Lecture 6

Cohomology

- So far we have talked about chain complexes & homology.
- Dually we have cochain complexes & cohomology.

Def) A cochain complex is a diagram

$$\cdots X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \cdots \text{ s.t.}$$

$$d^n \circ d^{n-1} = 0 \text{ all } n \in \mathbb{Z}.$$

Remark) Equivalently a chain complex in the opposite abelian cat $(R\text{-Mod})^{op}$ - see next week.

- Everything about chain complexes has a dual version for cochain complexes.

- The nth cohomology of a cochain complex X is defined as

$$H^n X := \ker d^n / \operatorname{im} d^{n-1}.$$

Examples

① If X is a ch. complex of R-modules & & A an R-module then

$$\cdots R\text{-}Mod(X_n, A) \xrightarrow{R\text{-}Mod(d^{n+1}, A)} R\text{-}Mod(X_{n+1}, A) \cdots$$

$$X_n \xrightarrow{f} A \longmapsto X_{n+1} \xrightarrow{d} X_n \xrightarrow{f} A$$

is a cochain complex $R\text{-}Mod(X, A)$ in Ab.

② Recall if X a top space, can form $SX \in Ch(Ab)$, whose homology is the singular homology of X.

If A is an abelian group, then $H_n(Ab(SX, A))$ is the so-called singular cohomology of X of X with coefficients in A.

Abelian categories

- What is correct context for homological algebra from a categorical perspective?

- Must talk about zero map, add and subtract morphisms & need to form kernels, images & quotients & these should behave as in Mod_R.

- Resulting notion : abelian category

Def^n) A pre-additive category (or Ab-enriched cat) $\mathcal{C}$ is a cat. in which hom-set $\mathcal{C}(a,b)$ has the structure of an abelian group -

$$\left(\text{i.e. we have } a \overset{f}{\underset{g}{\rightrightarrows}} b \mapsto a \xrightarrow{f+g} b,\ a \xrightarrow{-f} b,\ a \xrightarrow{0} b \right)$$

and moreover pre & postcomposition preserve the abelian group structure :

$$\left(\begin{array}{l} \text{given } x \xrightarrow{r} a \overset{f}{\underset{g}{\rightrightarrows}} b \xrightarrow{s} y \text{ we have} \\ (f+g)r = fr+gr \quad \& \quad s(f+g) = sf+sg. \\ 0.r = 0 \quad \& \quad s.0 = 0 \end{array} \right)$$

Example

- Mod_R is pre-additive :

Given $M \overset{f}{\underset{g}{\rightrightarrows}} N$, $f+g : M \longrightarrow N$ is **defined** by $(f+g)x = fx + gx$ which is an **abelian gp. homom.** **by** commut. **of** $+$ & $(f+g)(rx) = f(rx) + g(rx)$

$$= r(fx) + r(gx)$$
$$= r(fx + gx)$$
$$= r(f+g)(x).$$

- $M \xrightarrow{0} N$ is constant at 0.
- $M \xrightarrow{-f} N : x \longmapsto -(fx)$.

Given $A \xrightarrow{r} M \overset{f}{\underset{g}{\rightrightarrows}} N \xrightarrow{s} B$

we have

- $s(f+g) = sf + sg$ as s a **homomorphism**
$s(0) = 0$

whilst $(f+g)r = fr + gr$
$0r = 0$ is **trivial**.

Remark

$\mathcal{C}$ is pre-additive $\iff$ $\mathcal{C}^{op}$ is.
Therefore we can apply duality to pre-additive categories.

Proposition

Let $\mathcal{C}$ be a pre-additive category.

① Then $\mathcal{C}$ has a terminal object $\iff$ it has an initial object. In this case, they coincide.

② Then $\mathcal{C}$ has binary products $\iff$ it has binary coproducts. In this case, they coincide.

Proof

① Let t be terminal.

- Then $t \xrightarrow{0} t = t \xrightarrow{id_t} t$
- Now given x, we have $0 : t \longrightarrow x$ & must show it is unique, so consider $t \xrightarrow{f} x$.
- Then $f = f \circ id_t = f \circ 0 = 0$. Hence t is initial.
- The converse, starting from initial object, is dual.

note: often for identity, I will write 1_t or just 1

② let
$$\begin{tikzcd} & a \\ c \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & \\ & b \end{tikzcd}$$
be a product diagram.

- We have
$$\begin{tikzcd} & a \\ a \arrow[ur, "1"] \arrow[dr, "0"'] & \\ & b \end{tikzcd}$$
induces $a \xrightarrow{i_1 = \langle 1, 0 \rangle} c$ & sim.
$$\begin{tikzcd} & a \\ b \arrow[ur, "0"] \arrow[dr, "1"'] & \\ & b \end{tikzcd}$$
ind. $b \xrightarrow{i_2 = \langle 0, 1 \rangle} c$

So get
$$\begin{tikzcd} a \arrow[dr, "i_1"] & \\ & c \\ b \arrow[ur, "i_2"'] & \end{tikzcd}$$
which we must show is a coprod. diagram.

- The key point is to observe that the diagram
$$\begin{tikzcd} a \arrow[r, shift left, "i_1"] & c \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & b \arrow[l, shift right, "i_2"'] \end{tikzcd}$$
satisfies

- $p_1 i_1 = 1_a$, $p_2 i_1 = 0$
- $p_2 i_2 = 1_b$, $p_1 i_2 = 0$
- $i_1 p_1 + i_2 p_2 = 1_c$

Such a diagram is called a biproduct diagram.

To see $i_1 p_1 + i_2 p_2 = 1_c$, we use the univ. prop of the product. We check:

$$\begin{aligned} p_1(i_1 p_1 + i_2 p_2) &= p_1 i_1 p_1 + p_1 i_2 p_2 \\ &= 1_a p_1 + 0 \\ &= p_1 \end{aligned}$$
& similarly
$$p_2(i_1 p_1 + i_2 p_2) = p_2.$$

Any biproduct diagram

$$\begin{tikzcd} a \arrow[r, shift left, "i_1"] & c \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & b \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

$$\boxed{\begin{array}{l} \cdot\ p_1 i_1 = 1_a, \quad p_2 i_1 = 0 \\ \cdot\ p_2 i_2 = 1_b, \quad p_1 i_2 = 0 \\ \cdot\ i_1 p_1 + i_2 p_2 = 1_c \end{array}}$$

is in fact both a product & coproduct diagram. Let's show the latter.

- Indeed, given

$$\begin{tikzcd} a \arrow[dr, "f"] & \\ & d \\ b \arrow[ur, "g"'] & \end{tikzcd}$$

must find

$$\begin{tikzcd} a \arrow[dr, "i_1"] \arrow[drr, bend left, "f"] & & \\ & c \arrow[r, "t"] & d \\ b \arrow[ur, "i_2"'] \arrow[urr, bend right, "g"'] & & \end{tikzcd}$$

But then $t = t.1 = t(i_1p_1 + i_2p_2) = fp_1 + gp_2$.

let's check $fp_1 + gp_2$ has required props:

$(fp_1+gp_2)i_1 = fp_1i_1 + gp_2i_1 = f.1 + g.0 = f$

& sim. $(fp_1+gp_2)i_2 = g$, as required.

Converse is dual. $\square$

$\boxed{\text{Def}^n}$ A preadditive cat is <u>additive</u> if it has finite products (equivalently by above, finite coproducts).

$\boxed{\text{Example}}$ Mod_R is additive. Prop 3 generalises fact from Algebra 3 that in Mod_R finite products & coproducts coincide.

Notation) In an additive cat $\mathcal{C}$, let 0 be the terminal = initial ob.

Then
$$\begin{tikzcd} A \arrow[rr, "0_{A,B}"] \arrow[dr, "!"'] & {} \arrow[d, phantom, "="] & B \\ & 0 \arrow[ur, "!"'] & \end{tikzcd}$$

Kernels & quotients

- Let $\mathcal{C}$ be a (pre)additive category.
- Given $f : a \to b \in \mathcal{C}$ the <u>kernel of f</u> is an object $\ker f \in \mathcal{C}$ & arrow $i : \ker f \to a$ such that

$$\ker f \xrightarrow{i} a \xrightarrow{f} b = 0$$

& it is universal with this property: given $c \xrightarrow{j} a \xrightarrow{f} b = 0$ $\exists!\ c \xrightarrow{\bar{j}} \ker f$ such that

$$\begin{tikzcd} c \arrow[d, "\bar{j}"'] \arrow[dr, "j"] & \\ \ker f \arrow[r, "i"'] & a \end{tikzcd}$$

- The <u>cokernel of f</u> is dual: it is a map $b \xrightarrow{p} \operatorname{coker} f$ such that $a \xrightarrow{f} b \xrightarrow{p} \operatorname{coker} f = 0$ & if $a \xrightarrow{f} b \xrightarrow{g} c = 0$ then $\exists!\ \operatorname{coker} f \xrightarrow{\bar{g}} c$ such that

$$\begin{tikzcd} b \arrow[d, "p"'] \arrow[dr, "g"] & \\ \operatorname{coker} f \arrow[r, "\bar{g}"'] & c \end{tikzcd}$$

Remark:- Equivalently, $\ker f$ & $\operatorname{coker} f$ are the equaliser & coequaliser of $a \underset{0}{\overset{f}{\rightrightarrows}} b$.

- Note this implies that

$$i : \ker f \to a \text{ is mono}$$
$$\& \quad p : \operatorname{coker} f \to b \text{ is epi.}$$

Example

Mod$_R$ has kernels & cokernels.

- Given $f : A \to B$, $\ker f = \{x : fx = 0\}$
- $\operatorname{coker} f = B/\operatorname{im} f$
- If $a \overset{i}{\hookrightarrow} b$ is mono in $\mathcal{C}$, one often writes $b/a := \operatorname{coker}(i)$

Lemma

In an additive category $\mathcal{C}$,

1. $A \xrightarrow{f} B$ is mono $\iff \ker f = 0$
2. $A \xrightarrow{f} B$ is epi $\iff \operatorname{coker} f = 0$.

Proof

- By duality, it suffices to prove 1

- Let $A \xrightarrow{f} B$ be mono & consider $0 \xrightarrow{0} A \xrightarrow{f} B$.

- Given $C \xrightarrow{g} A$, if $f \circ g = 0_{0,B} = f \circ 0_{C,A}$ then as f is mono, then $g = 0 : C \to A$.
Then

$$\begin{tikzcd} C \arrow[rr, "g=0"] \arrow[dr, "0"'] \arrow[rr, phantom, shift right=3, "\shortparallel"] & & A \\ & 0 \arrow[ur, "0"'] & \end{tikzcd}$$

gives the factorisation, which is clearly unique, so $\ker f = 0$.

Conversely let top row be $0 \xrightarrow{0} A \xrightarrow{f} B$ be kernel.

$$\begin{tikzcd} & A \arrow[r, "f"] & B \\ C \arrow[ur, shift left, "g"] \arrow[ur, shift right, "h"'] & & \end{tikzcd}$$

If $fg = fh$, then $f(g-h) = 0$ so $\exists!\, t : C \to 0$ such that

$$\begin{tikzcd} 0 \arrow[r, "0"] & A \\ C \arrow[u, "t"] \arrow[ur, "g-h"'] \arrow[ur, phantom, shift left=3, "\shortparallel"] & \end{tikzcd}$$

but then $g - h = 0 \cdot t = 0$ so $g = h \implies f$ mono.

$\square$

In an additive category with kernels & cokernels, we can factor each arrow in 2 ways

(will write kf, cf instead of $\ker f$, $\operatorname{coker} f$)

① $$\begin{tikzcd} kf \arrow[r, "i"] & a \arrow[rr, "f"] \arrow[dr, "p"'] & & b \\ & & ckf \arrow[ur, "t"'] & \end{tikzcd}$$

where t is unique map induced by $fi = 0$ & universal prop of ckf.

② $$\begin{tikzcd} a \arrow[rr, "f"] \arrow[dr, "\ell"'] & & b \arrow[r, "q"] & cf \\ & kcf \arrow[ur, "m"'] & & \end{tikzcd}$$

where ℓ is unique map induced by $qf = 0$ & univ. prop of kcf.

As $0 = qf = qtp$ & p is epi, therefore $qt = 0$ so

we get unique map $ckf \xrightarrow{\alpha} kcf$

such that
$$\begin{tikzcd} a \arrow[rrr, "f"] \arrow[dr, "p"'] & & & b \\ & ckf \arrow[r, "\alpha"] \arrow[ur, phantom, "||"] & kcf \arrow[ur, "m"'] & \end{tikzcd}$$

Def^n) An abelian cat $\mathcal{C}$ is an additive cat w/ kernels & cokernels such that $\alpha : ckf \to kcf$ is an iso for all f.

$$\begin{tikzcd} a \arrow[rrr, "f"] \arrow[d, "p"'] & & & b \\ \operatorname{ck} f \arrow[r, "\alpha"] & \operatorname{kc} f \arrow[urr, "m"'] & & \end{tikzcd}$$

Def^n) An abelian cat $\mathcal{C}$ is an additive cat w/ kernels & cokernels such that $\alpha : \operatorname{ck} f \to \operatorname{kc} f$ is an iso for all f.

Example

Mod_R is abelian

Given $f : A \to B$,

- Recall $\ker f \overset{i}{\hookrightarrow} A$ the usual kernel & $\operatorname{coker} f = B/\operatorname{im} f$.
- Then $\operatorname{ck} f = c(i) = B/\ker f$.

$B/\ker f$ (as $\ker f \hookrightarrow A$ is mono)

- & $\operatorname{kc} f = k(B \to B/\operatorname{im} f) = \operatorname{im} f \hookrightarrow B$,

& induced map $\alpha : \operatorname{ck} f \to \operatorname{kc} f$ is

$$A/\ker f \xrightarrow{\alpha} \operatorname{im} f$$
$$[x] \longmapsto fx$$

which is iso

(First iso thm from Alg 3)

- In an abelian category, the two ways of factorising ① & ②, above coincide (up to unique iso) & we write

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\ & \operatorname{im} f \arrow[ur, "m"'] & \end{tikzcd}$$

$$\operatorname{im} f = \ker \operatorname{coker} f = \operatorname{coker} \ker f.$$

for this common factorisation.

In particular e is epi & m is mono.

- Below is <u>another characterisation of abelian cats</u> – we prove only 1 direction.

Note: ② implies epis are regular epis in an abelian cat!

Prop) Let $\mathcal{C}$ be additive w/ kernels & cokernels. Then $\mathcal{C}$ is abelian $\iff$

① Each mono is the kernel of its cokernel.
&
② Each epi is the cokernel of its kernel.

Proof) We will only show $\Longrightarrow$.

- Let $A \xrightarrow{f} B$ be mono, so $kf = 0$.
- Then

$$\begin{tikzcd} kf = 0 \arrow[r, "0"] & A \arrow[r, "f"] \arrow[d, "p=1"'] & B \arrow[r, "p"] & cf \\ & ckf = A \arrow[r, "\sim", "\alpha"'] \arrow[ur, "t=f"] & kcf \arrow[u, "m"'] & \end{tikzcd}$$

- This shows that f is iso to the kernel of its cokernel, as required.
- To show each epi is cokernel of kernel is <u>dual</u>.
- I leave $\Leftarrow$) to interested reader. We won't use it. □

Examples

① Mod_R

② If A is abelian, so is A^{op}.

This allows us to apply duality to abelian cats.

③ If A is abelian, we can consider chain complexes in A:

$$\begin{tikzcd} \cdots A_{n+1} \arrow[r, "d_{n+1}"] \arrow[rr, bend right, "0"'] & A_n \arrow[r, "d_n"] & A_{n-1} \arrow[r] & \cdots \end{tikzcd},$$

We have $\mathrm{im}(d_{n+1}) \hookrightarrow A_n$ & so can form

$$\begin{tikzcd} \mathrm{im}(d_{n+1}) \arrow[r, hook] \arrow[d, hook] & A_n \\ \mathrm{Ker}(d_n) \arrow[ur, hook] & \end{tikzcd}$$

<u>homology</u>

$H_n(A) := \mathrm{Ker}(d_n)/\mathrm{im}(d_{n+1})$ as before.

We can also speak of exact sequences, chain maps, homotopies in $\mathcal{A}$ & all the results from last week hold in this setting.

- We can form the cat $\underline{Ch(\mathcal{A})}$ of chain complexes in $\mathcal{A}$ & it is again $\underline{\text{abelian}}$: indeed $(f+g)_n = f_n + g_n$ for chain maps & kernels & cokernels are also constructed componentwise.
- Homology is a functor $H_n : Ch(\mathcal{A}) \longrightarrow \mathcal{A}$ as before.

Remark) A cochain complex in $\mathcal{A}$ is a chain complex in $\mathcal{A}^{op}$. In fact,

$$\mathrm{Cochain}(\mathcal{A}) = (Ch(\mathcal{A}^{op}))^{op} !$$

(4) If C is a small cat & A abelian, the functor cat $[C, A]$ is abelian with componentwise structure.

(5) If X is a top. space we can look at the poset $O(X)$ of open sets of X. A presheaf of R-modules is a functor $O(X)^{op} \longrightarrow Mod_R$.

- By the above the cat $[O(X)^{op}, Mod_R]$ is abelian.

- So is its full subcat $Sh(X, Mod_R)$ of sheaves & this is important in alg. geometry.