

Lecture 6

Cohomology

- So far we have talked about chain complexes & homology.
- Dually we have cochain complexes & cohomology.

Def) A cochain complex is a diagram

$$\dots X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \dots \text{ st.}$$
$$d^n \circ d^{n-1} = 0 \text{ all } n \in \mathbb{Z}.$$

Remark) Equivalently a chain complex in the opposite abelian cat $(R\text{-Mod})^{\text{op}}$ - see next week.

- Everything about chain complexes has a dual version for cochain complexes.
- The n th cohomology of a cochain complex X is defined as $H^n X := \ker d^n / \text{im } d^{n-1}$.

Examples

① IF X is a ch. complex of R -modules & A an R -module then

$$\begin{array}{ccc} \dots R\text{-Mod}(X_n, A) & \xrightarrow{R\text{-Mod}(d^{n+1}, A)} & R\text{-Mod}(X_{n+1}, A) \dots \\ X_n \xrightarrow{F} A & \longmapsto & X_{n+1} \xrightarrow{d} X_n \xrightarrow{f} A \end{array}$$

is a cochain complex $R\text{-Mod}(X, A)$ in Ab .

② Recall if X a top space, can form $SX \in \text{Ch}(\text{Ab})$, whose homology is the singular homology of X .

If A is an abelian group, then $H_n(\text{Ab}(SX, A))$ is the so-called

singular cohomology of X
of X with coefficients in A .

Abelian categories

- What is correct context for homological algebra from a categorical perspective?
- Must talk about zero map, add and subtract morphisms & need to form kernels, images & quotients & these should behave as in $\text{Mod } R$.
- Resulting notion: abelian category

Defⁿ) A pre-additive category (or Ab-enriched cat)

\mathcal{C} is a cat. in which hom-set $\mathcal{C}(a, b)$ has the structure of an abelian group -

(ie. we have $a \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} b \mapsto a \begin{array}{c} \xrightarrow{f+g} \\ \xrightarrow{0} \end{array} b, a \begin{array}{c} \xrightarrow{-f} \\ \xrightarrow{0} \end{array} b, a \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{0} \end{array} b$)

and moreover pre & postcomposition preserve the abelian group structure:

(given $x \xrightarrow{r} a \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} b \xrightarrow{s} y$ we have
 $(f+g)r = fr + gr$ & $s(f+g) = sf + sg$.
 $0 \cdot r = 0$ & $s \cdot 0 = 0$)

Example

- $\text{Mod } R$ is pre-additive :

Given $M \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} N$, $f+g : M \rightarrow N$ is defined by $(f+g)x = fx + gx$ which is an abelian gp. homom.

by commut. of $+$ & $(f+g)(rx) = f(rx) + g(rx)$
 $= r(fx) + r(gx)$
 $= r(fx + gx)$
 $= r(f+g)(x)$.

- $M \xrightarrow{0} N$ is constant at 0.

- $M \xrightarrow{-f} N : x \mapsto -(fx)$.

Given $A \xrightarrow{r} M \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} N \xrightarrow{s} B$

we have

- $s(f+g) = sf + sg$ as s a homomorphism
 $s(0) = 0$

whilst $(f+g)r = fr + gr$
 $Or = 0$

is trivial.

Remark

\mathcal{C} is pre-additive $\Leftrightarrow \mathcal{C}^{\text{op}}$ is.

Therefore we can apply duality to pre-additive categories.

Proposition

Let \mathcal{C} be a pre-additive category.

① Then \mathcal{C} has a terminal object \Leftrightarrow it has an initial object. In this case, they coincide.

② Then \mathcal{C} has binary products \Leftrightarrow it has binary coproducts. In this case, they coincide.

Proof

① let t be terminal.

- Then $t \xrightarrow{0} t = t \xrightarrow{\text{id}_t} t$

- Now given x , we have $0 : t \rightarrow x$ & must show it is unique, so consider $t \xrightarrow{f} x$.

- Then $f = f \circ \text{id}_t = f \circ 0 = 0$. Hence t is initial.

- The converse, starting from initial object, is dual.

note: often for identity, I will write 1_t or just 1

(2) let $\begin{array}{ccc} & p_1 \rightarrow a \\ c & \nearrow \\ & p_2 \rightarrow b \end{array}$ be a product diagram.

- We have

$\begin{array}{ccc} a & \xrightarrow{1} & a \\ & \searrow & \\ & & b \end{array}$ induces $a \xrightarrow{i_1 = \langle 1, 0 \rangle} c$ & sim. $\begin{array}{ccc} & \circ \rightarrow a \\ b & \nearrow \\ & & b \end{array}$ ind. $b \xrightarrow{i_2 = \langle 0, 1 \rangle} c$

So get $\begin{array}{ccc} a & \xrightarrow{i_1} & c \\ & \searrow & \\ b & \nearrow & \\ & & i_2 \end{array}$ which we must show is a coprod. diagram.

- The key point is to observe that the diagram $\begin{array}{ccc} & \xrightarrow{i_1} & c \\ a & \xleftarrow{p_1} & \\ & \xleftarrow{i_2} & b \end{array}$ satisfies

- $p_1 i_1 = 1_a, p_2 i_1 = 0$
- $p_2 i_2 = 1_b, p_1 i_2 = 0$
- $i_1 p_1 + i_2 p_2 = 1_c$

Such a diagram is called a biproduct diagram.

To see $i_1 p_1 + i_2 p_2 = 1_c$, we use the univ. prop of the product. We check:

$$\begin{aligned} p_1 (i_1 p_1 + i_2 p_2) &= p_1 i_1 p_1 + p_1 i_2 p_2 \\ &= 1_a p_1 + 0 \\ &= p_1 \quad \text{\& similarlv} \end{aligned}$$

$$p_2 (i_1 p_1 + i_2 p_2) = p_2.$$

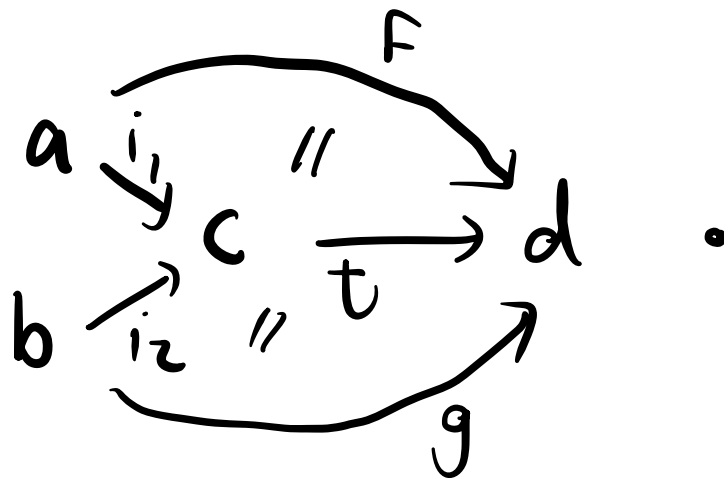
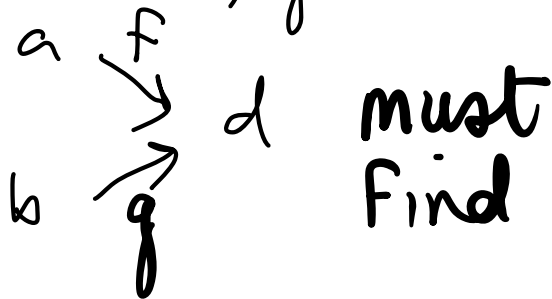
Any biproduct diagram

$$a \begin{array}{c} \xrightarrow{i_1} \\ \xleftarrow{p_1} \end{array} c \begin{array}{c} \xleftarrow{i_2} \\ \xrightarrow{p_2} \end{array} b$$

- $p_1 i_1 = 1_a$, $p_2 i_1 = 0$
- $p_2 i_2 = 1_b$, $p_1 i_2 = 0$
- $i_1 p_1 + i_2 p_2 = 1_c$

is in fact both a product & coproduct diagram. Let's show the latter.

- Indeed, given



But then $t = t \cdot 1 = t(i_1 p_1 + i_2 p_2) = f p_1 + g p_2$.

let's check $f p_1 + g p_2$ has required props:

$$(f p_1 + g p_2) i_1 = f p_1 i_1 + g p_2 i_1 = f \cdot 1 + g \cdot 0 = f$$

& sim. $(f p_1 + g p_2) i_2 = g$, as required.

Converse is dual. \square

Defⁿ A preadditive cat is additive if it has finite products (equivalently by above, finite coproducts).

Example $\text{Mod } R$ is additive. Prop 3 generalises fact from Algebra 3 that in $\text{Mod } R$ finite products & coproducts coincide.

Notation) In an additive cat \mathcal{C} , let 0 be the terminal = initial ob.

Then

$$\begin{array}{ccc} A & \xrightarrow{0_{A,B}} & B \\ & \searrow \downarrow & \nearrow \uparrow \\ & 0 & \end{array}$$

Kernels & quotients

- let \mathcal{C} be a (pre) additive category.

- Given $f: a \rightarrow b \in \mathcal{C}$ the kernel of f is an object $\ker f \in \mathcal{C}$ & arrow $i: \ker f \rightarrow a$ such that

$\ker f \xrightarrow{i} a \xrightarrow{f} b = 0$ & it is universal with this property:

given $c \xrightarrow{j} a \xrightarrow{f} b = 0$ $\exists! \bar{j}: c \rightarrow \ker f$ such that

$$\begin{array}{ccc} c & & \\ \bar{j} \downarrow & \searrow j & \\ \ker f & \xrightarrow{i} & a \end{array}$$

- The cokernel of f is dual:

it is a map $b \xrightarrow{p} \operatorname{coker} f$ such that $a \xrightarrow{f} b \xrightarrow{p} \operatorname{coker} f = 0$

& if $a \xrightarrow{f} b \xrightarrow{q} c = 0$

then $\exists! \bar{q}: \operatorname{coker} f \rightarrow c$ such that

$$\begin{array}{ccc} b & & \\ p \downarrow & \searrow q & \\ \operatorname{coker} f & \xrightarrow{\bar{q}} & c \end{array}$$

Remark: - Equivalently, $\ker f$ & $\operatorname{coker} f$ are the equaliser & coequaliser of $a \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{0} \end{array} b$.

- Note this implies that

$$i: \ker f \rightarrow a \text{ is } \underline{\text{mono}}$$
$$\& p: \operatorname{coker} f \rightarrow b \text{ is } \underline{\text{epi}}.$$

Example

$\operatorname{Mod} R$ has kernels & cokernels.

• Given $f: A \rightarrow B$, $\ker f = \{x: fx=0\}$

• $\operatorname{coker} f = B/\operatorname{im} f$

• If $a \xrightarrow{i} b$ is mono in \mathcal{C} , one often writes $b/a := \operatorname{coker}(i)$

Lemma

In an additive category \mathcal{C} ,

① $A \xrightarrow{f} B$ is mono $\Leftrightarrow \ker f = 0$

② $A \xrightarrow{f} B$ is epi $\Leftrightarrow \operatorname{coker} f = 0$.

Proof


• By duality, it suffices to prove ①

- let $A \xrightarrow{f} B$ be mono & consider $0 \rightarrow A \xrightarrow{f} B$.

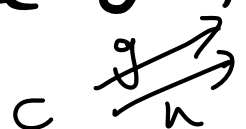
- Given $C \xrightarrow{g} A$, if $f \circ g = 0_{0,B} = f \circ 0_{C,A}$ then as f is mono,

then $g = 0 : C \rightarrow A$.

Then $C \xrightarrow{g=0} A$ gives the factorisation, which is clearly unique, so $\ker f = 0$.

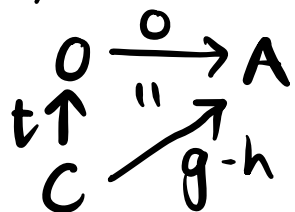


Conversely let top row be $0 \rightarrow A \xrightarrow{f} B$ be kernel.



If $f \circ g = f \circ h$, then $f(g-h) = 0$ so $\exists ! t : C \rightarrow 0$

such that $0 \xrightarrow{0} A$ but then $g-h = 0 \cdot t = 0$ so $g=h \Rightarrow$ f mono.



□

In an additive category with kernels & cokernels, we can factor each arrow in 2 ways

(we will write kf, cf instead of $\ker f, \operatorname{coker} f$)

① $kf \xrightarrow{i} a \xrightarrow{f} b$ where t is unique
 $\quad \quad \quad \downarrow p \quad \quad \quad \uparrow t$ map induced by
 $\quad \quad \quad ckf$ $f_i = 0$ & universal prop
of ckf .

② $a \xrightarrow{f} b \xrightarrow{q} cf$ where l is unique
 $\quad \quad \quad \downarrow e \quad \quad \quad \uparrow m$ map induced by
 $\quad \quad \quad kcf$ $qf = 0$ & univ. prop of
 kcf .

As $0 = qf = qt p$ & p is epi, therefore
 $qt = 0$ so

we get unique map $ckf \xrightarrow{\alpha} kcf$
such that

$$\begin{array}{ccc}
 a & \xrightarrow{f} & b \\
 \downarrow p & & \uparrow m \\
 ckf & \xrightarrow{\alpha} & kcf
 \end{array}$$

Defⁿ) An abelian cat \mathcal{C} is an additive cat
w' kernels & cokernels such that
 $\alpha: ckf \rightarrow kcf$ is an iso for all f .

$$\begin{array}{ccc}
 a & \xrightarrow{f} & b \\
 \downarrow p & & \downarrow m \\
 \text{ckf} & \xrightarrow{\alpha} & \text{kcf}
 \end{array}$$

Defⁿ) An abelian cat \mathcal{C} is an additive cat w' kernels & cokernels such that $\alpha: \text{ckf} \rightarrow \text{kcf}$ is an iso for all f .

Example

Mod R is abelian

Given $f: A \rightarrow B$,

• Recall $\text{ker} f \hookrightarrow A$ the usual kernel
& $\text{coker} f = B / \text{im} f$.

• Then $\text{ck} f = c(i) = B / \text{ker} f$.

$B / \text{ker} f$ (as $\text{ker} f \hookrightarrow A$
is mono)

• & $\text{kcf} = k(B \rightarrow B / \text{im} f) = \text{im} f \hookrightarrow B$,

& induced map $\alpha: \text{ck} f \rightarrow \text{kcf}$ is

$$A / \text{ker} f \xrightarrow{\alpha} \text{im} f$$

$$[x] \longmapsto fx \text{ which is } \underline{\text{iso}}$$

(First iso thm From Alg 3)

- In an abelian category, the two ways of factorising f (1) & (2), above coincide (up to unique iso) & we write

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 e \searrow & & \nearrow m \\
 & \text{im } f & \\
 & \text{ii} & \\
 & \text{ker coker } f & \\
 & \text{where ker } f &
 \end{array}$$

for this common factorisation.

In particular e is epi & m is mono.

- Below is another characterisation of abelian cats - we prove only 1 direction.

Note: (2) implies epis are regular epis in an abelian cat!

Prop) Let \mathcal{C} be additive w' kernels & cokernels. Then \mathcal{C} is abelian \iff

- (1) Each mono is the kernel of its cokernel.
- (2) Each epi is the cokernel of its kernel.

Proof) We will only show \implies

- Let $A \xrightarrow{f} B$ be mono, so $kf=0$.

- Then
$$\begin{array}{ccccc}
 0 & \xrightarrow{0} & A & \xrightarrow{f} & B & \xrightarrow{p} & cf \\
 & & \downarrow p=1 & \nearrow t=f & \parallel & \nearrow m & \\
 kf=0 & & A & \xrightarrow[\alpha]{\sim} & Kcf & &
 \end{array}$$

- This shows that f is iso to the kernel of its cokernel, as required.

- To show each epi is cokernel of kernel is dual.

- I leave \Leftarrow to interested reader.

We won't use it. \square

Examples

① $\text{Mod } R$

② If A is abelian, so is $A^{\mathcal{P}}$.

This allows us to apply duality to abelian cats.

③ If A is abelian, we can consider chain complexes in A :

$$\cdots A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \rightarrow \cdots$$

$\underbrace{\hspace{10em}}_{\parallel} \circ$

We have $\text{im}(d_{n+1}) \hookrightarrow A_n$ & so can form

$\begin{array}{c} \downarrow \\ \text{Ker}(d_n) \end{array} \nearrow$

homology

$H_n(A) := \text{Ker}(d_n) / \text{im}(d_{n+1})$ as before.

We can also speak of exact sequences, chain maps, homotopies in \mathcal{A} & all the results from last weeks hold in this setting.

- We can form the cat $\text{Ch}(\mathcal{A})$ of chain complexes in \mathcal{A} & it is again abelian: indeed $(f+g)_n = f_n + g_n$ for chain maps & kernels & cokernels are also constructed componentwise.

- Homology is a functor $H_n: \text{Ch}(\mathcal{A}) \rightarrow \mathcal{A}$ as before.

Remark) A cochain complex in \mathcal{A} is a chain complex in \mathcal{A}^{op} . In fact,
 $\text{Cochain}(\mathcal{A}) = (\text{Ch}(\mathcal{A}^{\text{op}}))^{\text{op}}$!

④ If C is a small cat & A abelian, the functor cat $[C, A]$ is abelian with componentwise structure.

⑤ If X is a top. space we can look at the poset $O(X)$ of open sets of X . A presheaf of R -modules is a functor $O(X)^{op} \rightarrow \text{Mod}_R$.

• By the above the cat $[O(X)^{op}, \text{Mod}_R]$ is abelian.

• So is its full subcat $\text{Sh}(X, \text{Mod}_R)$ of sheaves & this is important in alg. geometry.