Lecture 12 - Group representations ctd.

Recap from last time:

- G a group, k a field:
  - G-modules = vect spaces with action of G
    = kG-modules (kG the "group algebra")
  - Direct sums, kernels, quotients, images as usual.
- Reducible & irreducible G-modules
- Maschke's Theorem: under certain assumptions (G finite, $\mathrm{char}(k) \nmid |G|$ such as $k = \mathbb{R}, \mathbb{C}$) each proper G-submodule $U \leq V$ has a complement, in sense that $V = U \oplus W$)

$\Rightarrow$ under these assumptions, each non-zero fin. dim. G-mod V has a decomp

$$V = U_1 \oplus \dots \oplus U_n$$ as a direct sum of irreducibles.

(This week: finer results about this decomposition)

Prop) $k[G]$ is the free G-module on 1.

Proof] Follows from fact (last time) that

G-modules $\equiv$ $k[G]$-modules for $k[G]$ ring

& free $k[G]$-module on 1 is of course $k[G]$.

Explicitly,

$$\begin{tikzcd} k[G] \arrow[dr, dashed, "\exists! f\ G\text{-mod map}"] & \\ 1 \arrow[u, "e"] \arrow[r, "a"'] & U \end{tikzcd}$$

Must have $f(e) = a$.

Then need $f(g) = f(g.e) = g.f(e) = g.a$

& for linearity

$$f\Big(\sum_{i \in I} \lambda_i g_i\Big) = \sum_{i \in I} \lambda_i g_i . a . \quad \square$$

Schur's lemma

Let $\theta : U \longrightarrow W$ be a morph. of **irreducible** G-modules.

① Then $\theta = 0$ or θ is an **isomorphism**.

② If k is algebraically closed, & $U = W$ of finite dimension then $\exists \lambda \in k$ st

$$\theta(v) = \lambda v \text{ all } v \in V$$

~~Proof~~ ① As U is **irred**, $\ker\theta = 0$ or V.

As W **irred**, $\operatorname{im}\theta = 0$ or W.

If $\theta \neq 0$, then $\ker\theta \neq V$ & $\operatorname{im}\theta \neq 0$.

So $\ker\theta = 0$ & $\operatorname{im}\theta = W$.

Hence θ is inj & surj $\Rightarrow$ iso.

② Since k is alg. closed, the poly. in λ, $\det(\theta - \lambda.\mathrm{Id}) = 0$, has a solution (eigenvalue).

- But then as $\theta - \lambda.\mathrm{Id}$ **not invertible**, $\ker(\theta - \lambda.\mathrm{Id}) \neq 0$ so **contain** v (eigenvector).
- **But as** U is **irred**., $\ker(\theta - \lambda.\mathrm{Id}) = V$

Hence $\theta = \lambda.\mathrm{Id}$.

$\square$

Prop

Let G be finite gp st. $\mathrm{char}(K)$ does not divide $|G|$.

Let $K[G] = U_1 \oplus \dots \oplus U_n$ be a decomp. into irreducibles (last time).

Then every irreducible G-module U is iso. to one of the U_i.

Proof

- Let $v \neq 0 \in U$.
- By freeness of $K[G]$, $\exists! \, \theta : K[G] \to U$ st $\theta(e) = v$.
- Then $\mathrm{im}\,\theta \leq U$; since non-zero & U irr. $\mathrm{im}\,\theta = U$.
- Consider $\ker\theta \leq K[G]$.
- Have ses

$$0 \longrightarrow \ker\theta \xhookrightarrow{i} K[G] \xrightarrow{\theta} U \longrightarrow 0$$

which by Maschke's theorem splits:

$\exists \; K[G] \xrightarrow{p} \ker(\theta)$ st $pi = 1$ -

this implies $K[G] \cong \ker\theta \oplus U$, so U iso to a submodule of $K[G]$.

- (I will give a more elementary proof as we did not give lemma on split ses)

- By Maschke's theorem, $k[G] = \ker\theta \oplus W$.
- Consider $W \xhookrightarrow{j} \ker\theta \oplus W \xrightarrow{\theta} U$.
- Claim θj an iso, so

  $U \cong V \leq \ker\theta \oplus W = k[G]$.

- Let's show $\ker(\theta j) = 0$.
- If $\theta j v = \theta v = 0$ then $v \in \ker\theta \cap W$ $\Rightarrow v = 0$.
- As θ surj, given $u \in U$ $\exists\, a+b \in \ker\theta \oplus W$ such that

  $u = \theta(a+b) = \theta a + \theta b$

  but $\theta a = 0$ so $\theta b = u \Rightarrow \theta j$ surj.
- Hence θj an iso.

Therefore suffices to prove Theorem when U is a submodule of $K[G]$.

- For $K[G] = U_1 \oplus \cdots \oplus U_n$, consider

$$\begin{aligned} K[G] &\xrightarrow{p_i} U_i \\ u_1 + \cdots + u_n &\longmapsto u_i. \end{aligned}$$

- Since U is non-zero, one of the composites

$$U \hookrightarrow K[G] \longrightarrow U_i$$ must be non-zero, & so invertible by Schur's lemma Part 1. $\square$

Cor) Let G be finite gp st. char(K) does not divide $|G|$.
Then there are only finitely many irreducible G-modules up to iso.

Def^n) $\{U_1, \dots, U_n\}$ is a complete set of irred. G-modules if no two are iso. & every irreduc. G-module is iso to one of them.

Def^n) Let U, W be G-modules. Write $\mathrm{Hom}_{K[G]}(U, W)$ for vector space of G-module maps from U to W with pointwise operations.

Remark: $\mathrm{Hom}_{K[G]}(U, W)$ need not be a G-module unless G is commutative!

Finer results when $k = \mathbb{C}$

In this subsection, assume G is finite & $k = \mathbb{C}$.

Prop^n) Let V, W be irreducible finite-dimensional G-modules. Then

$$\dim \mathrm{Hom}_{\mathbb{C}(G)}(U, W) = \begin{cases} 1 & \text{if } U \cong W \\ 0 & \text{otherwise} \end{cases}$$

Proof

- $\dim \mathrm{Hom}_{\mathbb{C}(G)}(U, W) = 0 \iff$ only homomorphism $U \to W$ is zero $\overset{\text{(by Schur)}}{\iff}$ $U \not\cong W$.

- If $\dim \mathrm{Hom}_{\mathbb{C}(G)}(U, W) \neq 0$, $\exists$ non-zero hom. $U \to W$, which is an iso (by Schur)

- If $U \cong W$, we obtain iso of vector spaces

$$\mathrm{Hom}_{\mathbb{C}(G)}(U, U) \cong \mathrm{Hom}_{\mathbb{C}(G)}(U, W)$$

so it suffices to show lhs has dim 1.

But by Schur's lemma Part 2, each $f: U \to U$ equals $\lambda . \mathrm{Id}$ - thus lhs has basis $\{\mathrm{Id}: U \to U\}$, & so has dim. 1

$\square$

Theorem

Let $V \neq 0$ be a f.d. G-module. Then

① $V = V_1 \oplus \cdots \oplus V_n$ where the V_i are irreducible.

② Each irreducible G-module W appears in decomp., up to iso, $\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W))$ times.

③ In particular, let $U_1, \ldots, U_m$ is a complete set of irreducible G-modules. Then
$V \cong U_1^{d_1} \oplus \cdots \oplus U_m^{d_m}$ where
$d_i = \dim(\mathrm{Hom}_{\mathbb{C}G}(V, U_i))$.

Proof

- Proved ① last week.
- For ②, we have

$$\mathrm{Hom}_{\mathbb{C}G}(V, W) = \mathrm{Hom}_{\mathbb{C}G}(V_1 \oplus \cdots \oplus V_n, W)$$
$$\cong \mathrm{Hom}_{\mathbb{C}G}(V_1, W) \oplus \cdots \oplus \mathrm{Hom}(V_n, W)$$

since direct sum is a coproduct & restriction along each $V_i \hookrightarrow V$ is linear.

Taking dimensions,

$$\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W)) = \sum_{i=1}^{n} \dim(\mathrm{Hom}_{\mathbb{C}G}(V_i, W))$$
$$= \sum_{i : V_i \cong W} 1$$ by prev. proposition, ie.

the number of i st. $V_i \cong W$.

For ③, by ②,

$$U_1 \oplus \dots \oplus U_n =$$

$$\underbrace{(U_{11} \oplus \dots \oplus U_{1n_1}}_{\substack{\text{those } U_i \text{ iso to } U_1 - \\ \text{by ② there are} \\ d_1 \text{ of these}}}) \oplus \dots \oplus \underbrace{(U_{m1} \oplus \dots \oplus U_{mn_m}}_{\substack{\text{those } U_i \text{ iso to } U_m, \\ \text{of which there} \\ \text{are } d_m}})$$

$$\cong U_1^{d_1} \oplus \dots \oplus U_m^{d_m}. \quad \square$$

Corollary

Let $U_1, \dots, U_m$ be complete set of irreducibles. Then $\mathbb{C}[G] \cong U_1^{\dim(U_1)} \oplus \dots \oplus U_m^{\dim(U_m)}$.

Proof

By Part 3 of previous result,
we must show

$$d_i := \dim\left(\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i)\right) = \dim(U_i)$$

In fact, since $\mathbb{C}[G]$ is free G-module on 1, we have bij^n

$$\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], U_i) \cong U_i$$
$$F \longmapsto F(e)$$

& this evaluation map is clearly linear; hence an iso. of vector spaces.

Therefore lhs & rhs have same dimension. $\square$

Cor

$$|G| = \sum_{U_1,\dots,U_m} \dim(U_i)^2 .$$

Proof Since

$$\mathbb{C}[G] \cong U_1^{\dim(U_1)} \oplus \cdots \oplus U_m^{\dim(U_m)}$$

Taking dimensions of lhs & rhs proves claim as

$$\dim(\mathbb{C}[G]) = |G| . \quad \square$$

Remark: Above formula relating order of G with number of its irreducible reps is very useful in calculating all irreps. of a finite group.

Example

- Consider $D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$.

- By prev. Thm,

$$|D_8| = 8 = \sum_{U_1, \dots, U_m} \dim(U_i)^2$$
$$= 2^2 + 4 \cdot 1^2$$
$$= 8 \cdot 1^2$$

so 1 2-d irrep & 4 1 d irreps

or 8 1 d irreps.

- Recall 2-d real rep :

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \&$$

view as a complex 2-d rep - i.e. rep. on $\mathbb{C}^2$.

- As a 2-d rep, a non-triv. submodule must be 1-d subspace $\langle v \rangle$ s.t. $gv = \lambda v \in \langle v \rangle$ for each $g \in V$:
i.e. v should be eigenvector for both A & B.

- Can calc. eigenvectors of A which are $(1, i)$ & $(1, -i)$
& of B $(1, 0)$ & $(0, 1)$
but they have none in common.
Hence this is irreducible 2-d rep.

Therefore D_8 has one 2-d irrep & 4 1-d irreps: ie 4 1-d reps.

A 1-d rep is simply a homomorphism

$$D_8 \longrightarrow (\mathbb{C}, \cdot, 1)$$

& it is easy to see these are given by

$$a, b \longrightarrow (\pm 1, \pm 1)$$

so these are 1-d irreps. Hence we have calc. all complex irreps of D_8.