lecture 7

Additive & exact functors

Def^n) let $\mathcal{A}, \mathcal{B}$ be abelian cats. A functor
$F: \mathcal{A} \longrightarrow \mathcal{B}$ is additive if each function
$F_{X,Y} : \mathcal{A}(X,Y) \longrightarrow \mathcal{B}(FX, FY)$ is a homom. of abelian groups (ie. $F(f+g) = Ff + Fg$ & $F0_{X,Y} = 0_{FX,FY}$)

- From last week,

- terminal/initial ob are char. by diags $a \xrightarrow{0 = id} a$ which we call zero object diags & a a zero ob., denoted by 0.

- bin. products/coproducts are characterised by diagrams of form

$$a \underset{p_1}{\overset{i_1}{\rightleftarrows}} c \underset{p_2}{\overset{i_2}{\leftrightarrows}} b$$

$$\boxed{\begin{array}{l} \bullet\ p_1 i_1 = 1_a, \quad p_2 i_1 = 0 \\ \bullet\ p_2 i_2 = 1_b, \quad p_1 i_2 = 0 \\ \bullet\ i_1 p_1 + i_2 p_2 = 1_c \end{array}}$$

which are called biproduct diagrams, and often denote biprod. by $a \oplus b$.

Proposition

Additive functors preserve finite coproducts & products - in other words, they preserve biproducts & zero objects.

Proof) They preserve zero object & biproduct diagrams as additive.

ie. if we have product
$a \xleftarrow{p_1} c \xrightarrow{p_2} b$
we have a biprod. diagram

$$\begin{tikzcd} a \arrow[r, shift left, "i_1"] & c \arrow[l, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & b \arrow[l, shift right, "i_2"'] \end{tikzcd}$$

$$\boxed{\begin{array}{l} \bullet\ p_1 i_1 = 1_a, \quad p_2 i_1 = 0 \\ \bullet\ p_2 i_2 = 1_b, \quad p_1 i_2 = 0 \\ \bullet\ i_1 p_1 + i_2 p_2 = 1_c \end{array}}$$

& then

$$\begin{tikzcd} Fa \arrow[r, shift left, "Fi_1"] & Fc \arrow[l, shift left, "Fp_1"] \arrow[r, shift right, "Fp_2"'] & Fb \arrow[l, shift right, "Fi_2"'] \end{tikzcd}$$

$$\boxed{\begin{array}{l} \bullet\ Fp_1 Fi_1 = 1_{Fa}, \quad Fp_2 Fi_1 = 0 \\ \bullet\ Fp_2 Fi_2 = 1_{Fb}, \quad Fp_1 Fi_2 = 0 \\ \bullet\ Fi_1 Fp_1 + Fi_2 Fp_2 = 0 \end{array}}$$

since F additive

So $Fa \xleftarrow{Fp_1} Fc \xrightarrow{Fp_2} Fb$ a product.

Sim. zero objects. $\square$

Def^n) An additive functor $F: A \to B$ between abelian categories is

- left exact (lex) if it preserves kernels;
- right exact (rex) - - - - - - cokernels;
- exact if it preserves both.

Remark) Lex functors are those preserving finite limits (finite products & equalisers $\equiv$ kernels)

Rex functors preserve finite colimits & ex. functors preserve both.

Lemma

1. F is lex $\iff$ it preserves exactness of sequences $0 \to A \xrightarrow{f} B \xrightarrow{g} C$
2. F is rex $\iff$ it preserves exactness of sequences $A \xrightarrow{f} B \xrightarrow{g} C \to 0$
3. F is ex $\iff$ it preserves exactness everywhere $\iff$ it preserves short exact sequences.

Proof

First 1. Observe $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact

(*) $\iff$ $\ker f = 0$ (ie f is mono) & $\ker g = A$ ($\ker g = \operatorname{im} f = A$ as f mono).

So if F preserves kernels, it preserves exactness of such sequences.

Conversely consider exact sequence

$$0 \to \ker f \xrightarrow{i} A \xrightarrow{f} B.$$

Then $0 \to F\ker f \xrightarrow{Fi} FA \xrightarrow{Ff} FB$ is exact,

so by (*) $\ker Ff = F\ker f$.

2. is dual.

③ If F is exact, it preserves kernels & cokernels, and so images.
Therefore it preserves exactness everywhere & so short exact sequences.

Suppose F pres ses. We will show F pres. monos & epis.

If f is mono then
then $0 \to A \xrightarrow{f} B \to \operatorname{coker} f \to 0$ is ses.
Then F preserves it so by ex @ Fa, Ff is mono.

Dually F preserves epis.

Now show F preserves kernels (cokernels dual.)

At general $f: A \to B$ consider ses on horizontal row.

$$
\begin{tikzcd}
 & & & & B \\
0 \arrow[r] & \ker f \arrow[r, hook, "q"] & A \arrow[r, two heads, "p"] \arrow[urr, bend left, "f"] & \operatorname{im} f \arrow[r] \arrow[ur, hook, "i"] & 0
\end{tikzcd}
$$

Then Fq, Fi mono, Fp epi & hor. sequence exact after applying F, so

$\ker(Ff)$ = as Fi mono
$\ker(Fp)$ = as F pres ses
$\operatorname{Im}(Fq)$ = as Fq mono
$F(\ker f)$, so F pres. kernels. $\square$

Examples

① The forgetful functor $\mathrm{Mod}_R \xrightarrow{U} \mathrm{Ab}$ is additive & in fact left exact.

Indeed it preserves all limits, as limits in both cats are just as in Set, and so preserves kernels.

Indeed, we have

$$\begin{tikzcd} \mathrm{Mod}_R \arrow[rr, "U"] \arrow[dr, "U_1"'] & & \mathrm{Ab} \arrow[dl, "U_2"] \\ & \mathrm{Set} & \end{tikzcd}$$

comm diagram of forg. functors, so by Alg 3, U has a left adjoint F, which is then right exact.

In fact $FA = R \otimes_{\mathbb{Z}} A$.

(2) More generally, we have

$$\begin{tikzcd} \mathrm{Mod}_R(A,-) : \mathrm{Mod}_R \arrow[r] & \mathrm{Ab} \\ B \arrow[r, mapsto] \arrow[d, "g"'] & \mathrm{Mod}_R(A,B) \arrow[d, "g_*"'] & f \arrow[d, mapsto] \\ C & \mathrm{Mod}_R(A,C) & gf \end{tikzcd}$$

which is additive & left exact.

This has a left adjoint $A \otimes_{\mathbb{Z}} - : \mathrm{Ab} \to \mathrm{Mod}_R$ where $A \otimes_{\mathbb{Z}} B$ classifies functions

$K : A \times B \longrightarrow C$ ~ R-module

& $K(a,-) : B \to C$ is hom. of ab. groups

& $K(-,b) : A \to C$ is hom. of **R-modules**

& $A \otimes_{\mathbb{Z}} B$ is constructed as a quotient sim. to the tensor prod. of R-modules in Alg. 3.

In particular, as a left adjoint $A \otimes_{\mathbb{Z}} -$ preserves colimits & so is right exact.

- Note $U : \mathrm{Mod}_R \to \mathrm{Ab}$ has $UM \cong \mathrm{Mod}_R(R,M)$ so special case.

③ More generally, if $\mathcal{C}$ abelian cat & $A \in \mathcal{C}$, the functor

$$
\begin{array}{cccc}
\mathcal{C}(A,-) : \mathcal{C} & \longrightarrow & Ab & \\
X & & \mathcal{C}(A,X) & g \\
f\downarrow & & f_*\downarrow & \downarrow \\
Y & & \mathcal{C}(A,Y) & fg
\end{array}
$$

is additive & preserves all limits - therefore it is lex.
(It may not have a left adjoint.)

④ If $\mathcal{C}$ is abelian, so is $Ch(\mathcal{C})$ & then

$$
\begin{array}{ccc}
Ch(\mathcal{C}) & \xrightarrow{(-)_n} & \mathcal{C} \\
X & \longmapsto & X_n
\end{array}
$$

is exact since kernels & cokernels are componentwise in $Ch(\mathcal{C})$

To show localization is exact, I will use one more result - not sure how to fit it into previous chain of equivs.

Prop) let $F : A \to B$ be additive functor between abelian cats.

TFAE:

1. F is exact
2. F preserves monos & cokernels
3. -- -- -- epis & kernels.

Proof) Suffices to prove ① ⟺ ② as ②, ③ are dual.

① ⇒ ② is as in ③ of prev. prop, where showed exact functor preserves monos.

② ⇒ ①. Consider $A \xrightarrow{f} B$ &

$$\begin{tikzcd} kf \arrow[r, tail, "i"] & A \arrow[r, "f"] \arrow[dr, "p"'] & B \\ & & ckf \arrow[u, tail, "q"'] \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} Fkf \arrow[r, tail, "Fi"] & FA \arrow[r, "Ff"] \arrow[dr, "Fp"'] & FB \\ & & FckF = c(F(kf)) \arrow[u, tail, "Fq"'] \end{tikzcd}$$

where i, q are mono.

As B abelian, each mono is kernel of cokernel.

Since Fi mono, then

$FKF = k(Fp)$ but

$k(Fp) = k(Ff)$ since Fq mono □

5 R commutative ring & U multiplic. subset, with localization $U^{-1}R$.

Functor $U^{-1}: \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_R$ is left adjoint $U^{-1} \cong U^{-1}R \otimes_R - \dashv \mathrm{Mod}_R(U^{-1}R, -)$

so preserves cokernels & we showed it preserves monos.

By prev. proposition, it is exact.

Theorem (Freyd - Mitchell)

If $\mathcal{C}$ is a small abelian cat., $\exists$ a ring R & an exact fully faithful embedding

$$F : \mathcal{C} \longrightarrow \mathrm{Mod}_R .$$

- We will not prove it.

Consequence

When proving things about diagrams in an abelian category $\mathcal{C}$, we can assume we are working in a cat of R-modules, since the theorem lets us view $\mathcal{C}$ as a full subcategory of Mod_R closed under kernels, cokernels.

The snake lemma

Given a diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' & \end{tikzcd}$$

with exact rows in an abelian cat.,
there is an exact sequence

$$\ker(\alpha) \xrightarrow{f} \ker(\beta) \xrightarrow{g} \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \xrightarrow{f'} \operatorname{coker}(\beta) \xrightarrow{g'} \operatorname{coker}(\gamma).$$

Proof

All maps except δ are induced by universal props of kernel & cokernel.

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~~Proof~~

- All maps except δ are induced by universal props of kernel & cokernel.
- Exactness @ $k(\beta)$ & $c(\beta)$ are dual – we will prove it @ $k(\beta)$.

The snake lemma

Given a diagram
$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & & \end{array}$$
with exact rows in an abelian cat.,
there is an exact sequence
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- By FM-embedding thm, can suppose we are in Mod_R.

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- Exactness @ $k(\beta)$ & $c(\beta)$ are dual - we will prove it @ $k(\beta)$.
- By FM-embedding thm, can suppose we are in Mod_R.
- Let $x \in k(\beta)$. Then $\gamma g x = g'\beta x = g' 0 = 0$.

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- Let $b \in k(\beta)$ sat $gb = 0$.
- By ex@B, $\exists a \in A$ s.t. $fa = 0$.

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- By FM-embedding thm, can suppose we are in Mod_R.
- Let $b \in k(\beta)$ sat $gb = 0$.
- By ex@B, $\exists\, a \in A$ s.t. $fa = b$.
- Then $0 = \beta f a = f' \alpha a \Rightarrow \alpha a = 0$ as f' mono.

The snake lemma

Given a diagram

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' & \end{tikzcd}$$

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- All maps except δ are induced by universal props of kernel & cokernel.
- Exactness @ $k(\beta)$ & $c(\beta)$ are dual - we will prove it @ $k(\beta)$.
- By FM-embedding thm, can suppose we are in Mod_R.
- Let $b \in k(\beta)$ sat $gb = 0$.
- By ex@B, $\exists\, a \in A$ s.t. $fa = 0$.
- Then $0 = \beta f a = f' \alpha a \Rightarrow \alpha a = 0$ as f' mono.
- Hence $a \in \ker(\alpha)$ w/ $f(a) = b$, so exactness @ $\ker(\beta)$.

The snake lemma

Given a diagram
$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' & \end{tikzcd}$$
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Proof ctd Now construct $\delta : \ker(\gamma) \to \operatorname{coker}(\alpha)$

Consider $x \in k\gamma$. As g surj., $\exists x'$ st $gx' = x$.

The snake lemma

Given a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \end{tikzcd}$$

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Proof ctd

Now construct $\delta : \ker(\gamma) \to \operatorname{coker}(\alpha)$

Consider $x \in k\gamma$. As g surj., $\exists x'$ st $gx' = x$.

$$\begin{tikzcd} & & k\gamma \arrow[d, hook] & x \arrow[d, mapsto] \\ A \arrow[r, "f"] \arrow[d, "\alpha"] & B \ni x' \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] & x \\ A' \arrow[r, "f'"'] \arrow[d] & B' \ni \beta x' \arrow[r, "g'"'] & C' & \\ c\alpha & & & \end{tikzcd}$$

The snake lemma

Given a diagram

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' &
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with exact rows in an abelian cat.,

there is an exact sequence

$$\ker(\alpha) \xrightarrow{f} \ker(\beta) \xrightarrow{g} \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \xrightarrow{f'} \operatorname{coker}(\beta) \xrightarrow{g'} \operatorname{coker}(\gamma).$$

Proof ctd Now construct $\delta : \ker(\gamma) \to \operatorname{coker}(\alpha)$

Consider $x \in k\gamma$. As g surj., $\exists x'$ st $gx' = x$.

$$\begin{tikzcd}
& & k\gamma \arrow[d, hook] \\
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
A' \arrow[r, "f'"'] \arrow[d] & B' \arrow[r, "g'"'] & C' \\
c\alpha & &
\end{tikzcd}$$

$x' \mapsto x$, $x \mapsto x$, $x' \mapsto \beta x'$

Now $g'\beta x' = \gamma g x' = \gamma x = 0$, so
$\exists x'' \in A'$ st $f' x'' = \beta x'$.

The snake lemma

Given a diagram

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' &
\end{tikzcd}$$

with exact rows in an abelian cat.,
there is an exact sequence

$$\ker(\alpha) \xrightarrow{f} \ker(\beta) \xrightarrow{g} \ker(\gamma) \xrightarrow{\delta} \operatorname{coker}(\alpha) \xrightarrow{f'} \operatorname{coker}(\beta) \xrightarrow{g'} \operatorname{coker}(\gamma).$$

~~Proof ctd~~ **Now construct $\delta : \ker(\gamma) \to \operatorname{coker}(\alpha)$**

Consider $x \in k\gamma$. As g surj., $\exists x'$ st $gx' = x$.

$$\begin{tikzcd}
 & & k\gamma \arrow[d, hook] \\
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g"] & C' \\
c\alpha & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & x \arrow[d, mapsto] \\
 & x' \arrow[r, mapsto] \arrow[d, mapsto] & x \\
x'' \arrow[r, mapsto] \arrow[d, mapsto] & \beta x' & \\
x'' + \mathrm{im}\,\alpha & &
\end{tikzcd}$$

- Now $g'\beta x' = \gamma g x' = \gamma x = 0$, so $\exists x'' \in A'$ st $f'x'' = \beta x'$.
- Define $\gamma x = x'' + \mathrm{im}\,\alpha$!

Must show

$\delta x = x'' + \operatorname{im}\alpha$

<u>well defined!</u>

$$
\begin{tikzcd}
 & & \ker\gamma \arrow[d, hook] \\
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] & C' \\
\operatorname{coker}\alpha & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & x \arrow[d, mapsto] \\
 & x' \arrow[r, mapsto] \arrow[d, mapsto] & x \\
x'' \arrow[r, mapsto] \arrow[d, mapsto] & \beta x' & \\
x'' + \operatorname{im}\alpha & &
\end{tikzcd}
$$

Suppose $gy' = x$. Then $g(y' - x') = x - x = 0$.

Must show $\delta x = x'' + \operatorname{im}\alpha$ **well defined!**

$$a \longmapsto y'-x' \longmapsto 0$$

$$a \longmapsto \alpha a$$

$$\begin{tikzcd}
 & & \ker\gamma \arrow[d, hook] \\
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] & C' \\
\operatorname{coker}\alpha & &
\end{tikzcd}$$

Suppose $gy' = x$. Then $g(y'-x') = x - x = 0$.
So by exactness $\exists\, a \in A$ st $fa = y'-x'$.

Must show $\delta x = x'' + \operatorname{im}\alpha$ **well defined!**

$a \mapsto y'-x' \mapsto 0$, $\quad a \mapsto \alpha a$

$$
\begin{tikzcd}
 & & \ker\gamma \arrow[d, hook] \\
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] & C' \\
\operatorname{c}\alpha & &
\end{tikzcd}
$$

Suppose $gy' = x$. Then $g(y'-x') = x - x = 0$.

So by exactness $\exists\, a \in A$ st $fa = y' - x'$.

Then $f'\alpha a = \beta f a = \beta y' - \beta x'$

Must show $\delta x = x'' + \operatorname{im}\alpha$

well defined!

$$\begin{tikzcd}
 & & & \ker\gamma \arrow[d] \\
a \arrow[r, mapsto] \arrow[d] & & y'-x' \arrow[r, mapsto] & 0 \\
 & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
\alpha a & A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] & C' \\
 & \operatorname{coker}\alpha & &
\end{tikzcd}$$

Suppose $g\underline{y'} = x$. Then $g(y'-x') = x - x = 0$

So by exactness $\exists\, a \in A$ st $fa = y'-x'$.

Then $f'\alpha a = \beta f a = \beta y' - \beta x' = f'y'' - f'x''$

so as f' mono, $\alpha a = y'' - x''$.

Must show $\delta x = x'' + \operatorname{im}\alpha$ <u>well defined!</u>

$$a \longmapsto y'-x' \longmapsto 0$$

$$a \longmapsto \alpha a$$

$$\begin{tikzcd}
 & & \ker\gamma \arrow[d] \\
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] & C' \\
\operatorname{coker}\alpha & &
\end{tikzcd}$$

Suppose $g\underline{y'} = x$. Then $g(y'-x') = x - x = 0$.

So by exactness $\exists\, a \in A$ st $fa = y'-x'$.

Then $f'\alpha a = \beta f a = \beta y' - \beta x' = f'y'' - f'x''$

so as f mono, $\alpha a = y'' - x''$.

Therefore $y'' + \operatorname{im}\alpha = x'' + \operatorname{im}\alpha$,

as required.

So δ well defined.

Must show $\delta x = x'' + \operatorname{im}\alpha$ **well defined!**

$$\begin{tikzcd}
 & & K\gamma \arrow[d, hook] \\
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\
A' \arrow[r, "f'"] \arrow[d] & B' \arrow[r, "g'"] & C' \\
c\alpha & &
\end{tikzcd}$$

$a \longmapsto y'-x' \mapsto 0$

$a \mapsto \alpha a$

Suppose $gy' = x$. Then $g(y'-x') = x - x = 0$.
So by exactness $\exists\, a \in A$ st $fa = y'-x'$.
Then $f'\alpha a = \beta f a = \beta y' - \beta x' = f'y'' - f'x''$
so as f mono, $\alpha a = y'' - x''$.
Therefore $y'' + \operatorname{im}\alpha = x'' + \operatorname{im}\alpha$,
as required.
So δ well defined.

Easy to see δ a homomorph
and
exactness @ $\ker(\gamma)$, $\operatorname{coker}(\alpha)$ left
as *exercise*. $\square$

Theorem

let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a ses of chain complexes in an ab. cat. Then we obtain a long exact sequence of homology

$$\cdots H_{n+1}(A) \xrightarrow{H_{n+1}(f)} H_{n+1}(B) \xrightarrow{H_{n+1}(g)} H_{n+1}(C) \xrightarrow{\delta} H_n(A) \xrightarrow{H_n(f)} H_n(B) \xrightarrow{H_n(g)} H_n(C) \cdots$$

~~Proof~~ Suffices to prove in Mod_R as before.

- For ch. complex A:

$$\cdots A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}} \cdots\cdots$$

have
$$A_n/\mathrm{im}(d_{n+1}) \xrightarrow{d} \ker(d_{n-1}) = Z_{n-1}(A).$$
$$x + \mathrm{im}(d_{n+1}) \longmapsto d_n x$$

- whose kernel contains $x + \mathrm{im}(d_{n+1})$ such that $dx = 0$. In other words, $\ker(d) = \ker(d_n)/\mathrm{im}(d_{n+1}) = H_n(A)$.
- whose cokernel is $\ker(d_{n-1})/\mathrm{im}(d_n) = H_{n-1}(A)$.

Obtain diagram

$$\begin{tikzcd}
 & A_n/\mathrm{im}(d_{n+1}) \arrow[r, "f_n"] \arrow[d, "d"] & B_n/\mathrm{im}(d_{n+1}) \arrow[r, "g_n"] \arrow[d, "d"] & C_n/\mathrm{im}(d_{n+1}) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r, "f_{n-1}"] & Z_{n-1}(B) \arrow[r, "g_{n-1}"] & Z_{n-1}(C) &
\end{tikzcd}$$

& if we can show rows are exact, then by snake lemma, get the result.

- Exactness @ $C_n/\mathrm{im}(d_{n+1})$, $Z_{n-1}(A)$ easy as g_n epi, f_{n-1} mono.
- @ $Z_{n-1}(B)$: let $x \in Z_{n-1}(B)$ sat. $gx = 0$.

Then $\exists y \in A_{n-1}$ st $fy = x$ by ex@ B_{n-1}.

Need $dy = 0$, but $fdy = dfy = dx = 0$ so as f_{n-1} mono, $dy = 0$, so $y \in Z_{n-1}(A)$.

Obtain diagram

$$\begin{tikzcd}
& A_n/\mathrm{im}(d_{n+1}) \arrow[r, "f_n"] \arrow[d, "d"] & B_n/\mathrm{im}(d_{n+1}) \arrow[r, "g_n"] \arrow[d, "d"] & C_n/\mathrm{im}(d_{n+1}) \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & Z_{n-1}(A) \arrow[r, "f_{n-1}"] & Z_{n-1}(B) \arrow[r, "g_{n-1}"] & Z_{n-1}(C) &
\end{tikzcd}$$

- @ $B_n/\mathrm{im}(d_{n+1})$: let $x + \mathrm{im}(d_{n+1})$ sat $g_n x = 0 \in C_n/\mathrm{im}(d_{n+1})$

Then $gx \in \mathrm{im}(d_{n+1})$ so $gx = dc$, some $c \in C_{n+1}$

- Then $\exists\, y \in B_{n+1}$ st $dy = c$.
- Take $x - dy \in B_n$. Then $g(x-dy) =$

$gx - gdy = gx - dgy = gx - dc = 0$.

- So $\exists\, a \in A_n$ so $f(a) = x - dy$.
- Then $$\begin{aligned} F_n(a + \mathrm{im}\, d_{n+1}) &= f_n a + \mathrm{im}\, d_{n+1} \\ &= x - d_{n+1} y + \mathrm{im}\, d_{n+1} \\ &= x + \mathrm{im}\, d_{n+1} \end{aligned}$$

as required.