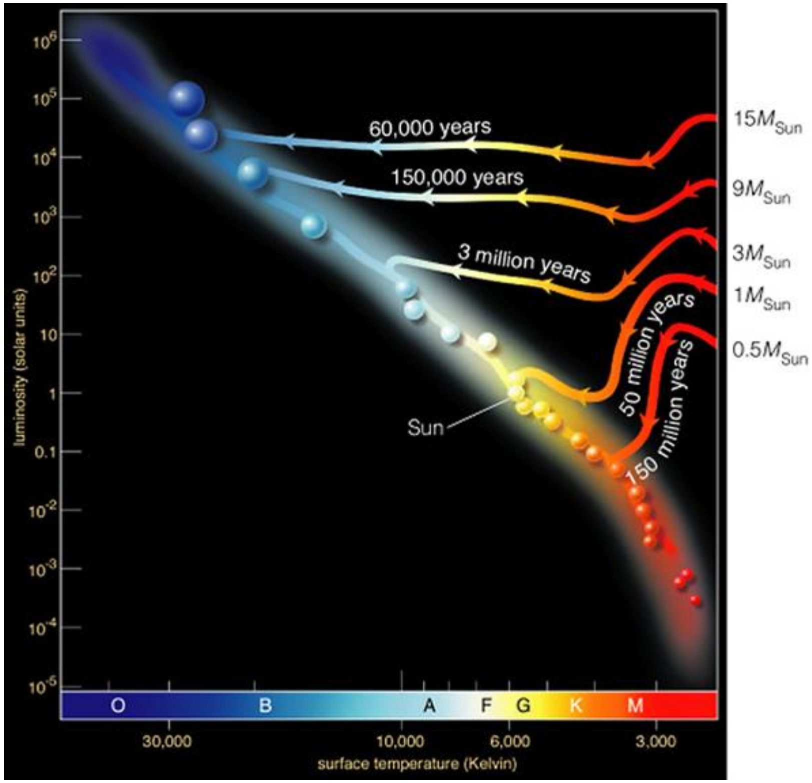
Different timescales

Mass ($M_\odot$)	Surface temperature (K)	Spectral class	Luminosity ($L_\odot$)	Main-sequence lifetime (10^6 years)
25	35,000	O	80,000	4
15	30,000	B	10,000	15
3	11,000	A	60	800
1.5	7000	F	5	4500
1.0	6000	G	1	12,000
0.75	5000	K	0.5	25,000
0.50	4000	M	0.03	700,000



At different stages of evolution, different timescales will be important

The characteristic timescales

There are three characteristic timescales that aid concepts in stellar evolution

The dynamical timescale

For the Sun $t_d \sim 2000$ s

$$t_d = \left(\frac{2r^3}{GM} \right)^{\frac{1}{2}}$$

The thermal timescale

Time for a star to emit its entire reserve of thermal energy upon contraction provided it maintains constant luminosity (Kelvin-Helmholtz timescale)

For the Sun $t_{th} \sim 30$ Myr

$$t_{th} \sim \frac{GM^2}{Lr}$$

The nuclear timescale

Time for star to consume all its available nuclear energy, ε is the typical nucleon binding energy/nucleon rest mass energy

For the Sun, t_{nuc} is larger than the age of Universe

$$t_{nuc} \sim \frac{\varepsilon M c^2}{L}$$

$$\Rightarrow t_d << t_{th} << t_{nuc}$$

The equation of radiative transport

We assume for the moment that the condition for convection is not satisfied, and we will derive an expression relating the change in temperature with radius in a star assuming all energy is transported by radiation. Hence we ignore the effects of convection and conduction.

The energy carried by radiation per square meter per second, i.e. the flux F_{rad}, can be expressed in terms of the temperature gradient and a coefficient of radiative conductivity, λ_{rad}, as follows:

$$F_{rad} = -\lambda_{rad} \frac{dT}{dr}$$

where the minus sign indicates that heat flows down the temperature gradient. Assuming that all energy is transported by radiation, we will now drop the subscript rad from the remainder of this discussion.

The radiative conductivity measures the readiness of heat to flow. Astronomers generally prefer to work with an inverse of the conductivity, known as the *opacity* κ, which measures the resistance of material to the flow of heat.

The equation of radiative transport

It can be show that the opacity, κ, is defined by the relation

$$\kappa(r) = \frac{4acT(r)^3}{3\rho(r)\lambda}$$

where a is the radiation density constant and c is the speed of light. Combining the above equations we obtain:

$$F(r) = -\frac{4acT(r)^3}{3\rho(r)\kappa(r)}\frac{dT}{dr}$$

Recalling that flux and luminosity are related by

$$L(r) = 4\pi r^2 F(r)$$

The equation of radiative transport

we can write:

$$L(r) = -\frac{16acr^2T(r)^3}{3}\frac{dT}{dr}$$

On rearranging, we obtain:

$$\frac{dT}{dr} = -\frac{3\rho(r)\kappa(r)}{16acr^2T(r)^3}L(r)$$

This is known as the ***equation of radiative transport*** and is the temperature gradient that would arise in a star if all the energy were transported by radiation. It should be noted that the above equation also holds if a significant fraction of energy transport is due to conduction, but in this case L refers to the luminosity due to radiative *and* conductive energy transport and refers to the opacity to heat flow via radiation *and* conduction.

Solving the equations of stellar structure

Hence we now have four differential equations, which govern the structure of stars (note – in the absence of convection).

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$$

$$\frac{dT(r)}{dr} = -\frac{3\rho(r)\kappa(r)}{16\pi acr^2 T(r)^3} L(r)$$

r … radius

P… pressure at r

M… mass of material within r

ρ … density at r

L … luminosity at r (rate of energy flow across sphere of radius r)

T … temperature at r

κ … Rosseland opacity at r

ε … energy release per unit mass per unit time

We will consider the quantities:

P = P(ρ, T, chemical composition) **The equation of state**

κ = κ(ρ, T, chemical composition)

ε = ε(ρ, T, chemical composition)

Boundary conditions

Two of the boundary conditions are fairly obvious, at the center of the star $M = 0, L = 0$ at $r = 0$.

At the surface of the star its not so clear. There is no sharp edge to the star, but for the Sun ρ(surface)~10^{-4} kg m^{-3}. Much smaller than the mean density ρ(mean)~1.4×10^{3} kg m^{-3} (which we derived). We know the surface temperature (5780K) is much smaller than its minimum mean temperature (2×10^{6} K).

Thus, we make two approximations for the surface boundary conditions: $\rho = T = 0$ at $r = r_s$

i.e. that the star does have a sharp boundary with the surrounding vacuum

Use of mass as the independent variable

The above formulae would (in principle) allow theoretical models of stars with a given radius. However from a theoretical point of view it is the mass of the star which is chosen, the stellar structure equations solved, then the radius and other parameters are determined. We observe stellar radii to change by orders of magnitude during stellar evolution, whereas mass appears to remain constant. Hence it is much more useful to rewrite the equations in terms of M rather than r.

If we divide the other three equations by the equation of mass conservation, and invert the latter:

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho} \qquad \frac{dL}{dM} = \varepsilon$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4} \qquad \frac{dT}{dM} = -\frac{3\kappa L}{16\pi^2 r^4 acT^3}$$

With boundary conditions:

$r = 0, L = 0$ at $M = 0$

$\rho = T = 0$ at $M = M_s$

We specify M_s and the chemical composition and now have a well defined set of relations to solve. It is possible to do this analytically if simplifying assumptions are made, but in general these need to be solved numerically.

Stellar evolution

We have a set of equations that will allow the complete structure of a star to be determined, given a specified mass and chemical composition. However what do these equations not provide us with?

In deriving the equation for hydrostatic support, we have seen that provided the evolution of star is occurring slowly compared to the dynamical time, we can ignore temporal changes (e.g. pulsations)

$$t_d = \left(\frac{2r^3}{GM} \right)^{\frac{1}{2}}$$

And for the Sun for example, this is t_d ~ 2000s, hence this is certainly true

And we have also made the assumption that time dependence can be omitted from the equation of energy generation, if the nuclear timescale (the time for which nuclear reactions can supply the stars energy) is greatly in excess of t_{th}

Stellar evolution

If there are no bulk motions in the interior of the star, then any changes of chemical composition are localised in the element of material in which the nuclear reactions occurred. So the star would have a chemical composition which is a function of mass M.

In the case of no bulk motions, the set of equations we derived must be supplemented by equations describing the rate of change of *abundances* of the different chemical elements. Let $C_{X,Y,Z}$ be the chemical composition of stellar material in terms of mass fractions of hydrogen (*X*), helium, (*Y*) and metals (*Z*)

$$\frac{\partial (C_{X,Y,Z})_M}{\partial t} = f(\rho, T, C_{X,Y,Z})$$

Now lets consider how we could evolve a model

$$(C_{X,Y,Z})_{M,t_0+\delta t} = (C_{X,Y,Z})_{M,t_0} + \frac{\partial (C_{X,Y,Z})_M}{\partial t}$$

Influence of convection

Ideally we would like to know exactly how much energy is transported by convection – but lack of a good theory makes it difficult to predict exactly.

Heat is convected by rising elements which are hotter than their surroundings and falling elements which are cooler. Suppose the element differs by δT from its surroundings, because an element is always in pressure balance with its surroundings, it has energy content per kg which differs from surrounding kg of medium of $c_\rho \delta T$ (where c_ρ *is the* specific heat at constant pressure).

If material is mono-atomic ideal gas then $c_\rho = 5k/2m$

Where m is the average mass of particles in the gas

Assuming a fraction α (≤1) of the material is in the rising and falling columns and that they are both moving at speed v [ms^{-1}] then the rate at which excess energy is carried across radius is:

$L_{conv} =$ surface area of sphere x rate of transport x excess energy

$$L_{conv} = 4\pi r^2 \alpha \rho v \frac{5k\delta T}{2m} = \frac{10\pi r^2 \alpha \rho v k \delta T}{m}$$

Hence putting in known solar values, at a radius halfway between surface and center, we get

$$L_{conv} = 10^{26}\alpha v \delta T \text{ [W]}$$

The surface luminosity of the Sun is $L(Sun) = 3.86 \times 10^{26}$ W, and at no point in the Sun can the luminosity exceed this value (see equation of energy production).

What can we conclude from this?

As the δT and v of the rising elements are determined by the difference between the actual temperature gradient and adiabatic gradient, this suggests that the actual gradient is not greatly in excess of the adiabatic gradient. To a reasonable degree of accuracy we can assume that the temperature gradient has exactly the adiabatic value in a convective region in the interior of a star and hence can rewrite the condition of occurrence of convection in the form

$$\frac{P}{T}\frac{dT}{dP} = \frac{\gamma - 1}{\gamma}$$

Thus ***in a convective region*** we must solve the four differential equations, together with equations for ε and P

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho} \qquad \frac{dL}{dM} = \varepsilon$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4} \qquad \frac{P}{T}\frac{dT}{dP} = \frac{\gamma - 1}{\gamma}$$

The equation for luminosity due to radiative transport is still true:

$$L_{rad} = \frac{16\pi^2 r^4 acT^3}{3\kappa}\frac{dT}{dM}$$

And once the other equations have been solved, L_{rad} can be calculated. This can be compared with L (from $dL/dM = \varepsilon$) and the difference gives the value of luminosity due to convective transport $L_{conv} = L - L_{rad}$

In solving the equations of stellar structure the equations appropriate to a convective region must be switched on whenever the temperature gradient reaches the adiabatic value, and switched off when all energy can be transported by radiation.