

lecture 7

Additive & exact Functors

Defⁿ) let A, B be abelian cats. A functor $F: A \rightarrow B$ is additive if each function $F_{x,y}: A(x,y) \rightarrow B(Fx, Fy)$ is a homom. of abelian groups (ie. $F(f+g) = Ff + Fg$ & $F_{x,y} \circ 0_{A(x,y)} = 0_{B(Fx, Fy)}$)

- From last week,
• terminal/initial ob are char. by diags $a \xrightarrow{0=id} a$ which we call zero object diags & a zero ob., denoted by 0 .

• bin. products/coproducts are characterised by diagrams of form

$$a \begin{array}{c} \xrightarrow{i_1} \\ \xleftarrow{p_1} \end{array} c \begin{array}{c} \xleftarrow{i_2} \\ \xrightarrow{p_2} \end{array} b$$

- $p_1 i_1 = 1_a$, $p_2 i_1 = 0$
- $p_2 i_2 = 1_b$, $p_1 i_2 = 0$
- $i_1 p_1 + i_2 p_2 = 1_c$

which are called biproduct diagrams, and often denote biprod. by $a \oplus b$.

Proposition

Additive functors preserve finite coproducts & products - in other words, they preserve biproducts & zero objects.

Proof) They preserve zero objects & biproduct diagrams as additive.

ie. if we have product

$$a \xleftarrow{p_1} c \xrightarrow{p_2} b$$

we have a

biproduct diagram

$$a \xrightleftharpoons[p_1]{i_1} c \xrightleftharpoons[p_2]{i_2} b$$

- $p_1 i_1 = 1_a$, $p_2 i_1 = 0$
- $p_2 i_2 = 1_b$, $p_1 i_2 = 0$
- $i_1 p_1 + i_2 p_2 = 1_c$

& then

$$F a \xrightleftharpoons[F_{p_1}]{F_{i_1}} F c \xrightleftharpoons[F_{p_2}]{F_{i_2}} F b$$

- $F_{p_1} F_{i_1} = 1_{F a}$, $F_{p_2} F_{i_1} = 0$
- $F_{p_2} F_{i_2} = 1_{F b}$, $F_{p_1} F_{i_2} = 0$
- $F_{i_1} F_{p_1} + F_{i_2} F_{p_2} = 0$

since F additive

$$\text{So } \begin{array}{ccc} & F_{p_1} & F_{p_2} \\ & \swarrow & \searrow \\ F a & & F b \end{array}$$

a product.

Sim. zero objects.

□

- Defⁿ) An additive functor $F: A \rightarrow B$ between abelian categories is
- left exact (lex) if it preserves kernels;
 - right exact (rex) - - - - okernels;
 - exact if it preserves both.

Remark) lex functors are those preserving finite limits (Finite products & equalisers \equiv kernels)

Rex functors preserve finite colimits & ex. functors preserve both.

Lemma

- (1) F is lex \Leftrightarrow it preserves exactness of sequences $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C$
- (2) F is rex \Leftrightarrow it preserves exactness of sequences $A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$
- (3) F is ex \Leftrightarrow it preserves exactness everywhere
 \Leftrightarrow it preserves short exact sequences.

Proof

First (1). Observe $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C$ is exact
 $\Leftrightarrow \ker f = 0$ (ie f is mono)
& $\ker g = \text{im } f = A$ (as f mono).

So if F preserves kernels, it preserves exactness of such sequences.

Conversely consider exact sequence

$$0 \rightarrow \ker f \xrightarrow{i} A \xrightarrow{f} B.$$

$$\text{Then } 0 \rightarrow F\ker f \xrightarrow{F_i} FA \xrightarrow{Ff} FB$$

is exact,

$$\text{so by } (*) \quad \ker Ff = F\ker f.$$

(2) is dual.

③ If F is exact, it preserves kernels & cokernels, and so images.

Therefore it preserves exactness everywhere & so short exact sequences.

Suppose F pres. ses. We will show F pres. monos & epis.

If f is mono then

then $0 \rightarrow A \xrightarrow{f} B \rightarrow \text{coker } f \rightarrow 0$ is ses.

Then F preserves it so by ex @ F , Ff is mono.

Dually F preserves epis.

Now show F preserves kernels (cokernels dual.)

At general $f: A \rightarrow B$ consider ses on horizontal row.

$$0 \rightarrow \ker f \xrightarrow{q} A \xrightarrow{f} \text{im } f \rightarrow 0$$

$\begin{array}{c} \xrightarrow{f} \\ \nearrow f_i \end{array} B$

Then Fq, Fi mono, Fp epi & hor. sequence exact after applying F , so

$$\ker(Ff) = \text{as } Fi \text{ mono}$$

$$\ker(Fp) = \text{as } F \text{ pres ses}$$

$$\text{Im}(Fq) = \text{as } Fq \text{ mono}$$

$$F(\ker f), \text{ so } F \text{ pres. kernels. } \square$$

Examples

① The forgetful Functor $\text{Mod}_R \xrightarrow{U} \text{Ab}$ is additive & in fact left exact.

Indeed it preserves all limits, as limits in both cats are just as in Set , and so preserves kernels.

Indeed, we have $\text{Mod}_R \xrightarrow{U} \text{Ab}$



comm diagram of forg. functors, so by Alg 3, U has a left adjoint F , which is then right exact.

In fact $FA = R \otimes_{\mathbb{Z}} A$.

② More generally, we have
 $\text{Mod}_R(A, -) : \text{Mod}_R \longrightarrow \text{Ab}$

$$\begin{array}{ccc} B & \longrightarrow & \text{Mod}_R(A, B) \\ \downarrow g & & \downarrow g_* \\ C & & \text{Mod}_R(A, C) \end{array} \quad \downarrow \text{gf}$$

which is additive & left exact.

This has a left adjoint $A \otimes_R - : \text{Ab} \rightarrow \text{Mod}_R$
 where $A \otimes_R B$ classifies functions

$$K : A \times B \longrightarrow \textcircled{C} \sim \text{R-module}$$

& $K(a, -) : B \rightarrow C$ is hom. of ab. groups

& $K(-, b) : A \rightarrow C$ is hom. of R-modules

& $A \otimes_R B$ is constructed as a quotient sim. to the tensor prod. of R-modules in Alg. 3.

In particular, as a left adjoint $A \otimes_R -$ preserves colimits & so is right exact.

• Note $U : \text{Mod}_R \rightarrow \text{Ab}$ has

$UM \cong \text{Mod}_R(R, M)$ so special case.

③ More generally, if \mathcal{C} abelian cat & $A \in \mathcal{C}$, the functor

$$\begin{array}{ccc} \mathcal{C}(A, -) : \mathcal{C} & \longrightarrow & \text{Ab} \\ & & \mathcal{C}(A, X) \\ & \downarrow f & \downarrow f_* \\ & \mathcal{C}(A, Y) & \downarrow fg \end{array}$$

is additive & preserves all limits - therefore it is lex.

(It may not have a left adjoint.)

④ If \mathcal{C} is abelian, so is $\text{Ch}(\mathcal{C})$ & then

$$\begin{array}{ccc} \text{Ch}(\mathcal{C}) & \xrightarrow{(-)_n} & \mathcal{C} \\ X & \longmapsto & X_n \end{array} \text{ is}$$

exact since kernels & cokernels are componentwise in $\text{Ch}(\mathcal{C})$

To show localization is exact, I will use one more result - not sure how to fit it into previous chain of equivs.

Prop) let $F: A \rightarrow B$ be additive functor between abelian cats.

TFAE

- ① F is exact
- ② F preserves mono & cokernels
- ③ F preserves epis & kernels.

Proof) Suffices to prove ① \Leftrightarrow ② as ②, ③ are dual.

① \Rightarrow ② is as in ③ of prev. prop, where showed exact functor preserves mono.

② \Rightarrow ①. Consider $A \xrightarrow{F} B$ &

$$\begin{array}{ccc}
 K F \xrightarrow{i} A \xrightarrow{F} B & & F K F \xrightarrow{F i} F A \xrightarrow{F F} F B \\
 \searrow p \quad \nearrow q & \mapsto & \searrow F p \quad \nearrow F q \\
 & & F C K F \\
 & & \text{"} \\
 & & \underline{c(F(KF))}
 \end{array}$$

where i, q are mono.

As B abelian, each mono is kernel of cokernel,

Since $F i$ mono, then

$$F K F = k(F p) \quad \text{but}$$

$$k(F p) = k(F F) \quad \text{since } F q \text{ mono } \square$$

⑤ R commutative ring & U multiplic. subset,
with localization $U^{-1}R$.

Functor $U^{-1}: \text{Mod}_R \rightarrow \text{Mod}_R$ is left
adjoint $U^{-1} \cong U^{-1}R \otimes_R - \dashv \text{Mod}_R(U^{-1}R, -)$
so preserves cokernels & we showed
it preserves monos.

By prev. proposition, it is exact.

Theorem (Freyd - Mitchell)

If \mathcal{C} is a small abelian cat.,
 \exists a ring R & an exact fully faithful embedding

$$F: \mathcal{C} \longrightarrow \text{Mod}_R$$

• We will not prove it.

Consequence

When proving things about diagrams
in an abelian category \mathcal{C} ,
we can assume we are working
in a cat of R -modules,

since the theorem lets us view \mathcal{C} as
a full subcategory of Mod_R closed
under kernels, cokernels.

The snake lemma

Given a diagram

$$\begin{array}{ccccccc} A & \xrightarrow{F} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

with exact rows in an abelian cat.,

there is an exact sequence

$$\ker(\alpha) \xrightarrow{F} \ker(\beta) \xrightarrow{g} \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \xrightarrow{f'} \operatorname{coker}(\beta) \xrightarrow{g'} \operatorname{coker}(\gamma).$$

~~Proof~~

All maps except δ are induced by universal props of kernel & cokernel.

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Proof

- All maps except δ are induced by universal props of kernel & cokernel.
- Exactness @ $k(\beta)$ & $c(\beta)$ are dual - we will prove it @ $k(\beta)$.

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- By FM-embedding thm, can suppose we are in $\text{Mod } R$.

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- By FM-embedding thm, can suppose we are in Mod_R .
- Let $x \in k(\beta)$. Then $\gamma g x = g' \beta x = g' 0 = 0$.

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- Let $b \in k(\beta)$ sat $gb = 0$.
- By ex @ B , $\exists a \in A$ s.t. $fa = 0$.

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- By ex @ B , $\exists a \in A$ s.t. $fa = b$.
- Then $0 = \beta fa = F' \alpha a \Rightarrow \alpha a = 0$ as F' mono.

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- By ex @ B , $\exists a \in A$ s.t. $fa = b$.
- Then $0 = \beta fa = f' \alpha a \Rightarrow \alpha a = 0$ as f' mono.
- Hence $a \in \ker(\alpha)$ w' $f(a) = b$, so exactness @ $k(\beta)$.

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~~Proof ctd~~

Now construct $\delta: \ker(\gamma) \rightarrow \operatorname{coker}(\alpha)$

Consider $x \in \ker(\gamma)$. As g surj., $\exists x'$ st $g x' = x$.

The snake lemma

Given a diagram

$$\begin{array}{ccccccc}
 A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
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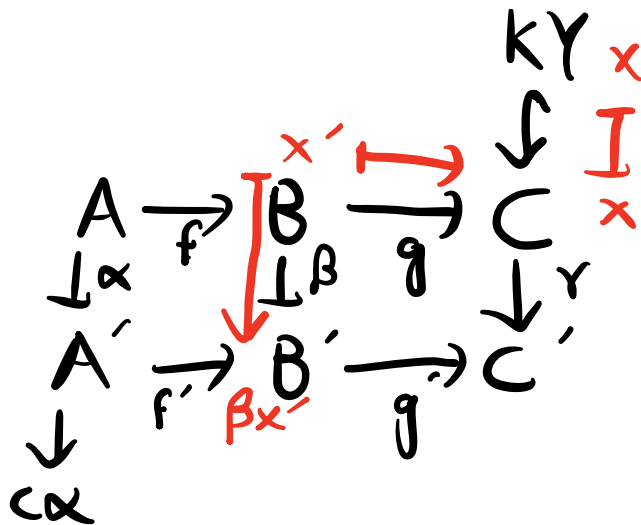
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$$\begin{array}{ccccccc}
 & & & & & \ker(\gamma) & x \\
 & & & & & \downarrow & \downarrow I \\
 A & \xrightarrow{f} & B & \xrightarrow{g} & C & & x \\
 \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
 A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \\
 \downarrow & & \downarrow \beta x' & & & & \\
 \operatorname{coker}(\alpha) & & & & & &
 \end{array}$$

Now $g'\beta x' = \gamma gx' = \gamma x = 0$, so $\exists x'' \in A'$ st $f'x'' = \beta x'$.

The snake lemma

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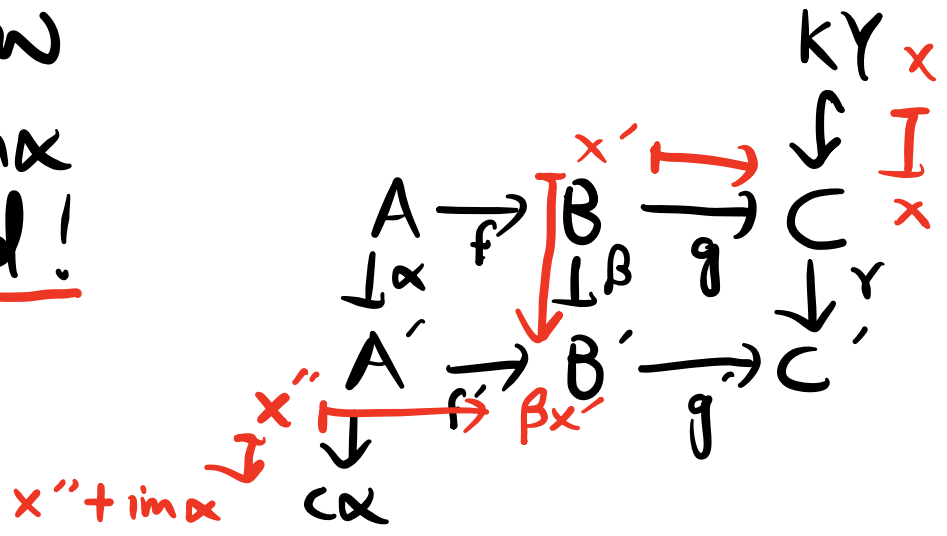
Consider $x \in \ker(\gamma)$. As g surj., $\exists x'$ st $gx' = x$.

$$\begin{array}{ccccccc} & & & & KY & x & \\ & & & & \downarrow & I & \\ & & & & \downarrow & x & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & & \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \\ \downarrow & & \downarrow & & & & \\ x'' + \operatorname{im} \alpha & & \beta x' & & & & \end{array}$$

Red arrows and labels in the diagram indicate the mapping of x' to $\beta x'$ and $x'' + \operatorname{im} \alpha$ to $\beta x'$ via the map f' .

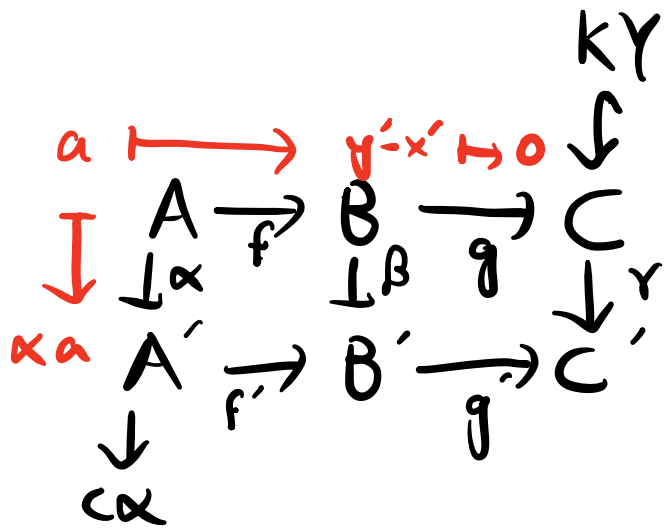
- Now $g'\beta x' = \gamma gx' = \gamma x = 0$, so $\exists x'' \in A'$ st $f'x'' = \beta x'$.
- Define $\delta x = x'' + \operatorname{im} \alpha$!

Must show
 $\delta x = x'' + \text{im} \alpha$
well defined!



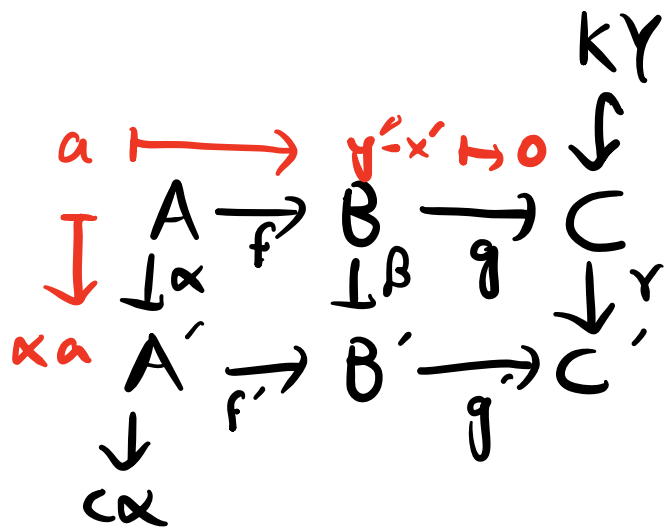
Suppose $g y' = x$. Then $g(y' - x') = x - x = 0$.

Must show
 $\delta x = x'' + \text{im} \alpha$
well defined!



Suppose $gy' = x$. Then $g(y'-x') = x - x = 0$.
 So by exactness $\exists a \in A$ st $fa = y'-x'$.

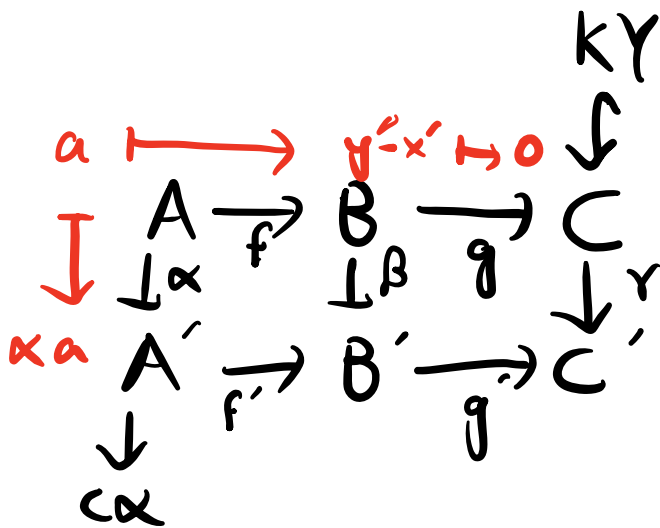
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Suppose $gy' = x$. Then $g(y' - x') = x - x = 0$.
 So by exactness $\exists a \in A$ st $fa = y' - x'$.

Then $f'xa = \beta fa = \beta y' - \beta x'$

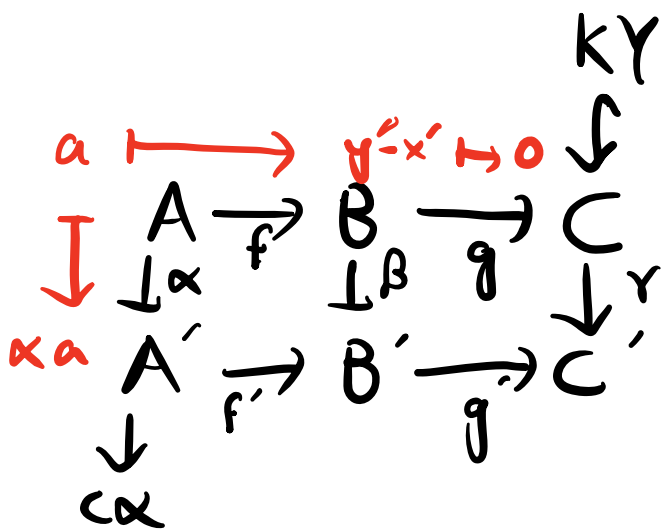
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Suppose $gy' = x$. Then $g(y' - x') = x - x = 0$
 So by exactness $\exists a \in A$ st $fa = y' - x'$.

Then $f' \alpha a = \beta fa = \beta y' - \beta x' = f' y'' - f' x''$
 so as f' mono, $\alpha a = y'' - x''$.

Must show
 $\delta x = x'' + \text{im } \alpha$
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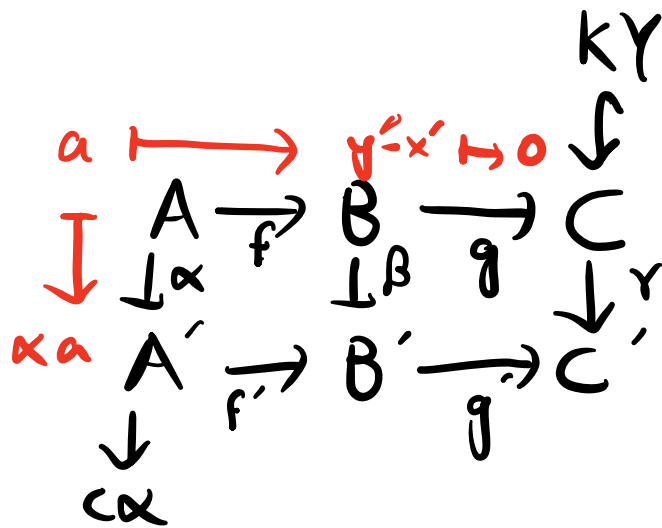
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Therefore $y'' + \text{im } \alpha = x'' + \text{im } \alpha$,
 as required.

So δ well defined.

Must show
 $\delta x = x'' + \text{im } \alpha$
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Suppose $gy' = x$. Then $g(y' - x') = x - x = 0$.
 So by exactness $\exists a \in A$ st $fa = y' - x'$.

Then $f' \alpha a = \beta fa = \beta y' - \beta x' = f' y'' - f' x''$
 so as f mono, $\alpha a = y'' - x''$.

Therefore $y'' + \text{im } \alpha = x'' + \text{im } \alpha$,
 as required.

So δ well defined.

Easy to see δ a homomorph
 and

exactness @ $\ker(\gamma), \ker(\alpha)$ left
 as exercise. \square

Theorem

let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a ses of chain complexes in an ab. cat. Then we obtain a long exact sequence of homology

$$\dots H_{n+1}(A) \xrightarrow{H_{n+1}(f)} H_{n+1}(B) \xrightarrow{H_{n+1}(g)} H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow{H_n(f)} H_n(B) \xrightarrow{H_n(g)} H_n(C) \dots$$

Proof Suffices to prove in Mod R as before.

• For ch. complex A :

$$\dots A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \dots$$

$$\text{have } A_n / \text{im}(d_{n+1}) \xrightarrow{d} \ker(d_{n-1}) = Z_{n-1}(A)$$
$$x + \text{im}(d_{n+1}) \longmapsto d_n x$$

- whose kernel contains $x + \text{im}(d_{n+1})$ such that $d_n x = 0$.
In other words, $\ker(d) = \ker(d_n) / \text{im}(d_{n+1}) = \underline{H_n(A)}$.
- whose cokernel is $\ker(d_{n-1}) / \text{im}(d_n) = \underline{H_{n-1}(A)}$.

Obtain diagram

$$\begin{array}{ccccccc} A_n / \text{im}(d_{n+1}) & \xrightarrow{f_n} & B_n / \text{im}(d_{n+1}) & \xrightarrow{g_n} & C_n / \text{im}(d_{n+1}) & \rightarrow & 0 \\ d \downarrow & & d \downarrow & & d \downarrow & & \\ 0 \rightarrow Z_{n-1}(A) & \xrightarrow{f_{n-1}} & Z_{n-1}(B) & \xrightarrow{g_{n-1}} & Z_{n-1}(C) & & \end{array}$$

& if we can show rows are exact, then by snake lemma, get the result.

- Exactness @ $C_n / \text{im}(d_{n+1})$, $Z_{n-1}(A)$ easy as g_n epi, f_{n-1} mono.
- @ $Z_{n-1}(B)$: let $x \in Z_{n-1}(B)$ sat. $gx = 0$.
Then $\exists y \in A_{n-1}$ st $fy = x$ by ex @ B_{n-1} .
Need $dy = 0$, but $fdy = dfy = dx = 0$
so as f_{n-1} mono, $dy = 0$, so $y \in Z_{n-1}(A)$.

Obtain diagram

$$\begin{array}{ccccccc}
 A_n / \text{im}(d_{n+1}) & \xrightarrow{f_n} & B_n / \text{im}(d_{n+1}) & \xrightarrow{g_n} & C_n / \text{im}(d_{n+1}) & \rightarrow & 0 \\
 d \downarrow & & d \downarrow & & d \downarrow & & \\
 0 \rightarrow Z_{n-1}(A) & \xrightarrow{f_{n-1}} & Z_{n-1}(B) & \xrightarrow{g_{n-1}} & Z_{n-1}(C) & &
 \end{array}$$

• @ $B_n / \text{im}(d_{n+1})$: let $x + \text{im}(d_{n+1})$ sat $g_n x = 0 \in \frac{C_n}{\text{im}(d_{n+1})}$

Then $g_n x \in \text{im}(d_{n+1})$ so $g_n x = d_n c$, some $c \in C_{n+1}$.

• Then $\exists y \in B_{n+1}$ st $d_n y = c$.

• Take $x - d_n y \in B_n$. Then $g_n(x - d_n y) = g_n x - g_n d_n y = g_n x - d_{n+1} g_{n+1} y = g_n x - d_n c = 0$.

• So $\exists a \in A_n$ so $f_n(a) = x - d_n y$.

• Then $f_n(a + \text{im}(d_{n+1})) = f_n a + \text{im}(d_{n+1})$
 $= x - d_n y + \text{im}(d_{n+1})$
 $= x + \text{im}(d_{n+1})$

as required.