Exercise 9 - Algebra 4

- In the course we introduced $\mathrm{Ext}^n(A,-)$ as the right nth right derived functor of $\mathrm{Hom}(A,-) : \mathrm{Mod}_R \to \mathrm{Ab}$, but then mentioned that elements of $\mathrm{Ext}^n(A,B)$ (for $n>0$) are equiv. classes of exact sequences
$$0 \to B \to X_1 \to \dots \to X_n \to A \to 0$$
- Here we explore some of the algebra of such exact sequences.

1 (Pullback & pushout)

- Consider

$$\begin{tikzcd} & & & A' \arrow[d, "\alpha"] & \\ 0 \arrow[r] & B \arrow[r, "f"'] & X \arrow[r, "g"'] & A \arrow[r] & 0 \end{tikzcd}$$

with exact row

Show that the top row is ex.

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "f'"] \arrow[d, "1"] & P \arrow[r, "g'"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A' \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & B \arrow[r, "f"'] & X \arrow[r, "g"'] & A \arrow[r] & 0 \end{tikzcd}$$

where g' is pullback of g & $f'x = (fx, 0)$.

- Try to relate it to les of cohomology for Ext.

(2) Explain the dual construction for pushouts,

& how the previous constructions give actions

$$\begin{aligned} \mathrm{Ext}_1(A,B) &\longrightarrow \mathrm{Ext}_1(A',B) \\ E &\longmapsto E\alpha \end{aligned}$$

&

&

$$\begin{aligned} \mathrm{Ext}_1(A,B) &\longrightarrow \mathrm{Ext}_1(A,B') \\ E &\longmapsto \beta E \end{aligned}$$

where $\beta : B \to B'$.

making $\mathrm{Ext}_1(-,-) : \mathrm{Mod}_R^{op} \times \mathrm{Mod}_R \to \mathrm{Ab}$ a functor.

5 (Baer sum)

Given $E : 0 \to B \to X \to A \to 0$

& $E' : 0 \to B \to Y \to A \to 0$

in $\mathrm{Ext}_1(A, B)$ we will define the Baer sum

$$E + E' \in \mathrm{Ext}_1(A, B).$$

Firstly, take

$$E \oplus E' : 0 \to B \oplus B \to X \oplus Y \to A \oplus A \to 0$$

& then define

$$E + E = \nabla_A (E \oplus E') \Delta_B$$

where $\nabla_A : A \oplus A \to A$ is codiagonal

$$(x, y) \longmapsto x + y$$

& $\Delta_B : B \longrightarrow B \oplus B$

$$x \longmapsto (x, x)$$

is diagonal.

- Show that this makes $\mathrm{Ext}_1(A, B)$

into an abelian group.

- What is the unit?