Exercises 2 - Algebra 4

Reminder

- A pre-additive category $\mathcal{C}$ is one in which each hom $\mathcal{C}(a,b)$ is an abelian group & pre & post composition preserves the abelian group structure.
- In a pre-additive category $\mathcal{C}$, binary products & coproducts are the same thing - biproducts. These are diagrams

$$\begin{tikzcd} a \arrow[r, shift left, "i_1"] & c \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & b \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

satisfying

- $p_1 i_1 = 1_A$, $p_2 i_1 = 0$
- $p_1 i_2 = 0$, $p_2 i_2 = 1_B$
- $i_1 p_1 + i_2 p_2 = 1_C$.

- Sim. terminal & initial obs coincide, & are captured by the property that $c \xrightarrow{0 = \mathrm{id}} c$ and we call them a zero object.
- A pre-additive category is additive if it has biproducts & a zero ob.

- Given $f : A \to B \in \mathcal{C}$ its kernel
$\ker f \xrightarrow{i} A \xrightarrow{f} B$ is the universal ob. such that $f \circ i = 0$. Cokernels are dual.

It is abelian if $\operatorname{coker} \ker f \to \ker \operatorname{coker} f$ is invertible, or equiv: each mono is kernel of its cokernel, each epi the cokernel of its kernel.

1. Check that in Mod_R, given $f: A \to B$, we have $\mathrm{coker} f = B/\mathrm{im} f$.

2. In a pre-additive cat $\mathcal{C}$, show directly that $\ker f \xrightarrow{i} A$ is mono & $B \xrightarrow{p} \mathrm{coker} f$ is epi.

3. Show that a pre-additive $\mathcal{C}$ has kernels $\iff$ it has equalisers.

4. Recall the cohomology functor
$$H^n : \mathrm{Cochain}(\mathrm{Mod}_R) \longrightarrow \mathrm{Mod}_R.$$
Using duality, construct this using the homology functor on an abelian cat.

5. Suppose that $\mathcal{A}$ is abelian. Describe kernels and cokernels in the cat $\mathrm{Ch}(\mathcal{A})$ in detail.

(6) - An element $x \in A$ an abelian group has torsion if $\exists n \in \mathbb{Z}/0$ st $nx = 0$.
- An abelian group is torsion-free if it has no non-trivial torsion elements.

- Show that the category TFreeAb is additive with kernels & cokernels, but not abelian by finding a mono which is not the kernel of its cokernel.

(7) A cat $\mathcal{C}$ is pointed if it has an object 0 which is both terminal & initial.
Show that $\forall a, b$ $\exists$ a morphism
$0_{a,b} : a \longrightarrow b$ preserved by pre & post composition.

(8) Show that in a pointed cat there always exists a canonical map
$\lambda_{a,b} : a + b \longrightarrow a \times b$.
Using this, define biproducts in a pointed category.

(9) Show that in a pointed cat. $\mathcal{C}$ with biproducts, each homset $\mathcal{C}(a,b)$ has the str. of a commutative monoid.