Group cohomology & related topics

- G a group, can form group (co)homology via derived functors, using Ext & Tor.

- Consider category G-Mod of G-modules:
  - a G-module is an abelian group A + associative action $\cdot : G \times A \to A$ with each $g\cdot - : A \to A$ a homomorph. of abelian groups.
  - In other words, a monoid homom. $G \to Ab(A,A)$ or

(equivalently, a functor

$$\Sigma G \xrightarrow{M} Ab$$

1 ob. cat.

$$\bullet \circlearrowleft g \longmapsto M \circlearrowleft g\cdot -$$
)

(will look at these more closely in group representation theory!!)

- These are just $\mathbb{Z}G$-modules where $\mathbb{Z}G$ the group ring, whose elements are sums

$$n_1 g_1 + \dots + n_k g_k, \quad n_i \in \mathbb{Z}, g_i \in G.$$

& whose multiplication extends that of G.

- Inclusion $Ab \xrightarrow{i} \mathbb{Z}G\text{-Mod}$ sends A to abelian group A with trivial action $g.a = a$ all $a \in A$.

- It has left & right adjoints

$$\begin{tikzcd} Ab \arrow[r, "i" description] & \mathbb{Z}G\text{-Mod} \arrow[l, bend right, "\mathrm{Coinv}"'] \arrow[l, bend left, "\mathrm{Inv}"] \end{tikzcd}$$

where $\mathrm{Inv}(M) = \{x \in M : gx = x \text{ all } g \in G\}$

Equally, the limit of the diagram

$$\Sigma G \xrightarrow{M} Ab$$

1 ob. cat.

$$\cdot \circlearrowleft g \longmapsto M \circlearrowleft g.-$$

ie.

$$\begin{tikzcd} \mathrm{Invar}(M) \arrow[r, hook] \arrow[dr] & M \arrow[d, "g.-"] \\ & M \end{tikzcd} \quad \forall g.$$

- It's colimit is $\mathrm{Coinv}(M) = M / \langle gx - x : x \in M, g \in G \rangle$.

Remark) Viewing $\mathbb{Z}G\text{-Mod} = [\Sigma G, Ab]$, then $i : Ab \longrightarrow [\Sigma G, Ab]$ is just the diagonal/constant functor, so it is standard that it is left & right adjoints, taking colims & lims.

- In particular,

the right adj. $\mathrm{Inv} : \mathbb{Z}G\text{-Mod} \to \mathrm{Ab}$ is lex & Coinv is rex, so we can take their derived functors:

$R_n\mathrm{Inv},\ L_n\mathrm{Coinv} : \mathbb{Z}G\text{-Mod} \to \mathrm{Ab}$ & then

the cohomology of G with coeff in $M \in \mathbb{Z}G\text{-Mod}$ is defined as $H^n(G;M) = R_n\mathrm{Inv}(M)$

whilst

homology of G with coeff in $M \in \mathbb{Z}G\text{-Mod}$ defined as $H_n(G;M) = L_n\mathrm{Coinv}(M)$

Lemma

$\mathrm{Inv}(M) \cong \mathbb{Z}G\text{-Mod}(i\mathbb{Z}, M)$ &

$\mathrm{Coinv}(M) \cong i\mathbb{Z} \otimes_{\mathbb{Z}G} M$ viewed as triv right G-module

Proof

- A homomorphism $i\mathbb{Z} \to M$ is a hom. of ab. groups $\mathbb{Z} \xrightarrow{f} M$ such that

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{f} & M \\ {\scriptstyle 1\cdot -}\downarrow & = & \downarrow{\scriptstyle g\cdot -} \\ \mathbb{Z} & \xrightarrow{f} & M \end{array}$$

but as $\mathbb{Z}$ is free, $f(n) = n.a$ for unique $a \in M$ & the condition then just says

$n.ga = g(n.a) = n.a \quad \forall n \in \mathbb{Z}$

ie $g.a = a$,

which is to say $a \in \mathrm{Inv}(M)$.

- I will leave $\mathrm{Coinv}(M) \cong i\mathbb{Z} \otimes_{\mathbb{Z}G} M$ as an exercise. $\square$

Therefore, the group cohomology is

$H^n(G;M) \cong \underline{\mathrm{Ext}_n(i\mathbb{Z}, M)}$

whilst group homology is

$H_n(G;M) \cong \underline{\mathrm{Tor}_n(i\mathbb{Z}, M)}$.

- We will focus on group cohomology & so $\mathrm{Ext}_n(\mathbb{Z}, M)$.
- This can be calculated by taking a projective resolution, called the (unreduced) Bar resolution, which we will now describe.
- To describe it, consider the free ab. group $\mathbb{Z}[G^n]$ with pointwise G-module structure $h.(g_1, \dots, g_n) = (hg_1, \dots, hg_n)$.
- Much as for simplicial complexes, we obtain maps

$$\mathbb{Z}[G^{n+1}] \xrightarrow{d_i} \mathbb{Z}[G^n] :$$

$$(g_0, \dots, g_n) \longmapsto (g_0, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$$

$\in \mathbb{Z}G\text{-Mod}$ which omit the i'th element, for $i \in \{0, \dots, n\}$

- Taking alternating sums, we obtain a chain complex in $\mathbb{Z}G$-Mod.

$$\cdots \to \mathbb{Z}[G^{n+1}] \xrightarrow{d = \sum_{i=0}^{n} (-1)^i d_i} \mathbb{Z}[G^n] \longrightarrow \cdots$$

$$\cdots \to \mathbb{Z}[G^1] \xrightarrow{d} \mathbb{Z} = \mathbb{Z}[G^0]$$

$$\sum n_i g_i \longmapsto \sum n_i .$$

- Must show

$$\cdots \mathbb{Z}[G^2] \xrightarrow{d} \mathbb{Z}[G^1] \xrightarrow{d} \mathbb{Z}[G^0] \to 0$$

is exact in $Ch(\mathbb{Z}G\text{-Mod})$, $'' X$

& as $u : \mathbb{Z}G\text{-Mod} \to Ab$ is exact

(kernels & cokernels of modules are as for abelian groups),

suffices to show it is exact in $Ch(Ab)$.

- In fact, in $Ch(Ab)$, we have a htpy

$$\begin{tikzcd} X \arrow[r, bend left, "id"] \arrow[r, bend right, "0"'] \arrow[r, phantom, "h \, S"] & X \end{tikzcd}, \text{ so } \begin{tikzcd} & 0 \arrow[rr, "0"] \arrow[dr, "0"] & & 0 \\ X \arrow[ur, "0"] \arrow[rr, "1"'] \arrow[r, phantom, "S1"] & & X \arrow[ur, "0"'] \arrow[u, phantom, "||"] \end{tikzcd}$$

$\Rightarrow X \to 0$ a htp equiv $\Rightarrow H_n X = 0$ all n.

• The htpy $X \overset{\text{id}}{\underset{0}{\rightrightarrows}} X$ (h, s) consists of maps $\mathbb{Z}[G^n] \longrightarrow \mathbb{Z}[G^{n+1}]$

$$(g_1, \ldots, g_n) \overset{h_n}{\longmapsto} (1, g_1, \ldots, g_n)$$

(which are $\mathbb{Z}$-linear but not $\mathbb{Z}G$-linear!)

• Must show they satisfy

$$dh + hd = 1_{\mathbb{Z}[G^n]} \quad \text{or}$$

$$dh = 1 - hd$$

Now $dh(g_1, \ldots, g_n) = d(1, g_1, \ldots, g_n)$

$= (g_1, \ldots, g_n) -$

$(1, g_2, \ldots, g_n) + (1, g_1, g_3, \ldots, g_n) \ldots$

$= (g_1, \ldots, g_n) - hd(g_1, \ldots, g_n) = 1 - hd\,(g_1, \ldots, g_n)$

as required.

• Now, in fact this is a free resolution (each $\mathbb{Z}[G^{n+1}]$ is a free $\mathbb{Z}G$-module (& so projective).

- Indeed, we have iso of $\mathbb{Z}G$-modules
$\underline{\mathbb{Z}G[G^n] \overset{\theta_n}{\cong} \mathbb{Z}[G^{n+1}]}$ on basis elements

$(g_0, \ldots, g_{n-1}) \longmapsto (1, g_0, \ldots, g_{n-1})$

(Note: $\theta_n^{-1}(g_0, g_1, \ldots, g_{n-1}) =$
$\theta_n^{-1} g_0.(1, g_0^{-1}g_1, \ldots, g_0^{-1}g_{n-1}) = g_0 \theta_n^{-1}(1, g_0^{-1}g_1, \ldots, g_0^{-1}g_{n-1})$
$= g_0.(g_0^{-1}g_1, \ldots, g_0^{-1}g_{n-1})$)

(On lhs, $\mathbb{Z}G$-lin combos $2h.(g_0, \ldots, g_n) + 3h'(g_0', \ldots, g_n')$
on rhs, $\mathbb{Z}$-lin combos $2(h, g_0, \ldots, g_n) + 3(h', g_0', \ldots, g_n')$ etc)

- Therefore, we have constructed a <u>free resolution</u> of $i\mathbb{Z}$.

- $\ldots \longrightarrow \mathbb{Z}[G^2] \longrightarrow \mathbb{Z}[G] \longrightarrow i\mathbb{Z}$

but in order to better <u>understand the cohomology</u>, it is preferable to have it in the isomorphic form

- $\ldots \longrightarrow \mathbb{Z}G[G] \longrightarrow \underset{\mathbb{Z}G[0]}{\overset{\mathbb{Z}G}{\shortparallel}} \longrightarrow i\mathbb{Z}$

• Can transport chain complex structure along these isos to a chain complex

$$\begin{array}{ccc} \mathbb{Z}[G^{n+1}] & \xrightarrow{d} & \mathbb{Z}[G^n] \\ \theta_n \uparrow \wr & " & \wr\downarrow \theta_n^{-1} \\ \cdots\ \mathbb{Z}G[G^n] & \xrightarrow{d} & \mathbb{Z}G[G^{n-1}]\ \cdots \end{array}$$

though, disturbingly, we don't do this - instead we use a different iso

$$\varphi_n : \mathbb{Z}G[G^n] \longrightarrow \mathbb{Z}[G^{n+1}]$$

$$(g_0, \dots, g_{n-1}) \longmapsto (1, g_0, g_0 g_1, \dots, g_0 g_1 \cdots g_n)$$

which I leave as an exercise to check is an iso.

(Reason is that resulting degeneracy maps are a bit nicer, I believe.)

• Now transporting along these isos (in same way) obtain Free resolution

$$\mathbb{Z}G[G^2] \xrightarrow{d} \mathbb{Z}G[G^1] \xrightarrow{d} \underset{\mathbb{Z}G}{\overset{\shortparallel}{\mathbb{Z}G[G^0]}} \longrightarrow i\mathbb{Z}$$

$$\sum n_i g_i \longmapsto \sum n_i$$

where $\mathbb{Z}G[G^{n+1}] \xrightarrow{d} \mathbb{Z}G[G]$

$$(g_1, \ldots, g_{n+1}) \longmapsto g_1(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i (g_1, \ldots, g_{i+1}g_i, \ldots, g_{n+1}) + (-1)^{n+1}(g_1, \ldots, g_n)$$

which is the so-called <u>Bar resolution</u> B_n.

Examples)
- $d : \mathbb{Z}G[G] \longrightarrow \mathbb{Z}G$

$$g \longmapsto g - 1$$

- $d : \mathbb{Z}G[G^2] \longrightarrow \mathbb{Z}G[G]$

$$(g_1, g_2) \longmapsto g_1(g_2) - (g_1g_2) + g_1$$

- $\mathbb{Z}G[G^3] \longrightarrow \mathbb{Z}G[G^2]$

$$(g_1, g_2, g_3) \longmapsto g_1(g_2g_3) - (g_1g_2, g_3) + (g_1, g_2g_3) - (g_1, g_2)$$

Then, given a G-module A, we obtain a cochain complex $\mathbb{Z}G\text{-Mod}(B_\bullet, A)$ with values

$$\mathbb{Z}G\text{-Mod}(\mathbb{Z}G, A) \xrightarrow{d^0} \mathbb{Z}G\text{-Mod}(\mathbb{Z}G[G], A) \xrightarrow{d^1} \cdots$$

$$\begin{array}{ccccc} \cong & & \cong & & \\ \mathrm{Set}(1, A) & \xrightarrow{d^0} & \mathrm{Set}(G, A) & \xrightarrow{d^1} & \mathrm{Set}(G^2, A) \cdots \end{array}$$

elements of A — functions $G \to A$ — functions $G^2 \to A$

- d^0 sends $a \in A \longmapsto \begin{array}{l} G \longrightarrow A \\ g \longmapsto ga - a \end{array}$ so

where maps $G \to A$ of this form are called principal homomorphisms

- So $H^0(G : A) = \ker(d_0) = \{a : ga = a\}$
$= \mathrm{Inv}(A)$, as expected.

- Given $X : G \longrightarrow A$, $d^0 X : G \times G \longrightarrow A$ has values

$$d^1 X(a,b) = a.Xb - X(ab) + Xa.$$

Then $X \in \operatorname{Ker}(d^1) \Leftrightarrow X(a,b) = a.Xb + Xa.$

Maps with this property are called crossed homomorphisms

- If A has trivial action, crossed hom $\equiv$ homomorphism.

$$H^1(G : A) = \frac{\text{crossed homs } G \longrightarrow A}{\text{principal homs } G \longrightarrow A}.$$

What about the higher cases $H^n(G:A)$?

- Well, using the formulae for d^i, $H^2(G:A)$

$$= \frac{G\times G \xrightarrow{X} A \text{ st } a.X(b,c)+X(a,bc) = X(ab,c)+X(a,b)}{G\times G \xrightarrow{X} A \text{ st } \Psi: G\to A \text{ st } X(a,b) = a.\Psi b - \Psi(ab) + \Psi a}$$

which is a bit mysterious.

- Actually, they classify so-called group extensions - ses of groups
$$0\to A \xrightarrow{i} H \xrightarrow{p} G \longrightarrow 0$$ up to iso of ses.

- More precisely, if we have an ses where A is an ab. gp, H, G groups,

- More precisely, if this is a ses, p is surj, so $\exists s: G\to H$ a function such that $ps=1$; now we can define an action of G on A by $g.a = s(g).i(a).s(g)^{-1} \in A \leq H$.

Indeed
$$p(s(g).i(a).s(g)^{-1}) = g.pi(a).g^{-1} = pi(a) = 0$$
so $g.a \in \ker(p) = \mathrm{im}(i) = A$.

Exercise ① Check it is a G-module.

② Check it is independent of the choice of section.

- Now given a G-module A, let $\mathrm{Ext}(G,A)$ = iso classes of extensions

$$0 \to A \xrightarrow{i} H \xrightarrow{p} G \longrightarrow 0$$ for which the induced G-module structure on A is the original one.

- We will see that $\mathrm{Ext}(G,A) \cong H^2(G:A)$.

- Indeed, given a ses as above, consider a section $s : G \longrightarrow H$; we measure how far it is from being a homom. by considering

$$\varphi_s : G \times G \longrightarrow A : (g,h) \mapsto s(gh)s(h)^{-1}s(g)^{-1}$$

which lives in $\ker p = A$.

- This is an elt of $H^2(G:A)$, independent up to a choice of section

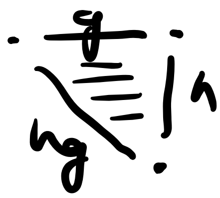
& induces the iso

$$\mathrm{Ext}(G,A) \cong H^2(G:A).$$

Via spaces

- Each group G has an associated topological space BG,

  which is gen. by - a point •

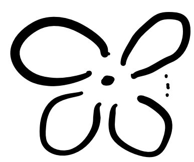
  - a loop $\bullet \xrightarrow{g} \bullet$ For $g \in G$
  - a simplex 

    For $g, h \in G$

  & so on,

  called its <u>classifying space</u>.

- E.g., if F_n is free group on n-elts,

  BF_n is "<u>bouquet</u>" 

- If A is an abelian group, then

$$H^n(BG;A) = H^n(G;A)$$

viewed as trivial G-module

cohomology of space

cohomology of group

- In this case, cocycles simpler because G-action disappears: $X(ab)-X(bc)=X(ab,c)-X(a,bc)$. etc.

General machine of simplicial resolutions

- Δ = cat of non-empty finite ordinals $[n] = \{0<1<..n\}$ & order-preserving maps.
- Diagram $\Delta^{op} \xrightarrow{X} \mathcal{C}$ is simplicial object in $\mathcal{C}$: it involves "face maps"

$$X[n] \xrightarrow{d_i} X[n-1]$$

which "remove i'th face".

- If $\mathcal{C}$ = Set, we call it a simplicial set.

Example

- If G is a group, obtain simplicial set NG, where

$$NG[n] = \{(g_0, g_1, \dots, g_n) : g_i \in G\}$$

Face maps

$$\begin{tikzcd} NG[2] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & NG[1] \end{tikzcd}$$

$$(g_0, g_1) \longmapsto g_1, g_1g_0, g_0$$

Called nerve of G.

- Given a simplicial ob $X : \Delta^{op} \longrightarrow \mathcal{C}$ in abelian cat, obtain a chain complex in $\mathcal{C}$

$$\cdots\; X[n] \xrightarrow{d = \sum_{i=0}^{n} (-1)^i d_i} X[n-1] \;\cdots$$

- Eg. If A an abelian group, $\mathrm{Set}(NG, A) : \Delta^{op} \longrightarrow \mathrm{Ab}^{op}$ & the corresponding chain complex in Ab^{op} (so cochain complex)

$$\mathrm{Set}(\cdot, A) \longrightarrow \mathrm{Set}(G, A) \longrightarrow \mathrm{Set}(G^2, A) \longrightarrow \cdots$$

is again that from Bar resolution.

- More importantly, if $\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} D$, $F \dashv U$, obtain @ $c \in \mathcal{C}$ a diagram

$$\begin{tikzcd} \cdots & (FU)^3 c \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & (FU)^2 c \arrow[r, shift left, "\varepsilon_{FUc}"] \arrow[r, shift right, "FU\varepsilon_c"'] & FUc \arrow[r, "\varepsilon_c"] & c \end{tikzcd}$$

simplicial object called "simplicial resolution"

- Altogether, the above is an augmented simplicial object, which if $\mathcal{C}$ is abelian induces chain complex

$$\cdots \to (FU)^3 c \xrightarrow{d} (FU)^2 c \xrightarrow{d} FUc \xrightarrow{d} c$$

(Bar resolution of c)

which is always exact after applying U.

- Have adjunction $R\text{-Mod} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Ab}$ where $FA = R \otimes_{\mathbb{Z}} A$ →

@ R-module M, obtain

$$\circledast \quad R\otimes R\otimes R\otimes A \xrightarrow{d} R\otimes R\otimes A \xrightarrow{d} R\otimes A \xrightarrow{d} A.$$

- It is always exact in R-Mod, since U reflects kernels & cokernels.

- When $R = \mathbb{Z}G$ & $A = i\mathbb{Z}$, this gives

$$\begin{array}{ccccc}
\cdots \mathbb{Z}G\otimes\mathbb{Z}G\otimes\mathbb{Z} & \longrightarrow & \mathbb{Z}G\otimes\mathbb{Z} & \longrightarrow & i\mathbb{Z} \\
\text{can iso} \wr\| & & \wr\| \text{ can iso} & & \\
\cdots \mathbb{Z}G\otimes\mathbb{Z}G & & \mathbb{Z}G & & \\
\wr\| & \nearrow & & & \\
\cdots \mathbb{Z}G[G] & & & &
\end{array}$$

which is the Bar resolution from earlier (this time it captures exactly those face maps)

Hochschild cohomology

- Recall the $\mathbb{R}$-algebra $A = C^\infty(\mathbb{R},\mathbb{R})$ of inf. differentiable functions.
- We have derivative $d/dx : A \longrightarrow A$

$$f \longmapsto df/dx$$

which is R-linear & satisfies chain rule

$$d/dx(fg) = df/dx \cdot g + f\, dg/dx.$$

- let k be a field & A a k-algebra: a derivation is a k-linear map $\delta : A \to A$ such that $\delta(ab) = \delta(a).b + a.\delta(b)$.
- The set $Der(A) \subseteq k\text{-Vect}(A,A)$ is a k-module (or vector space) but not a k-algebra:

$\delta \circ \delta'$ not a derivation, but

$$[\delta,\delta'] = \delta \circ \delta' - \delta' \circ \delta \quad \in Der(A).$$

this gives $Der(A)$ the str of a Lie-algebra.

- Are there higher-dim derivations?
Yes, they appear in Hochschild cohomology.

- Consider the vector space $C^n(A)$ of k-multilinear maps $A^n \longrightarrow A$.
- Cochain complex $C^*(A)$:

$$\cdots \quad C^n(A) \xrightarrow{d^n} C^{n+1}(A) \quad \cdots$$

$$\delta \longmapsto d^n\delta : A^{n+1} \longrightarrow A$$

$$(a_1, \ldots, a_{n+1}) \longmapsto a_1\delta(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \delta(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n \delta(a_1, \ldots, a_n) a_{n+1}.$$

where $C^0(A) = 0$.

- Its cohomology is the Hochschild cohomology.
- In particular, $H^1(C^*(A)) = \mathrm{Der}(A)$, the vector space of derivations.

- Special case of Ext :
  - the k-alg A is an (A,A)-bimodule
  - (A,A)-bimodules $\equiv$ left-(A^e)-modules

where $A^e = A^{op} \otimes_k A$

- Now if M an A-bimodule, can consider

$\mathrm{Ext}_n(A, M)$ which for $A = M$ gives Hoschild cohomology.