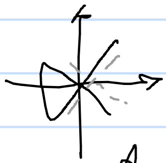
① Dimension

Recall :
The tangent cone of $X = V(f)$ at p is defined as

$C_p(X) = V(f_{in})$ where f_{in} denotes the initial terms of f.

Find $\dim C_p(X)$.

i)



$X = V(\boxed{y^2} - x^3 - \boxed{x^2})$

non-irr

$\dim V = \max\{\dim V_i\}$

Ans : $C_p(X) = V(y^2 - x^2) = V(y-x) \cup V(y+x)$

$\dim C_p(X) = 1$, coincide with $\dim X$

ii) Let X, Y be 2 irr aff var.

Show that $\dim X \times Y = \dim X + \dim Y$

Pf. Note that $\mathbb{k}(X \times Y) \cong \mathbb{k}(X) \otimes \mathbb{k}(Y)$.

So if $\mathbb{k}(X) = \mathbb{k}(x_1, \dots, x_n)$

and $\mathbb{k}(Y) = \mathbb{k}(y_1, \dots, y_m)$

then $\mathbb{k}(X \times Y)$

$= \mathbb{k}(x_1, \dots, x_n, y_1, \dots, y_m)$

iii) Calculate the dim of the grassmanian variety of $G(k,n)$.

Ans : Recall we have

$\gamma : V(k,n) \to G(k,n)$

$(v_1, \dots, v_k) \mapsto [v_1, \dots, v_k]$

We have $\dim V(k,n) = nk$

Now consider the fibre of each point Λ in $G(k,n)$, i.e. $\gamma^{-1}(\Lambda)$, which consists of all bases for the vector space Λ, which is $GL(k)$, so $\dim \gamma^{-1}(\Lambda) = k^2$.

Now,

$\dim V(k,n) = \dim G(k,n) + \dim \gamma^{-1}(\Lambda)$ — collapse

$nk = \dim G + k^2$

$\Rightarrow \dim G = nk - k^2 = k(n-k)$

By thm, $f: X \to Y$

$\dim X = \dim Y + \dim f^{-1}(y)$

② Blow-up

Idea: turn a non-smooth curve (with singularities) into a smooth one.

Let $X \subseteq \mathbb{A}^n$ be an irr aff var of dim at least 1.
WLOG, let $P_0 = (0,0) \in X$.

In general, the blow-up variety of X at P_0 is defined as

$$B_{P_0}(X) := \{ (p, \ell) \mid p \in X \cap \ell, \ p \neq P_0 \} \subseteq \mathbb{A}^n \times \mathbb{P}^{n-1}$$

In $n=2$ case:

Let $f: U_0 \to \mathbb{A}^1$ — recall $U_0 = \{ (x,y) \in \mathbb{A}^2 \mid x \neq 0 \}$

$(x,y) \mapsto \frac{y}{x}$

Then $\Gamma_f = \{ (x,y,t) \in \mathbb{A}^3 \mid y = tx, \ x \neq 0 \}$

$\because y - tx$ irr

Define $B = \{ (x,y,t) \in \mathbb{A}^3 \mid y = tx \}$ which is an irr var

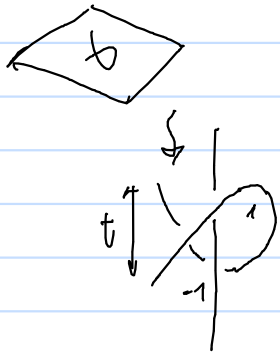
and $\pi : B \to \mathbb{A}^2$ be proj.
$(x,y,t) \mapsto (x,y)$

Then $\pi(B) = U_0 \cup \{P_0\}$, and $B = \overline{\Gamma_f}$.

i) $X = V(y^2 - x^3 - x^2)$

Solving $\begin{cases} y^2 - x^3 - x^2 = 0 \\ y = tx \end{cases}$

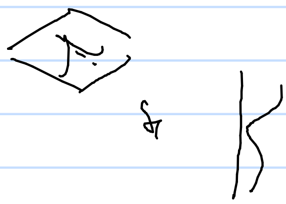
gives $B = \{ (t^2 - 1, t^3 - t, t) \}$, the blowup var of X at $(0,0)$



ii) $X = V(x^3 - y^2)$

Solving $\begin{cases} x^3 - y^2 = 0 \\ y = tx \end{cases}$

gives $B = \{ (t^2, t^3, t) \}$, the blowup var of X at $(0,0)$



iii) Denote by $B_p(X)$ the blow-up of X at P.

Show that $\dim B_p(X) = \dim X$.

Pf. Note that removing a point on a curve gives an open dense subset of the curve.

Hence $B_p(X) \simeq_{b.e.} X$.

So they have the same coordinate ring, hence by previous tutorials, $\dim B_p(X) = \dim X$.

iv) Let $X \subseteq \mathbb{A}^2$ be irr. Show $\dim C_p X = \dim X$

Pf. $X = V(f)$, $C_pX = V(f_{in})$ are plane curves.

By last tutorial, dim of plane curve is always 1

v) A point P is smooth $\Leftrightarrow$ $C_pX = T_pX$

Pf. By def, P is smooth $\Leftrightarrow$ $\dim T_p(X) = \dim X$

Hence the result follows.

③ Sheaves

- Recall that a presheaf on a cat $\mathrm{Ouv}(X)$, the poset of open subsets of X ordered by inclusion, is a functor
$$\mathcal{F} : \mathrm{Ouv}(X)^{op} \to \mathrm{Set}.$$

- A morphism between 2 presheaves is a nat tran
$$\varphi : \mathcal{F}_1 \to \mathcal{F}_2$$
i.e. $\forall\ V \subset U$ open subsets of X,
$$\begin{array}{ccc} \mathcal{F}_1(U) & \xrightarrow{\varphi_U} & \mathcal{F}_2(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_1(V) & \xrightarrow[\varphi_V]{} & \mathcal{F}_2(V) \end{array}$$

- A presheaf $\mathcal{F}$ is called a sheaf precisely when for all union of open subsets $U = \bigcup_{i \in I} U_i$, there is an equaliser diagram in Set
$$\mathcal{F}(U) \overset{e}{\rightarrowtail} \prod_{i \in I} \mathcal{F}(U_i) \underset{\psi_2}{\overset{\psi_1}{\rightrightarrows}} \prod_{i,j \in I} \mathcal{F}(U_i \cap U_j)$$

idea : matches on intersections 'gluing'

where for $s \in \mathcal{F}(U)$, $(s_i) \in \prod \mathcal{F}(U_i)$,
$$e(s) = \{ s|_{U_i} \mid i \in I \}$$
$$\psi_1((s_i)) = \{ s_i|_{U_i \cap U_j} \}$$
$$\psi_2((s_i)_i) = \{ s_j|_{U_i \cap U_j} \}$$

- The stalk of presheaf $\mathcal{F}$ at $x \in X$ is
$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$$

i.e. for $x \in U_1 \subset U_2 \subset U_3 \dots$
we have
$$\mathcal{F}(U_1) \leftarrow \mathcal{F}(U_2) \leftarrow \mathcal{F}(U_3) \leftarrow \cdots$$
$$\varinjlim_{x \in U} \mathcal{F}(U)$$

idea: shrinking $U \ni x$ smaller to be like $\{x\}$

- $\mathcal{F}(U) \to \mathcal{F}_x$, $s \mapsto s_x$ is called the germ of s at x

- Let $\mathcal{F}$ be a presheaf.

Define $\mathcal{F}^+$ by $\mathcal{F}^+(U) := \{ f : U \to \coprod_{x \in U} \mathcal{F}_x \mid f \text{ satisfies (i) \& (ii)} \}$

where (i) $f(x) \in \mathcal{F}_x \quad \forall x \in U$

(ii) $\forall x \in U, \quad \exists x \in V \subset U$ and $g \in \mathcal{F}(V)$, s.t. $f(y) = g_y$ $\forall y \in V$

i) Sheafification

Prove that $\mathcal{F}^+$ is a sheaf

Pf. We want to show that

$$\mathcal{F}^+(U) \overset{e}{\rightarrowtail} \prod_{i \in I} \mathcal{F}^+(U_i) \underset{\gamma_2}{\overset{\gamma_1}{\rightrightarrows}} \prod_{i,j \in I} \mathcal{F}^+(U_i \cap U_j)$$

is an equaliser diag in Set

First, let $f \in \mathcal{F}^+(U)$.

$e(f) = \{ f|_{U_i} : U_i \to \coprod_{x \in U_i} \mathcal{F}_x \}$

Then $\gamma_1 e(f) = \{ (f|_{U_i})|_{U_i \cap U_j} : U_i \cap U_j \to \coprod_{x \in U_i \cap U_j} \mathcal{F}_x \}$

$\gamma_2 e(f) = \{ (f|_{U_j})|_{U_i \cap U_j} : U_i \cap U_j \to \coprod_{x \in U_i \cap U_j} \mathcal{F}_x \}$

are equal.

Second, it suffices to show e is injective.

Suppose

$e(f_1) = e(f_2)$

$\{ f_1|_{U_i} \} = \{ f_2|_{U_i} \}$

$\Rightarrow f_1 = f_2$

(i) We have the following adjunction:

$$\begin{tikzcd} \mathrm{PreSh}(X) \arrow[r, shift left, "(-)^+"] \arrow[r, phantom, "\perp"] & \mathrm{Sh}(X) \arrow[l, shift left, "\mathrm{inc}"] \end{tikzcd}$$

$PS(\mathcal{F}, \mathrm{inc}\,\mathcal{G}) \cong S(\mathcal{F}^+, \mathcal{G})$

i.e. there is a universal property:

Let $\mathcal{F} \in \mathrm{PreSh}(X)$, $\mathcal{G} \in \mathrm{Sh}(X)$, $\varphi : \mathcal{F} \to \mathcal{G}$.

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr, "\varphi"'] & \mathcal{F}^+ \arrow[d, dashed, "\exists!\, \varphi^+"] \\ & \mathcal{G} \end{tikzcd}$$

$\mathcal{F}_x$

Pf.

Let $s \in \mathcal{F}^+(U)$, $U = \bigcup_{i \in I} U_i$ where $s|_{U_i} = s_i \in \mathcal{F}(U_i)$

Then $\varphi_{U_i}(s_i) \in \mathcal{G}(U_i)$

$\mathcal{F}(U_i) \xrightarrow{\varphi_{U_i}} \mathcal{G}(U_i)$

Now $\forall\, x \in U_i \cap U_j$, $s(x) = (s_i)_x = (s_j)_x$ by (ii)

So $\varphi_{U_i}(s_i)|_{U_i \cap U_j} = \varphi_{U_j}(s_j)|_{U_i \cap U_j}$ $\forall\, i, j \in I$. (agree at x $\Rightarrow$ agree at $U_i \cap U_j$)

Finally, since $\mathcal{G}$ is a sheaf, $\mathcal{G}(U) \overset{e}{\rightarrowtail} \prod \mathcal{G}(U_i)$

$\exists!\, t \in \mathcal{G}(U)$ s.t. $t|_{U_i} = \varphi_{U_i}(s_i)$ $\forall\, i \in I$

Setting $\varphi^+(s) = t$, we are done.