# L12 — Tensor products & a glance at characters

- Since $k$ is a field, the category $k\text{-Mod} = \text{Vect}$ is symmetric monoidal closed:

— given vector spaces $V, W$ we can form their tensor product $V \otimes W$ (Alg 3) whose elements are sums $\sum \lambda_i (v_i \otimes w_i)$: there is bilinear map $\theta : U \times W \longrightarrow U \otimes W : (u,w) \mapsto u \otimes w$ through which each $F : U \times W \longrightarrow A$ factors uniquely as

$$\begin{tikzcd} & U \otimes W \arrow[dr, dashed, "\bar{F}"] & \\ U \times W \arrow[ur, "\theta"] \arrow[rr, "F"'] & & A \end{tikzcd}$$

~ linear

This gives functor $\text{Vect} \times \text{Vect} \xrightarrow{\otimes} \text{Vect}$.

— (Unit $k$) $\quad k \otimes V \overset{\ell}{\cong} V \overset{r}{\cong} V \otimes k$.

— (Assoc.) $\quad (U \otimes V) \otimes W \overset{\alpha}{\cong} U \otimes (V \otimes W)$

— (Symmetry) $\quad U \otimes V \overset{s}{\cong} V \otimes U$

— (Internal hom) $\quad [U,V] = \text{Vect}(U,V)$ is a vect. space & have $\text{Vect}(W, [U,V]) \cong \text{Vect}(W \otimes U, V)$.

- Above holds generally if $k$ a commutative ring.

What about G-modules? They are modules over $kG$, which is not commutative.

- If $U, W$ are G-modules, the tensor product $U \otimes W$ becomes a G-module if we define
$$g.(v \otimes w) = gv \otimes gw.$$

Indeed, $g.- : U \otimes W \longrightarrow U \otimes W$ is unique linear map such that
$$\begin{tikzcd} U \times W \arrow[r, "g.- \times g.-"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "="] & U \times W \arrow[d, "\otimes"] \\ U \otimes W \arrow[r, "g.-"'] & U \otimes W \end{tikzcd}.$$

Ex) The tensor product of G-modules $U \otimes W$ classifies bilinear maps $U \times W \xrightarrow{F} A$ such that $g.F(v,w) = F(gv, gw)$ all $g, v, w$.

- $k$ becomes a G-module with trivial G-module str:
$$g.v = v.$$

- It is easy to see that the isos $\alpha, l, r, s$ lift to isos of G-modules, so that G-Mod is also a symmetric monoidal cat.

# The internal hom

- We would like to put G-module str on $[V,W]$ such that

$$
\begin{tikzcd}
U\otimes V \arrow[r, "F"] \arrow[d, "g\cdot-\otimes g\cdot-"'] \arrow[dr, phantom, "="] & W \arrow[d, "g\cdot-"] \\
U\otimes V \arrow[r, "F"'] & W
\end{tikzcd}
\iff
\begin{tikzcd}
U \arrow[r, "\bar F"] \arrow[d, "g\cdot-"'] \arrow[dr, phantom, "="] & {[V,W]} \arrow[d, "g\cdot-?"] \\
U \arrow[r, "\bar F"'] & {[V,W]}
\end{tikzcd}
$$

- Here $\bar F(u) : V \to W \ : \ v \mapsto F(u,v)$.
- The commut. of lhs equally becomes

$$
\begin{tikzcd}
U \arrow[r, "\bar F"] \arrow[dd, "g\cdot-"'] & {[V,W]} \arrow[d, "{[V,g\cdot-]}"] \\
 & {[V,W]} \\
U \arrow[r, "\bar F"'] & {[V,W]} \arrow[u, "{[g\cdot-,W]}"']
\end{tikzcd}
$$

i.e. $u \mapsto v \mapsto F(u,v)$

$v \mapsto g.F(u,v)$

&

$u \mapsto gu \mapsto$

$v \mapsto F(gu,v) \mapsto$

$v \mapsto F(gu,gv)$.

or

$$
\begin{tikzcd}
U \arrow[r, "\bar F"] \arrow[dd, "g\cdot-"'] & {[V,W]} \arrow[d, "{[V,g\cdot-]}"] \\
 & {[V,W]} \arrow[d, "{([g^{-1}\cdot-,W])}"] \\
U \arrow[r, "\bar F"'] & {[V,W]}
\end{tikzcd}
$$

$V \xrightarrow{f} W$

$\mapsto$

$V \to W : v \mapsto g.f(v)$

$\mapsto$

$V \to W : \underline{\underline{v \mapsto g.f(g^{-1}(v))}}$

so we are forced to define

$$[V,W] \xrightarrow{g\cdot -} [V,W]$$
$$f \longmapsto g.f(v) = g.f(g^{-1}(v))$$

It is easy to see this makes $[V,W]$ a $G$-module & above analysis then shows we have

$$G\text{-Mod}(U \otimes V, W) \cong G\text{-Mod}(U, [V,W]).$$

- This makes $G$-Mod a symmetric monoidal closed cat & $U : G\text{-Mod} \longrightarrow \text{Vect}$ a (strict) symmetric monoidal closed functor.

Remark

- Each $G$-module has a vector space $V^G$ of fixpoints $V^G = \{ v \in V : gv = v \}$.
- Observe that $[V,W]^G = \text{Hom}_{kG}(V,W)$ since

$$g.f = f \iff$$
$$g.f(g^{-1}(v)) = f(v) \ \forall g,v \iff$$
$$f(g^{-1}(v)) = g^{-1}(f(v)) \ \forall g,v \iff$$
$$f(g(v)) = g(f(v)) \text{ all } g,v \iff$$

$f$ a $G$-module homomorphism.

# Homs & duals

- In Vect, can form dual vector space
  $V^* = [V,K]$.
- If $V$ is fin. dim. w/ basis $E = e_1, \dots, e_n$ then $V^*$ has dual basis $E^* = e_1^*, \dots, e_n^*$ where
  $e_i^* : V \to K : e_i \mapsto 1$
  0 otherwise.
- There is a canonical linear map

$$U^* \otimes W \xrightarrow{\varphi} [V,W]$$
$$f \otimes w \longmapsto v \mapsto \underbrace{f(v)}_{\text{scalar}}.w$$

(Ex: construct using univ props)

Prop) • $U^* \otimes W \xrightarrow{\varphi} [V,W]$ is iso if $V, W$ are fin. dimensional.

- If $U, W$ are G-modules, then $\varphi$ is an iso of G-modules.

~~Proof~~ ① $(v_i)_{i \in I}$, $(w_j)_{j \in J}$ bases of $U$ & $W$.

- Then $U^* \otimes W$ has basis $(v_i^* \otimes w_j)_{(i,j) \in I \times J}$.
- Elements $x_j^i : U \to W : x_j^i(v_i) = w_j$ & 0 otherwise are basis
- $\varphi(v_i^* \otimes w_j) = x_j^i$ so $\varphi$ induces bij on basis elements $\to$ bij$^n$.

② Straightforward.

# A glance at Hopf algebras

# Characters of groups

- We assume $k = \mathbb{C}$,
- Given an $n\times n$-matrix $A = (a_{ij})$, its trace is $tr(A) = \sum_{i=1}^{n} a_{ii}$.

Lemma) (Standard props of trace)

1) $tr(A+B) = trA + trB$
2) $tr(AB) = tr(BA)$
3) $tr(B^{-1}AB) = tr(A)$
4) $tr(A\oplus B) = tr(A) + tr(B)$
5) $tr(A\otimes B) = trA \cdot trB$.

Pf) 1, 2 elementary. $2 \Rightarrow 3$.

4) $A\oplus B = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$ 5) $A\otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{n1}B \\ a_{12}B & a_{22}B & \vdots \\ \vdots & \ddots & \vdots \\ a_{1n}B & \cdots & a_{nn}B \end{pmatrix}$.

- Trace can be defined abstractly as

$$[V,V] \cong V^*\otimes V \xrightarrow{ev} k$$

Def$^n$) Let $V$ be a f.d. $G$-module, with basis $B = \{v_1, \dots, v_n\}$. Then each $g.- : V \to V$ yields a matrix $[g]_B$.
The character $\chi_V : G \to \mathbb{C}$ is the function $g \mapsto \operatorname{tr}[g]_B$.

Remark) • By $\operatorname{tr}(B^{-1}AB) = \operatorname{tr}(A)$, $\chi_V$ is indep. of choice of basis.

Def) $\chi_V$ is irreducible if $V$ is.

Prop) Isomorphic $G$-modules $V$ & $W$ have same character.

Proof) Let $V, W$ have bases $B, C$.
Then $[g]_B = T^{-1}[g]_C T$ as matrix reps are equiv.
So $\chi_V(g) = \chi_W(g)$. $\square$

We will prove that, in fact, $V$ is determined, up to iso, by its character!

Will prove it as follows:

1. Introduce inner product $\langle -,- \rangle$ on $\mathrm{Fun}(G, \mathbb{C})$, vector space of functions $G \to \mathbb{C}$.

2. $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_{G}(V,W)$.

(By Schur's lemma, implies irreducible characters are orthonormal.)

3. Now $U \cong \bigoplus_{i=1,\dots,n} V_i^{d_i}$ where $U_1, \dots, U_n$ is a complete set of irreducibles & $d_i = \dim(\mathrm{Hom}(V, V_i)) = \langle \chi_V, \chi_{V_i} \rangle$.

But then $U$ can be recovered up to iso from its character!

# Need a few technicalities:

Prop) $G$ a fin. gp, $V$ a f.d. $G$-mod over $\mathbb{C}$.

① Let $g$ have order $n$. $\exists$ basis $B$ such that $[g]_B$ is diagonal, with values $n$'th roots of unity. In partic, $\chi(g)$ is sum of $n$'th roots of unity.

② $\chi(g^{-1}) = \overline{\chi(g)}$, the complex conjugate.

Proof) ① Consider $\langle g \rangle \leq G$. $V$ a $\langle g \rangle$-module.

Since $\langle g \rangle$ abelian,

$V = V_1 \oplus \dots \oplus V_n$ with $V_i$ 1-d submods.

Taking basis $v_1, \dots, v_n$ we have $g v_i = \lambda_i v_i$ where $\lambda_i^n = 1$.

Then $\chi(g) = \sum_{i=1}^{n} \lambda_i$.

② In above basis, $g^{-1} v_i = \overline{\lambda_i} v_i$ so

$$\chi(g) = \overline{\lambda_1} + \dots + \overline{\lambda_n}$$
$$= \overline{\lambda_1 + \dots + \lambda_n} = \overline{\chi(g)}$$

# Characters of sums, tensor prods & homs

Prop) ① $\chi_{V\oplus W} = \chi_V + \chi_W$

② $\chi_{V\otimes W} = \chi_V \cdot \chi_W$

③ $\chi_{V^*} = \overline{\chi_V}$

④ $\chi_{[V,W]} = \overline{\chi_V} \cdot \chi_W$ .

Proof) ① & ② by descriptions of sum & tensor prod of matrices.

③ ⟹ ④ using ② & $[V,W] \cong V^* \otimes W$.

- For ③, recall $V^* = \mathrm{Vect}(V, \mathbb{C})$ with action $[g^{-1}, \mathbb{C}] : [V, \mathbb{C}] \longrightarrow [V, \mathbb{C}]$, which in dual basis $B^*$ has matrix $[g^{-1}]_B^T$ the transpose.
- But trace of a matrix & it's transpose are equal
so $\chi_{V^*}(g) = \mathrm{tr}[g^{-1}]_B^T = \mathrm{tr}[g^{-1}]_B = \chi_V(g^{-1})$
so $\chi_{V^*}(g) = \overline{\chi_V(g)}$ .

## Prop) Projection Formula

Let $G$ be finite a $V$ a $G$-mod over $\mathbb{C}$.

① The linear map
$$p : v \longmapsto \frac{1}{|G|}\sum_{g\in G} g(v)$$
is a projection with image $V^G$, the vector subspace of fixpoints.

② In particular, $\dim(V^G) = \frac{1}{|G|}\sum \chi_V(g)$.

**Proof** ① The first part is similar to Maschke's theorem - in fact, generalises it (Exercise) & I will not repeat it.

② Since $p$ a projn, can write $V = V^G \oplus W$ & $p = \mathrm{id}_{V^G} \oplus 0_W$. Then $\mathrm{tr}(p) = \mathrm{tr}(\mathrm{id}_{V^G}) + 0 = \dim(V^G)$.

Recalling $\mathrm{Hom}_{\mathbb{C}G}(V,W) = [V,W]^G$ &

$$\chi_{[V,W]} = \overline{\chi_V} \cdot \chi_W$$

the above results give

$$\underline{\dim(\mathrm{Hom}_{\mathbb{C}G}(V,W))} = \frac{1}{|G|}\sum_g \chi_{[V,W]}(g)$$

$$= \frac{1}{|G|}\sum_g \overline{\chi_V(g)} \cdot \chi_W(g) \,. \qquad (*)$$

## The standard inner product

- Given $f_1, f_2 : G \to \mathbb{C}$ functions, define $\langle f_1, f_2 \rangle = \frac{1}{|G|} \sum_g f_1(g)\overline{f_2(g)}$.

This is inner prod on $\mathrm{Fun}(G,\mathbb{C})$:
- linear in 1st var
- conjug. symmetric ($\langle f,g\rangle = \overline{\langle g,f\rangle}$)
- positive definite.

Cor) $\langle \chi_V, \chi_W \rangle = \dim(\mathrm{Hom}_{\mathbb{C}G}(V, W))$

Pf) $\dim(\mathrm{Hom}_{\mathbb{C}G}(V, W)) =$

$$= \frac{1}{|G|} \sum_g \overline{\chi_V(g)} \cdot \chi_W(g) = \underbrace{\phantom{x}}_{\langle \chi_W, \chi_V \rangle}$$

$$= \overline{\langle \chi_V, \chi_W \rangle}$$

$= \langle \chi_V, \chi_W \rangle$ since a natural no!

$\square$

This completes the proof that for $U_1, \ldots, U_n$ a complete set of irreducibles,

$d_i = \dim(\mathrm{Hom}(V, V_i)) = \underline{\langle \chi_V, \chi_{V_i} \rangle}$.

Thm) ① $U \cong W \iff \chi_V = \chi_W$.

② Irreducible characters are orthormal.

Proof) ① $U \cong \bigoplus U_i^{d_i}$ where $d_i = \langle \chi_V, \chi_{V_i} \rangle$

so if $\chi_U = \chi_W$, then

$U \cong W$.

② $\langle \chi_V, \chi_W \rangle = \dim(\mathrm{Hom}_G(V, W)) = \begin{cases} 1 & \text{if } U \cong W \\ 0 & \text{otherwise} \end{cases}$

but $U \cong W \iff \chi_V = \chi_W$ by preceding.

In Fact,

Thm) $V$ is irreducible $\iff \langle \chi_V, \chi_V \rangle = 1$.

Pf) One dir holds by above.

Suppose $\langle \chi_V, \chi_V \rangle = 1$.

Write $U = \bigoplus U_i^{d_i}$.

Then $\chi_V = \sum d_i \chi_{U_i}$ so

$\langle \chi_V, \chi_V \rangle = \sum d_i^2 = 1$.

As a sum of nat. nos., this can be 1

$\iff$ one $d_i = 1$ & others are 0

$\Rightarrow V \cong V_i$.

Makes it easy to test for irreducibility!

Exercise: apply to $D_8$ from last week.

- Many other interesting things about characters:
  - basis for class functions $G \to \mathbb{C}$: those invariant on conjugacy classes

  → no. of irr chars = no of conj. classes of $G$.

  - character tables - -