Lecture 12 - The symmetric group

- Goal: glance at irreps of symmetric group S_n
- Last week mentioned that irreducible characters form basis for class functions
$G \to \mathbb{C}$ $\Longrightarrow$

Theorem

For G a finite group,
no of iso classes of complex irreps of G
= no of conjugacy classes of G

Recall $a, b \in G$ are conjugate $(a \sim b)$ if
$\exists g \in G$ st $g^{-1} a g = b$.
E-classes of $\sim$ are called conjugacy classes.

- The symmetric group S_n is the group of permutations of the set $\{1, \dots, n\}$.
  Each $g \in S_n$ can be written as a product of disjoint cycles:
  eg. $(45)(132)(6) \in S_6$ & its cycle type is the list of orders of its cycles
  in this example $\{2, 3, 1\}$.

- Moreover $g, h \in S_n$ are conjugate $\iff$ they have the same cycle type.

So cycle types $\sim$ conjugacy classes

- Cycle types are parametrised by partitions of n:
  sequences $(\lambda_1, \dots, \lambda_t)$ with $\lambda_i \geq \lambda_{i+1}$
  st $\sum_{i=1}^{t} \lambda_i = n$.

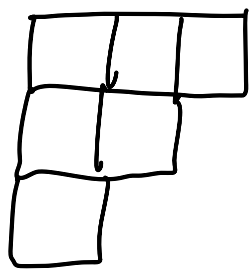
  E.g. $(3,2,1)$

- We write $\lambda \vdash n$ to indicate λ is a partition of n.

- By theorem, irreps of S_n are parametrised by partitions $\lambda \vdash n$.

- Partition $\lambda = (\lambda_1, \dots, \lambda_t) \vdash n$ can be represented by an array with t rows where i'th row has length λ_i.

  E.g. $(3,2,1) \sim$



Array called shape of the partition λ.

- A **Young tableau** t of shape $\lambda \vdash n$ (or **λ-tableau**) is an array of shape λ whose entries are bij. filled with $\{1, \dots, n\}$.

- E.g. $\begin{array}{|c|c|c|} \hline 4 & 6 & 5 \\ \hline 3 & 2 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$ is λ-tableau for $\lambda = (3,2,1)$

- Observe there is bij^n between λ-tableau & elements of S_n.

E.g. above λ-tableaux $\sim$
$1,2,3,4,5,6 \mapsto 4,6,5,3,2,1$
so **$n!$ λ-tableaux** for each $\lambda \vdash n$.

- S_n **acts** on the set of λ-tableaux in obvious way:
$(gt)_{i,j} = g(t_{i,j})$ by applying permutations g to entries of tableaux:

eg.

$$(1\,3)\ \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

- Two λ-tableaux s, t are row equivalent if entries of each row of s, t coincide.

E.g. $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}$ & $\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$ are row equiv.

- Row equivalence classes $\{t\}$ are called λ-tabloids : diagramatically remove boxes from rows

eg. $\begin{array}{cc}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array}$, $\begin{array}{cc}\hline 3 & 2 \\ \hline 1 & \\ \cline{1-1}\end{array}$ each represent the above two λ-tableau.

Lemma

The action of S_n on λ-tableaux respects row equivalence & so induces an action of S_n on set of λ-tabloids.

Proof/ Let s & t be row equiv. $(s \sim t)$.
Must show for $g \in S_n$

that $gs \sim gt$ & suffices to do this for generators – transp, $\sigma = (i\ j) \in S_n$.

Suppose $i \in \text{row}_n$ of s & t
-- $j \in \text{row}_m$ of s & t
Then $i \in \text{row}_m$ of $\sigma s, \sigma t$
$j \in \text{row}_n$ of $\sigma s, \sigma t$
which are otherwise unchanged;
hence $\sigma s \sim \sigma t$. $\square$

Let $\{t_1\}, \ldots, \{t_m\}$ be the complete set of λ-tabloids.

Def^n) We define
$$M^\lambda = \mathbb{C}(\{t_1\}, \ldots, \{t_m\})$$
to be the corresponding permutation representation (i.e. w/ basis elements $\{t_1\}, \ldots, \{t_m\}$.

Typical element of M^λ:

e.g. $$7 \begin{array}{|cc|}\hline 2 & 3 \\ \hline 1 & \\ \cline{1-1}\end{array} + \begin{array}{|cc|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array} .$$

Examples

(1) $\lambda = (n)$, only one λ-tabloid $\overline{\underline{1\,2 - - - - - - n}}$ so $M^{(n)} = \mathbb{C}(\overline{\underline{1\,2 - - - n}})$ with trivial action of S_n - ie. trivial rep of S_n.

(2) $\lambda = (1, 1, ---, 1)$ no 2 tableaux are row equiv. as rows have length 1, so a tabloid $\sim$ tableau $\sim$ elt of S_n; hence $M^{(1,1,\dots,1)} \cong \mathbb{C}\{S_n\}$ the regular representation (ie. free S_n-module on 1 {.

(3) $\lambda = (n-1, 1)$:

λ-tabloid $\sim$ choice of elt on second row.

Write $\bar{i} = \begin{array}{l} \hline - - - - - - - \\ \hline i \\ \hline \end{array}$. Hence

$M^\lambda = \mathbb{C}\{\bar{1}, ----, \bar{n}\}$ which is iso to perm. rep. ind. by action of S_n on $\{1, --, n\}$.

Polytabloids & Specht modules

Def^n) Let t be a λ-tableau.
The column stabiliser $C_t \leq S_n$ consists of those $g \in S_n$ which permute elements within each column of t.

- If t has columns $c_1, \dots, c_k$ then
$$C_t = S_{c_1} \times \dots \times S_{c_k}.$$

eg. $t = \begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array}$ then

$$C_t = S_{4,3} \times S_{1,5} \times S_2 = \{(4\,3), (1\,5), (4\,3)(1\,5), e\}$$

For t a λ-tableau, the associated polytabloid $e_t \in M^\lambda$ is the element
$$e_t = \sum_{g \in C_t} \operatorname{sign}(g).\, g\{t\} \in M^\lambda, \text{ where}$$
$\{t\}$ is λ-tabloid associated to t.

Example

In above case, $e_t =$

$$\begin{array}{|c|c|c|}\hline 4 & 1 & 2 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 3 & 1 & 2 \\ \hline 4 & 5 & \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline 4 & 5 & 2 \\ \hline 3 & 1 & \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 3 & 5 & 2 \\ \hline 4 & 1 & \\ \cline{1-2}\end{array}.$$

$\boxed{\text{Def}^n}$ The Specht module S^λ is S_n-submodule
$\langle e_t : t \text{ a } \lambda\text{-tableau} \rangle \leq M^\lambda$

Remark

One can show $g e_t = e_{gt}$: hence S^λ consists of linear combinations of the ass. polytabs e_t.

Theorem (see eg. notes on my webpage)

The Specht modules S^λ are irreducible & form a complete set of irreducible S_n-modules for $\lambda \vdash n$.

Examples

① $\lambda = (n)$, one λ-tabloid $\overline{\underline{1\,2\,\cdots\,n}}$.

For each tableau t, C_t is trivial, hence $e_t = \overline{\underline{1\,2\,\cdots\,n}}$, the unique λ-tabloid. Then $S^{(n)} = M^{(n)} = \mathbb{C}(\overline{\underline{1\,2\,\cdots\,n}})$ the trivial S_n-module.

② $\lambda = (1,1,\dots,1)$. $M^\lambda \cong \mathbb{C}\{S_n\}$.

Let $t = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \vdots \\ \hline n \\ \hline \end{array}$. Then $C_t = S_n$. Will show $e_{\pi t} = \mathrm{sgn}(\pi)\, e_t$ - hence $S^\lambda = \langle e_t \rangle \leq M^\lambda$ so $S^\lambda \cong \mathbb{C}$ with so-called sign representation $g.a = \mathrm{sgn}(g).a$.

Proof of claim:

$$e_{\pi t} = \pi e_t = \pi \sum_\sigma \mathrm{sign}(\sigma)\, \sigma\{t\}$$

not prove, but true for all λ, t

$$= \sum_\sigma \mathrm{sign}(\sigma)\, \pi\sigma\{t\}$$

$$= \sum_\sigma \mathrm{sign}(\pi^{-1}\sigma)\, \pi(\pi^{-1}\sigma\{t\})$$

$$= \mathrm{sign}\,\pi^{-1} \sum_\sigma \mathrm{sign}\,\sigma\ \sigma\{t\}$$

$$= \mathrm{sign}(\pi)\, e_t.$$

The above are two irred. 1-d reps.

③ $\lambda = (n-1,1)$, $M^\lambda = \mathbb{C}\{\bar{1}, \bar{2}, \dots, \bar{n}\}$

let $t = \begin{array}{|c|c|c|c|c|}\hline i & - & - & - & \cdots \\ \hline k \\ \cline{1-1}\end{array}$ so $\{t\} = \bar{k}$.

Then $C_t = \{e, (ik)\}$ so $e_t = \bar{k} - \bar{i}$.

- Thus $S^\lambda = \underline{\langle \bar{i} - \bar{j} : i \neq j \rangle} \leq M^\lambda$ & this spans subspace $\{c_1\bar{1} + \dots c_n\bar{n} : \sum_{i=1}^{n} c_i = 0\}$ & has basis the vectors $\{\bar{i} - \bar{1} : i \neq 1\}$ & so is of dim. $\underline{n-1}$.

- Saw this example for S_3 as $\langle 3-2, 2-1 \rangle \leq \mathbb{C}\{1,2,3\}$ in **homework**

Final note on basis

A λ-tableau is <u>standard</u> if its rows & columns form increasing sequences:

eg
$$\begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 4 & \\ \cline{1-2} 5 & & \\ \cline{1-1} \end{array} \quad \text{but not} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 4 & 3 & \\ \cline{1-2} 5 & & \\ \cline{1-1} \end{array}.$$

Theorem

The set $\{e_t : t \text{ standard } \lambda\text{-tableau}\}$ form a basis for S^λ.

Remark:

Lots of connections between reps of symmetric group & other areas:

- combinatorics, probability (eg. card shuffling) ...

eg. see "The symm group: reps, combinatorial algo & symm. functions".