

Exercises 3

- ① Show that $0 \rightarrow A \rightarrow 0$ is exact $\Leftrightarrow A = 0$
& $0 \rightarrow A \rightarrow B \rightarrow 0$ is exact $\Leftrightarrow A \rightarrow B$ is an iso.

- ① Recall that if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a seq of chain complexes, then we get the long exact sequence of homology

$$\dots H_{n+1}(A) \rightarrow H_{n+1}(B) \rightarrow H_{n+1}(C) \xrightarrow{\delta} H_n(A) \rightarrow H_n(B) \rightarrow \dots$$

Using this,

show that if any 2 of A, B & C are exact, then so is the third.

- ② Homology groups of spheres

- If X is a top. space, & A, B subspaces whose interiors jointly cover X , then by the les of homology one can get a les

$$\dots H_1(A) \oplus H_1(B) \rightarrow H_1(X) \rightarrow H_0(A \cap B) \rightarrow H_0(A) \oplus H_0(B) \rightarrow H_0(X) \rightarrow 0$$

called the Mayer-Vietoris sequence.

- For example, when $X = S^n$ the n -sphere, we can write it as union of its subspaces A, B obtained by removing north & south pole.

$$S^1 = \bigcirc, A = \bigcirc, B = \bigcirc, A \cap B = \bigcirc \stackrel{\text{htpy eq.}}{\simeq} \cdot = S^0$$

Then $A \cap B \simeq S^{n-1}$ in general, &
 $A, B \simeq \cdot$

Using this, & that $H_n(S^1) = \mathbb{Z}$ if $n=1$
else 0 ,

show $H_n(S^n) = \mathbb{Z}$ all $n \geq 1$.

③ Show that there is a category whose objects are chain complexes & whose morphisms are homotopy classes of chain maps.

④ Consider the categories $SES(\mathcal{C})$ $LES(\mathcal{C})$ whose objects are short exact & long exact sequences in an ab. cat \mathcal{C} . Show that the long exact sequence of homology gives a

functor

$$SES(\mathcal{C}h(\mathcal{C})) \longrightarrow LES(\mathcal{C}).$$