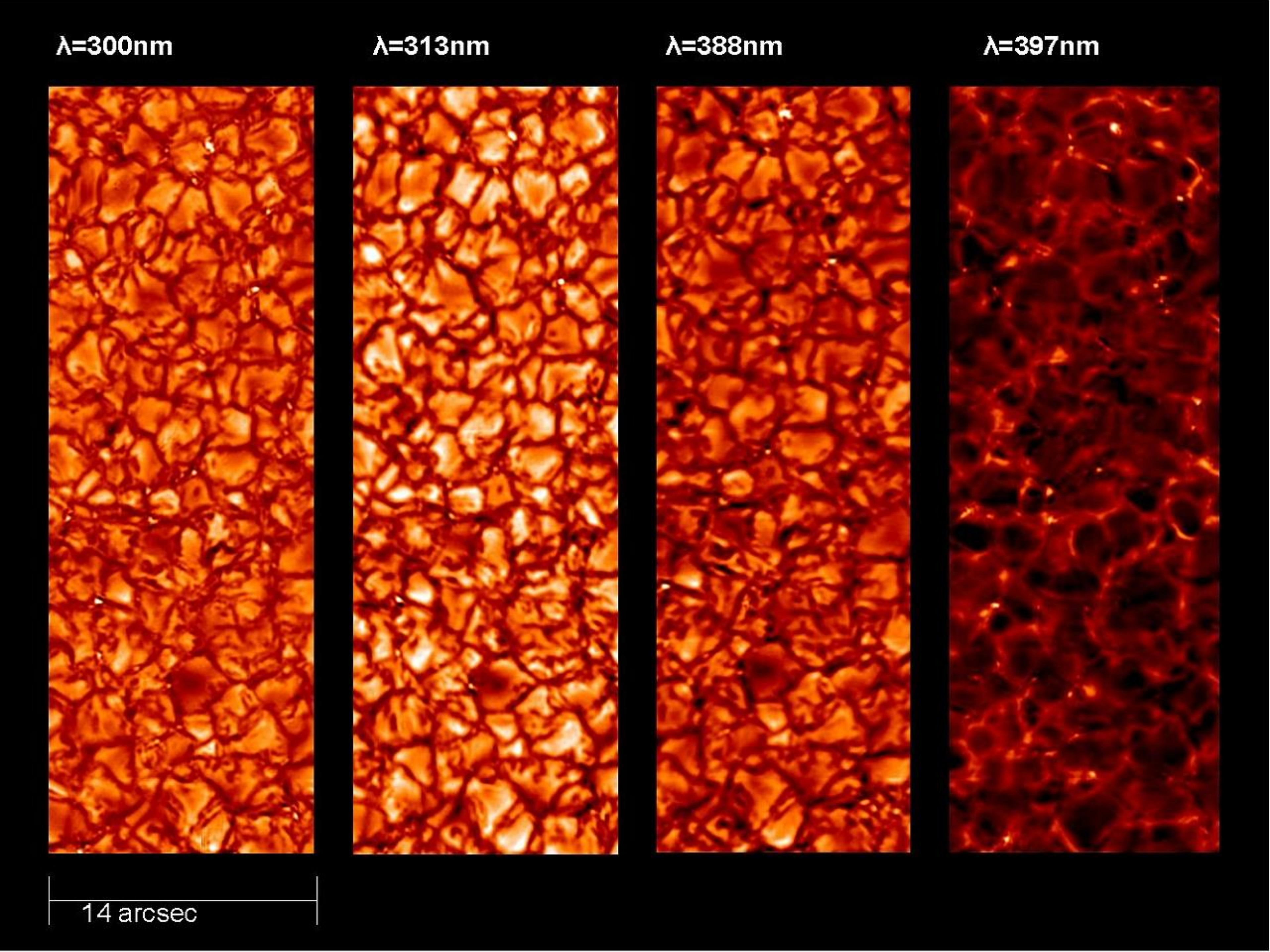
Energy carried by a typical particle $\sim 3\,kT/2$ is comparable to energy carried by typical photon $\sim hc/\lambda$

But number density of particles is much greater than that of photons. This would imply conduction is more important than radiation.

Mean free path of photon ~ 10^{-2}m

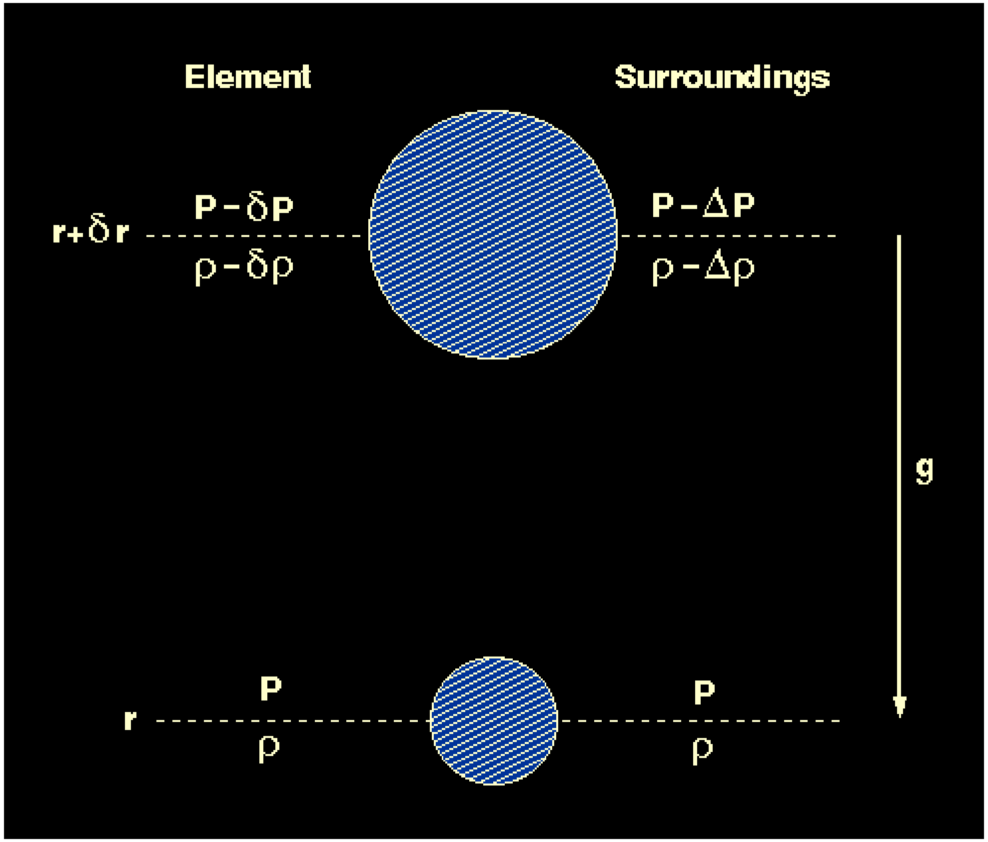
Mean free path of particle ~ 10^{-10}m

Photons can move across temperature gradients more easily, hence larger transport of energy. Conduction is negligible, radiation transport in dominant


λ=300nm
λ=313nm
λ=388nm
λ=397nm
14 arcsec

Convection

Convective element of stellar material



Convection is the mass motion of gas elements which only occurs when temperature gradient exceeds some critical value. We can derive an expression for this.

Consider a convective element at distance r from centre of star. Element is in equilibrium with surroundings.

Now let's suppose it rises to $r + \delta r$. It expands, $P(r)$ and $\rho(r)$are reduced to $P - \delta P$ and $\rho - \delta\rho$

But these may not be the same as the new surrounding gas conditions. Define those as $P - \delta P$ and $\rho - \delta\rho$

If gas element is denser than surroundings at $r + \delta r$ then will sink (i.e. stable)

If it is less dense then it will keep on rising – ***convectively unstable***

The condition for instability is therefore

$$\rho - \delta\rho < \rho - \Delta\rho$$

Whether or not this condition is satisfied depends on two things:

- The rate at which the element expands due to decreasing pressure
- The rate at which the density of the surroundings decreases with height

Let's make two assumptions

1. The element rises *adiabatically*
2. The element rises at a speed much less than the sound speed. During motion, sound waves have time to smooth out the pressure differences between the element and the surroundings. Hence $\delta P = \Delta P$ *at all times*

The first assumption means that the element must obey the adiabatic relation between pressure and volume

$$PV^{\gamma} = \text{constant}$$

Where $\gamma = c_p/c_v$ is the specific heat (i.e. the energy in J to raise temperature of 1kg of material by 1K) at constant pressure c_p, divided by specific heat at constant volume c_v

Given that V is inversely proportional to ρ, we can write

$$\frac{P}{\rho^{\gamma}} = \text{constant}$$

Hence equating the term at r *and* $r + \delta r$:

$$\frac{P - \delta P}{(\rho - \delta\rho)^{\gamma}} = \frac{P}{\rho^{\gamma}}$$

If $\delta\rho$ is small we can expand $(\rho - \delta\rho)^{\gamma}$ using the binomial theorem as follows

$$(\rho - \delta\rho)^{\gamma} \sim \rho^{\gamma} - \gamma\rho^{\gamma\text{-}1}\ \delta\rho \quad \text{and combining last two expressions}$$

$$\delta\rho = \frac{\rho}{\gamma \mathrm{P}}\delta P$$

Now we need to evaluate the change in density of the surroundings, $\delta\rho$

Lets consider an infinitesimal rise of δr

$$\Delta\rho = \frac{d\rho}{dr}\delta \mathrm{r}$$

And substituting these expressions for $\delta\rho$ and $\Delta\rho$ into the condition for convective instability derived above:

$$\frac{\rho}{\gamma P}\delta P < \frac{d\rho}{dr}\delta \mathrm{r}$$

And this can be rewritten by recalling our 2nd assumption that element will remain at the same pressure as it surroundings, so that in the limit

$$\delta r \to 0, \quad \frac{\delta P}{\delta r} = \frac{dP}{dr}$$

$$\frac{\rho}{\gamma P}\frac{dP}{dr} < \frac{d\rho}{dr}$$

The LHS is the density gradient that would exist in the surroundings if they had an adiabatic relation between density and pressure. RHS is the actual density in the surroundings. We can convert this to a more useful expression, by first dividing both sides by dP/dr. Note that dP/dr is negative, hence the inequality sign must change.

$$\frac{\rho}{\gamma P} < \frac{d\rho}{dr} \Big/ \frac{dP}{dr}$$

$$\Rightarrow \quad \frac{\rho}{\gamma P} > \frac{d\rho}{dP}$$

$$\left(\frac{P}{\rho}\right) \frac{d\rho}{dP} < \frac{1}{\gamma}$$

And for an ideal gas in which radiation pressure is negligible (where m is the mean mass of particles in the stellar material)

$$P = \frac{\rho k T}{m}$$

$$\ln P = \ln \rho + \ln T + \text{constant}$$

And can differentiate to give

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

And combining this with the equation above gives ….

Condition for occurrence of convection

$$\frac{P}{T}\frac{dT}{dP} > \frac{\gamma - 1}{\gamma}$$

Which is the condition for the occurrence of convection (in terms of the temperature gradient). A gas is convectively unstable if the actual temperature gradient is steeper than the adiabatic gradient.

If the condition is satisfied, then large scale rising and falling motions transport energy upwards.

The criterion can be satisfied in two ways. The ratio of specific heats γ is close to unity or the temperature gradient is very steep.

For example if a large amount of energy is released at the centre of a star, it may require a large temperature gradient to carry the energy away. Hence where nuclear energy is being released, convection may occur.

Condition for occurrence of convection

Alternatively in the cool outer layers of a star, gas may only be partially ionised, hence much of the heat used to raise the temperature of the gas goes into ionisation and hence the specific heat of the gas at constant V is nearly the same as the specific heat at constant P, and $\gamma \sim 1$.

In such a case, a star can have a cool outer convective layer.

Convection is an extremely complicated subject and it is true to say that the lack of a good theory of convection is one of the worst defects in our present studies of stellar structure and evolution. We know the conditions under which convection is likely to occur but don't know how much energy is carried by convection.