Lecture 8

- So far, we have constructed the homology functors $H_n : Ch(\mathcal{C}) \to \mathcal{C}$ & studied some of their props.

- Also, have constructed functors

$$\begin{array}{ccc} Top & \longrightarrow & Ch(Ab) \\ X & \longmapsto & SX \end{array}$$

& used these to define homology & cohomology of spaces

- If we wish to form (co)homology of more general sorts of structures, we need to understand how to turn them into chain complexes — via projective & injective resolution.

We do that today, & study the resulting left & right derived functors, which next week we'll use to look at Ext, Tor & group cohomology.

Notation: In ab. cat, $\twoheadrightarrow$ for epi, $\rightarrowtail$ or $\hookrightarrow$ for mono.

Projectives (see Alg 3 - we will do it quickly here)

Def^n) An ob. $A \in \mathcal{C}$ in an abelian cat is projective if given any diagram

$$\begin{tikzcd} & A \arrow[d, two heads, "\alpha"] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd} \quad \exists A \xrightarrow{\alpha'} B \text{ st } \begin{tikzcd} & A \arrow[dl, "\alpha'"'] \arrow[d, two heads, "\alpha"] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd} \; = .$$

Remark: This says $\mathcal{C}(A,B) \xrightarrow{f_*} \mathcal{C}(A,C) \in Ab$

$$g \longmapsto fg$$

is surjective, so

A is proj. $\iff$ $\mathcal{C}(A,-) : \mathcal{C} \longrightarrow Ab$ preserves epis.

Def^n) $\mathcal{C}$ has enough projectives if for each $A \in \mathcal{A}$ $\exists$ X projective & epi $X \twoheadrightarrow A$.

Prop^n) Mod_R has enough projectives. (The projectives are the retracts of free modules.)

$\boxed{\text{Proof}}$ See Alg. 3. Given A we take counit map $FUA \xrightarrow{\varepsilon_A} A$ for adjunction $\mathrm{Mod}_R \underset{U}{\overset{F}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{Set}$ - it takes formal sums to sums in A.

- Each free module is projective, so FUA is.
- Also ε_A is surjective, as required. $\square$

Proposition (Properties of projectives)

Let $\mathcal{C}$ be an abelian cat.

① Direct sums / biproducts and retracts of projectives are projective.

② $A \in \mathcal{C}$ is proj. $\iff$ $\mathcal{C}(A,-) : \mathcal{C} \to Ab$ is exact.

~~Proof~~ For ①, let A, B be proj. & consider direct sum $A \oplus B$.

Given
$$\begin{tikzcd} & A\oplus B \arrow[d, "\alpha"] \\ C \arrow[r, two heads, "f"'] & D \end{tikzcd}$$
we have
$$\begin{tikzcd} & A \arrow[d, "i_1"] \\ B \arrow[r, "i_2"'] & A\oplus B \end{tikzcd}$$
& so

$$\begin{tikzcd} & A \arrow[dl, dashed, "\exists\theta_1"'] \arrow[d, "\alpha i_1"] \\ C \arrow[r, two heads, "f"'] & D \end{tikzcd} \quad \& \quad \begin{tikzcd} & B \arrow[dl, dashed, "\exists\theta_2"'] \arrow[d, "\alpha i_2"] \\ C \arrow[r, two heads, "f"'] & D \end{tikzcd}$$
& then by u.p. of coprod $A \oplus B$

$\exists!\, \theta : A \oplus B \to C$ s.t. $\theta i_1 = \theta_1$, $\theta i_2 = \theta_2$.

Then
$$\begin{tikzcd} & A\oplus B \arrow[dl, "\theta"'] \arrow[dr, "\alpha"] \arrow[d, phantom, "="] & \\ C \arrow[rr, two heads, "f"'] & {} & D \end{tikzcd}$$
commutes using u.p. of coproduct.

Hence $A \oplus B$ projective.

For retracts, see proof in Alg 3 (won't use it.)

For ②, recall last wk we showed that an additive functor is exact $\iff$ preserves kernels & epis.

- Now $\mathcal{C}(A,-) : \mathcal{C} \to Ab$ always preserves kernels, so it is exact $\iff$ it preserves epis (that is, if A is projective) $\square$

Injectives

- For $\mathcal{C}$ abelian, we say that $A \in \mathcal{C}$ is <u>injective</u> if it is <u>projective</u> in $\mathcal{C}^{op}$.
- In elementary terms,

  A is <u>injective</u> if given
  $$\begin{tikzcd} B \arrow[r, tail, "\text{mono}"] \arrow[dr, "f"'] & C \\ & A \end{tikzcd} \quad \exists \quad \begin{tikzcd} B \arrow[r, tail, "\text{mono}"] \arrow[dr, "f"'] & C \arrow[d, "f'"] \\ & A \end{tikzcd}.$$

- By duality, anything about projectives has a <u>dual version</u> about injectives – eg. it follows that injectives are also closed under biproducts & retracts.

- In Ab, A is inj $\Leftrightarrow$ it is <u>divisible</u> ($\forall a \in A$, $n \in \mathbb{N} - \{0\}$, $\exists b \in A$ st $n.b = a$)
- $\mathcal{C}$ has <u>enough injectives</u> if $\forall A \in \mathcal{C}$, $\exists A \rightarrowtail A'$ with A' injective.
- Mod_R has enough injectives (hard result following from <u>Baer criterion</u> - see Alg 3)

Projective resolutions

Notation) A chain complex X s.t. $X_n = 0$ all $n<0$ can be identified with a positive chain complex

$$\cdots\cdots X_2 \xrightarrow{d} X_1 \xrightarrow{d} X_0 .$$

Write $Ch(\mathcal{C})_{\geq 0}$ for cat of positive chain complexes in $\mathcal{C}$.

Def) Let $\mathcal{C}$ be an abelian cat and $A \in \mathcal{C}$.

A projective resolution of A is a chain complex $C \in Ch(\mathcal{C})_{\geq 0}$ with a map $C_0 \xrightarrow{\varepsilon} A$ such that

① $\cdots\cdots C_2 \xrightarrow{d} C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} A \longrightarrow 0$ is exact

② each C_i is projective.

We can give more conceptual def, as below. Firstly,

- a chain map $f: A \longrightarrow B$ is a quasi-iso if $H_n f : H_n A \longrightarrow H_n B$ is isomorphism in $\mathcal{C}$ $\forall n$.
- For $A \in \mathcal{C}$, let $A[0] = \cdots \to 0 \to 0 \longrightarrow A$ be positive chain complex, degree 0.

Proposition

Projective resolution C of A $\equiv$ quasi-iso $C \longrightarrow A[0]$ with each C_i projective.

Proof

- A chain map $C \longrightarrow A[0]$:

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ C_2 \arrow[r] \arrow[d, "d"'] & 0 \arrow[d] \\ C_1 \arrow[r] \arrow[d, "d"'] & 0 \arrow[d] \\ C_0 \arrow[r, "\varepsilon"] & A \end{tikzcd}$$

is specified by a single map $C_0 \xrightarrow{\varepsilon} A$ such that $\varepsilon d = 0$

- It is a quasi-iso $\Longleftrightarrow$

a) For $n \geq 1$, $H_n(C) \cong H_n(A[0]) = 0$ (ie. $H_n(C) = 0$)

& b) $H_0(C) \xrightarrow{H_0(\varepsilon)} H_0(A[0])$

$$\begin{array}{ccc} H_0(C) & \xrightarrow{H_0(\varepsilon)} & H_0(A[0]) \\ \| & & \| \\ C_0/\operatorname{im} d & \xrightarrow{\bar{\varepsilon}} & A \\ x + \operatorname{im} d & \longmapsto & \varepsilon x \end{array}$$

induced map is invertible

Now $\bar{\varepsilon}$ is invertible $\Longleftrightarrow$

b1) $\bar{\varepsilon}$ is inj. (it's kernel $\ker \varepsilon / \operatorname{im} d = 0$)

b2) $\bar{\varepsilon}$ is surj. (equiv, ε is surj) so

(a), (b1), (b2) $\Longleftrightarrow$ $\cdots C_1 \xrightarrow{d} C_0 \xrightarrow{\varepsilon} A \longrightarrow 0$ is exact

where (b2) corresp. to ex@A,
(b1) - - - - - - ex@C_0,
(a) - - - - - - other positions. $\square$

Proposition

If $\mathcal{C}$ has enough projectives, then each object has a projective resolution.

Proof. Let $A \in \mathcal{C}$ & consider $C_0 \xrightarrow{\varepsilon} A$ with C_0 proj.

- Then $C_0 \xrightarrow{\varepsilon} A \to 0$ is exact.

- Now form

$$\begin{tikzcd}
 & \ker\varepsilon \arrow[dr, hook, "i"] & & & \\
C_1 \arrow[ur, "p_0"] \arrow[rr, "d"] & & C_0 \arrow[r, "\varepsilon"] & A \arrow[r] & 0
\end{tikzcd}$$

with C_1 projective.

- Then $\ker\varepsilon = \operatorname{im} p_0 \leq \operatorname{im} d \leq \ker\varepsilon$ so $\ker\varepsilon = \operatorname{im} d$.

- Now continuing in this way

$$\begin{tikzcd}
 & & & \ker\varepsilon \arrow[dr, hook, "i"] & & & \\
\cdots \; C_2 \arrow[rr, "d"] \arrow[dr] & & C_1 \arrow[ur, "p_0"] \arrow[rr, "d"] & & C_0 \arrow[r, "\varepsilon"] & A \arrow[r] & 0 \\
 & \ker p_0 \arrow[ur] & & & & &
\end{tikzcd}$$

we obtain a projective resolution. $\square$

Derived Functors

let $F: \mathcal{C} \to \mathcal{D}$ be rex functor between abelian cats.

The n'th left derived functor

$L_nF: \mathcal{C} \longrightarrow \mathcal{D}$ is defined as follows:

@ $x \in \mathcal{C}$, let $X_\bullet \xrightarrow{d} X[0]$ be proj. resolution of X

$$(X_\bullet = \cdots \cdots X_2 \xrightarrow{d} X_1 \xrightarrow{d} X_0)$$

so can form

$$FX_\bullet = \cdots \to FX_2 \xrightarrow{d} FX_1 \xrightarrow{d} FX_0 .$$

We define $L_nF(X) = H_n(FX_\bullet)$

Still have to define L_nF on morphisms.

Remark: Strange definition, since pr. resol. not functorial nor unique up to iso. However they are so, up to homotopy, & we will use this to define L_nF on morphisms.

- Let $[0] : \mathcal{C} \to Ch(\mathcal{C})_{\geq 0}$ be the Functor sending $X \mapsto X[0]$ &

$f : X \to Y \mapsto f[0] : X[0] \longrightarrow Y[0]$

f in degree 0, else 0.

Lemma

Consider $f : A \to B \in \mathcal{C}$ & proj. resolutions $A_\bullet, B_\bullet$ of A & B.
Then $\exists$ a chain map $f_\bullet$ st the square

$$\begin{tikzcd}
A_\bullet \arrow[r, "f_\bullet"] \arrow[d, "d"'] & B_\bullet \arrow[d, "d"] \\
A[0] \arrow[r, "f[0]"'] & B[0]
\end{tikzcd}$$

commutes. Moreover, $f_\bullet$ is unique with this property up to homotopy.

~~Proof~~ We construct $f_\bullet$ inductively.

- Firstly,

$$\begin{tikzcd}
\cdots \arrow[r] & A_1 \arrow[r, "d"] & A_0 \arrow[r, "d"] \arrow[d, "f_0"] \arrow[dr, phantom, "//"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & B_1 \arrow[r, "d"'] & B_0 \arrow[r, "d"'] & B \arrow[r] & 0
\end{tikzcd}$$

as A_0 proj, $B_0 \twoheadrightarrow B$ epi obtain f_0.

- Now $d f_0 d = f d d = 0$, so obtain

$$\begin{tikzcd}
\cdots \arrow[r] & A_1 \arrow[rr, "d"] \arrow[dr, red, "\exists!"'] & & A_0 \arrow[r, "d"] \arrow[dd, "f_0"] \arrow[dr, phantom, "="] & A \arrow[r] \arrow[dd, "f"] & 0 \\
 & & \ker d \arrow[dr] & & & \\
\cdots \arrow[r] & B_1 \arrow[rr, "d"'] \arrow[ur] & & B_0 \arrow[r, "d"'] & B \arrow[r] & 0
\end{tikzcd}$$

Moreover, by exactness, $\operatorname{im}(d : B_1 \to B_0) = \ker(d)$, so $B_1 \longrightarrow \ker d$ is epi.

Hence as A_1 proj., $\exists f_1$ as below.

$$\begin{tikzcd}
\cdots \arrow[r] & A_1 \arrow[rr, "d"] \arrow[dr, red, "\exists!"'] \arrow[dd, red, "\exists f_1"'] & & A_0 \arrow[r, "d"] \arrow[dd, "f_0"] \arrow[dr, phantom, "="] & A \arrow[r] \arrow[dd, "f"] & 0 \\
 & & \ker d \arrow[dr] & & & \\
\cdots \arrow[r] & B_1 \arrow[rr, "d"'] \arrow[ur] & & B_0 \arrow[r, "d"'] & B \arrow[r] & 0
\end{tikzcd}$$

Now continue inductively.

- For uniqueness, suppose
$$\begin{tikzcd} A_0 \arrow[r, shift left, "f_0"] \arrow[r, shift right, red, "g"'] \arrow[d, "d"'] & B_0 \arrow[d, "d"] \\ A[0] \arrow[r, "{f[0]}"'] & B[0] \end{tikzcd}$$

Then $d(f_0 - g_0) = fd - fd = 0$, so

Then we have

$$\begin{tikzcd} \cdots A_2 \arrow[r, "d"] & A_1 \arrow[r, "d"] & A_0 \arrow[r, "d"] \arrow[dl, red, "\exists!"'] \arrow[dd, "f_0 - g_0"] & A \arrow[r] \arrow[dd, "f"] & 0 \\ & \ker d \arrow[dr, hook] & & & \\ \cdots B_2 \arrow[r, "d"'] & B_1 \arrow[u] \arrow[r, "d"'] & B_0 \arrow[r, "d"'] & B \arrow[r] & 0 \end{tikzcd}$$

but as A_1 proj, $B_1 \twoheadrightarrow \ker d$ epi,

$$\begin{tikzcd} \cdots A_2 \arrow[r, "d"] & A_1 \arrow[r, "d"] & A_0 \arrow[r, "d"] \arrow[dl] \arrow[ddl, red, "\exists h_0"'] \arrow[dd, "f_0 - g_0"] & A \arrow[r] \arrow[dd, "f"] & 0 \\ & \ker d \arrow[dr, hook] & & & \\ \cdots B_2 \arrow[r, "d"'] & B_1 \arrow[u] \arrow[r, "d"'] & B_0 \arrow[r, "d"'] & B \arrow[r] & 0 \end{tikzcd}$$

satisfying $dh_0 = f_0 - g_0$.

Next we need a map $h_1 : A_1 \to B_2$ s.t.

$dh_1 + h_0 d = f_1 - g_1$

or equiv. st. $dh_1 = f_1 - g_1 - h_0 d := k$

$$\begin{tikzcd} \cdots A_2 \arrow[r, "d"] & A_1 \arrow[r, "d"] \arrow[d, red, "k"] & A_0 \arrow[r, "d"] \arrow[d, "f_0 - g_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\ \cdots B_2 \arrow[r, "d"'] & B_1 \arrow[r, "d"'] & B_0 \arrow[r, "d"'] & B \arrow[r] & 0 \end{tikzcd}$$

Now $dk = df_1 - dg_1 - dh_0 d = f_0 d - g_0 d - (f_0 - g_0)d = 0$

so

$$\begin{tikzcd}
 & & A_1 \arrow[r, "d"] \arrow[d, "k"] \arrow[dl, "\exists!"'] \arrow[dll, bend right, "\exists h_1"'] & A_0 \arrow[r, "d"] \arrow[d, "f_0 - g_0"] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots B_2 \arrow[r, "d"'] \arrow[r, "\ker d"] & \ker d \arrow[r] & B_1 \arrow[r, "d"'] & B_0 \arrow[r, "d"'] & B \arrow[r] & 0
\end{tikzcd}$$

k factors through $\ker d$, & now using proj. of A_2, we obtain $h_1, \ldots$
& so on inductively $\square$

With this in place, given $f: A \to B \in \mathcal{C}$, we obtain $f_\bullet : A_\bullet \to B_\bullet$ as above, & so $Ff_\bullet : FA_\bullet \to FB_\bullet$ & we define

$$L_nF(f) = H_n(Ff_\bullet) : H_n(FA_\bullet) \longrightarrow H_n(FB_\bullet)$$

Proposition

$L_nF : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor.

~~Proof~~ Firstly, observe that if

$$\begin{tikzcd}
A_\bullet \arrow[r, shift left, "f_\bullet"] \arrow[r, shift right, "f'_\bullet"'] \arrow[d, "d"'] & B_\bullet \arrow[d, "d"] \\
A[0] \arrow[r, "f[0]"'] & B[0]
\end{tikzcd}$$

, then $f_\bullet \sim f'_\bullet$ so $Ff_\bullet \sim Ff'_\bullet$ so $H_n(Ff_\bullet) = H_n(Ff'_\bullet)$ as homology identifies homotopic maps.

- Hence L_nF is well defined on **morphisms**.

Consider $A \xrightarrow{f} B \xrightarrow{g} C$. Then

$$\begin{tikzcd}
A_\bullet \arrow[r, "f_\bullet"] \arrow[d, "d"'] \arrow[dr, phantom, "="] & B_\bullet \arrow[r, "g_\bullet"] \arrow[d, "d"] \arrow[dr, phantom, "="] & C_\bullet \arrow[d, "d"] \\
A[0] \arrow[r, "f[0]"] \arrow[rr, bend right, "gf{[0]}"'] & B[0] \arrow[r, "g[0]"] \arrow[d, phantom, "="] & C[0] \\
 & {} &
\end{tikzcd}$$

so

by **lemma**, have $g_\bullet f_\bullet \sim (gf)_\bullet$, so $F(g_\bullet)F(f_\bullet) \sim F(gf_\bullet)$ so

$$\begin{array}{ccc}
H_nF(g_\bullet)H_nF(f_\bullet) & = & H_nF(gf_\bullet) \\
\| & & \| \\
L_nF(g)L_nF(f) & & L_nF(gf).
\end{array}$$

Similarly, L_nF pres. identities & so is a functor. $\square$

Right derived functors

There is a dual story of right derived functors.

- An inj resolution of $X \in \mathcal{C}$ is an exact sequence $0 \to X \to X^0 \to X^1 \to \dots$ with each X^i injective.
- This gives a positive cochain comp. $X^\bullet$ & morphism $X[0] \to X^\bullet$ which induces an iso on cohomology.

- If $\mathcal{C}$ has enough injectives, each ob X has an injective resolution.
- If $F : \mathcal{C} \to \mathcal{D}$ is lex functor & has enough injectives, we can form its n'th right derived functor $R^n F : \mathcal{C} \to \mathcal{D}$, which has value $R^n F(X) = H^n(F X^\bullet)$.

- Through duality, we can look at it as -
  from $F : \mathcal{C} \to \mathcal{D}$ rex, we get
  $$F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op} \text{ lex so form}$$
  $$L_nF : \mathcal{C}^{op} \to \mathcal{D}^{op} \text{ \& then}$$
  $$R_nF = (L_nF)^{op} : \mathcal{C} \to \mathcal{D},$$

- Next time - props of derived functors & examples (Ext, Tor & group cohomology)