Lecture 9

This time we look at props of derived functors before turning to the examples of Tor and Ext, which are obtained by

- left deriving tensor prod. functors
- right deriving hom functors.

Split short exact sequences

- Recall that given $A, B \in \mathcal{C}$ an abelian cat, we can form the biproduct diagram

$$\begin{tikzcd} A \arrow[r, shift right, "i"'] & A \oplus B \arrow[l, shift right, "p"'] \arrow[r, shift left, "q"] & B \arrow[l, shift left, "j"] \end{tikzcd}.$$

Lemma

$$0 \longrightarrow A \xrightarrow{i} A \oplus B \xrightarrow{q} B \longrightarrow 0 \text{ is short exact.}$$

Proof Recall $i = \langle 1_A, 0 \rangle : A \longrightarrow A \oplus B$ (induced map to product)

$q = \langle 0, 1_B \rangle : A \oplus B \longrightarrow B$ (product projⁿ)

As $pi = 1$, i is mono. As $qj = 1$, q is epi.

So exact at A, B.

Given $C \xrightarrow{\langle f, g \rangle} A \oplus B$ st $0 = q \langle f, g \rangle = g$,

so

$$\begin{tikzcd} C \arrow[r, "{\langle f, g \rangle}"] \arrow[dr, "f"'] & A \oplus B \\ & A \arrow[u, "{\langle 1_A, 0 \rangle}"'] \end{tikzcd} \Rightarrow \ker q = A.$$

Also $\operatorname{im}(i) = A$ as A mono $\Rightarrow$ exact @ $A \oplus B$.

(Such a sequence is called split exact.)

Lemma (Horseshoe lemma)

$\mathcal{C}$ an abelian cat with enough projectives. Given a ses $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, & projective resolutions $A_\bullet \xrightarrow{d_A} A[0]$, $C_\bullet \xrightarrow{d_C} C[0]$, we can find a comm. diagram

$$\begin{tikzcd}
A_\bullet \arrow[r, "i"] \arrow[d, "d_A"'] & B_\bullet \arrow[r, "q"] \arrow[d, "d"] & C_\bullet \arrow[d, "d_C"] \\
A[0] \arrow[r, "f(0)"'] & B[0] \arrow[r, "g(0)"'] & C[0]
\end{tikzcd}$$

whose central column is a proj. resolution of B, whose top row is levelwise split exact.

~~Proof~~

Need

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots & \vdots \\
A_1 \arrow[r, "i"] \arrow[d, "d_A"'] & A_1 \oplus C_1 \arrow[r, "q"] \arrow[d, "d_1 \, ?"] & C_1 \arrow[d, "d_C"] \\
A_0 \arrow[r, "i"] \arrow[d, "d_A"'] & A_0 \oplus C_0 \arrow[r, "q"] \arrow[d, "d_0 \, ?"] & C_0 \arrow[d, "d_C"] \\
A \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots & \vdots \arrow[d] & \\
 & A_1 \arrow[r, "i"] \arrow[d, "d_A"'] & A_1 \oplus C_1 \arrow[r, "q"] & C_1 \arrow[d, "d_C"] & \\
 & A_0 \arrow[r, "i"] \arrow[d, "d_A"'] & A_0 \oplus C_0 \arrow[r, "q"] & C_0 \arrow[d, "d_C"] \arrow[dl, "\exists h"'] & \\
0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0
\end{tikzcd}
$$

as C_0 projective

& then take

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots & \vdots \arrow[d] & \\
 & A_1 \arrow[r, "i"] \arrow[d, "d_A"'] & A_1 \oplus C_1 \arrow[r, "q"] & C_1 \arrow[d, "d_C"] & \\
 & A_0 \arrow[r, "i"] \arrow[d, "d_A"'] & A_0 \oplus C_0 \arrow[r, "q"] \arrow[d, "d_0 = (fd, h)"'] & C_0 \arrow[d, "d_C"] \arrow[dl, "h"'] & \\
0 \arrow[r] & A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r] & 0
\end{tikzcd}
$$

as C_0 proj.

- By snake lemma, get

$$\operatorname{coker} d_A \longrightarrow \operatorname{coker} d_0 \longrightarrow \operatorname{coker} d_C \text{ exact}$$

but of form

$$0 \longrightarrow \operatorname{coker} d_0 \longrightarrow 0$$ as d_A, d_C epi $\Rightarrow$ $\operatorname{coker} d_0 = 0$ so d_0 epi.

We need

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & & \\
& A_1 \arrow[r, "i"] \arrow[d, "d_A"'] \arrow[dr, phantom, "="] & A_1 \oplus C_1 \arrow[r, "q"] \arrow[d, "d_1"] \arrow[dr, phantom, "="] & C_1 \arrow[d, "d_c"] & \\
& A_0 \arrow[r, "i"] \arrow[d, "d_A"'] & A_0 \oplus C_0 \arrow[r, "q"] \arrow[d, "{(fd, h)}"'] & C_0 \arrow[d, "d_c"] & \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

& the two commutative squares force it to be of the form

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & & \\
& A_1 \arrow[r, "i"] \arrow[d, "d_A"'] & A_1 \oplus C_1 \arrow[r, "q"] \arrow[d, "{\begin{pmatrix} d_A & x \\ 0 & d_c \end{pmatrix} = d_1}"] & C_1 \arrow[d, "d_c"] & \\
& A_0 \arrow[r, "i"] \arrow[d, "d_A"'] & A_0 \oplus C_0 \arrow[r, "q"] \arrow[d, "{(fd, h) = d_0}"] & C_0 \arrow[d, "d_c"] & \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

as C_0 proj

where $x : C_1 \to A_0$.

Then $d_0 d_1 = 0$ ~~says~~

$$(fd, h)\begin{pmatrix} d_A & x \\ 0 & d_c \end{pmatrix} = (0, fdx + hd_c) = (0, 0)$$

or $fdx = -hd_c$

but we can find such an x in

2 steps:

$$
\begin{tikzcd}
 & & C_1 \arrow[dl, "\exists!"'] \arrow[d, "d_c"] & \\
A_0 \arrow[d, "d"'] & \ker g \arrow[dr] & C_0 \arrow[d, "-h"'] \arrow[dr, "d_c"] & \\
A \arrow[ur] \arrow[rr, "f"'] & & B \arrow[r, "g"'] & C
\end{tikzcd}
$$

& then by exactness at B:

$$
\begin{tikzcd}
 & & C_1 \text{ - proj} \arrow[dll, bend right, "\exists x"'] \arrow[dl, "\exists!"'] \arrow[d, "d_c"] & \\
A_0 \arrow[d, "d"'] & \ker g \arrow[dr] & C_0 \arrow[d, "-h"'] \arrow[dr, "d_c"] & \\
A \arrow[ur] \arrow[rr, "f"'] & & B \arrow[r, "g"'] & C
\end{tikzcd}
$$

etc.

Now one continues inductively to get the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "d_A"'] & \vdots \arrow[d, "d_2"] & \vdots \arrow[d, "d_C"] & \\
0 \arrow[r] & A_1 \arrow[r, "i"] \arrow[d, "d_A"'] & A_1 \oplus C_1 \arrow[r, "q"] \arrow[d, "d_1"] & C_1 \arrow[r] \arrow[d, "d_C"] & 0 \\
0 \arrow[r] & A_0 \arrow[r, "i"] \arrow[d, "d_A"'] & A_0 \oplus C_0 \arrow[r, "q"] \arrow[d, "d_0"] & C_0 \arrow[r] \arrow[d, "d_C"] & 0 \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

- This is ses of chain complexes as all rows are ses.
- Hence as left & right ch. comps are exact, so is central one (ex last wk) so it is projective resolution.

$\square$

Theorem

① Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an additive functor between abelian cats & suppose $\mathcal{C}$ has enough projs so that $L_nF : \mathcal{C} \to \mathcal{D}$ exists $\forall n$.

Then if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an ses is $\mathcal{C}$, we obtain a les

$$\cdots \; L_1FA \to L_1FB \to L_1FC \xrightarrow{\delta} L_0FA \xrightarrow{L_0F(f)} L_0FB \xrightarrow{L_0F(g)} L_0FC$$

② If F is rex, then $L_0F(A) \cong FA$ (naturally in A)

③ If F is rex,

then $L_nF(A) = 0$ all A, n $\iff$

$L_1F(A) = 0$ all A $\iff$

F is exact.

Remark

③ shows that the higher left derived functors measure the failure of F to be exact.

~~Proof~~ ① We have

$$\begin{tikzcd}
0 \arrow[r] & A_\bullet \arrow[r, "f_\bullet"] \arrow[d] & B_\bullet \arrow[r, "g_\bullet"] \arrow[d] & C_\bullet \arrow[r] \arrow[d] & 0 \\
 & A[0] \arrow[r, "{f[0]}"'] & B[0] \arrow[r, "{g[0]}"'] & C[0] &
\end{tikzcd}$$

levelwise split exact.

Then applying F levelwise, obtain

$$0 \longrightarrow FA_\bullet \xrightarrow{Ff_\bullet} FB_\bullet \xrightarrow{Fg_\bullet} FC_\bullet \longrightarrow 0$$

which is <u>still levelwise split exact</u> & so by the les of homology obtain a les

$$\cdots\cdots \to L_1FC \xrightarrow{\delta} L_0FA \longrightarrow L_0FB \longrightarrow L_0FC \longrightarrow 0$$

② At $A_\bullet \xrightarrow{d} A[0]$, we get

$$L_0FA = H_0FA_\bullet \xrightarrow{H_0(Fd)} H_0FA[0] = A,$$

which give the components of a natural transformation $L_0F \longrightarrow F$.

As F is rex,

$$FA_1 \xrightarrow{Fd_0} FA_0 \xrightarrow{Fd} FA \longrightarrow 0 \text{ is exact}$$

so
$$\begin{aligned} H_0FA_\bullet &= FA_0/\operatorname{im} Fd_0 \\ &\cong FA_0/\ker Fd \quad \text{by exactness} \\ &\cong FA \quad \text{as } Fd \text{ epi.} \end{aligned}$$

③ If F is exact, then

$$\cdots \longrightarrow FA_1 \xrightarrow{Fd} FA_0 \xrightarrow{Fd} FA \to 0$$

is exact so $L_nF(A) = H_n(FA_\bullet) = 0$ all $n > 0$

$\Rightarrow$ $L_1F(A) = 0$.

Now if $L_1F(A)$ all A,

Δ $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ a ses,

then by ①, ②, get les

$$\cdots L_1FC \longrightarrow L_0FA \cong FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \to 0$$
$$L_1FC = 0$$

so that $0 \to FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \longrightarrow 0$

a ses.

Therefore F is exact.

$\square$

Tensor product of bimodules

~~Def^n~~ An (R,S)-bimodule M is an abelian group M with
- a left R-module structure
- a right S-module - - - -

such that $(rm)s = r(ms)$
all $r \in R$, $m \in M$, $s \in S$.

Notation: can write $R \xrightarrow{M} S$.

Example

- Abelian groups $\equiv$ $\mathbb{Z}$-modules, left or right.
  $2.a = a + a = a.2$, etc.

  Then $(R, \mathbb{Z})$-bimods $\equiv$ left R-mods
  $(\mathbb{Z}, R)$-bimods $\equiv$ right R-mods

- If R is commutative, each R-module (left or right) has canonical (R,R)-bimodule structure.

- (R,S)-bimodules form a cat ${}_R\mathrm{Mod}_S$.

- The notation suggests that given

$$R \xrightarrow{M} S \xrightarrow{N} T,$$ we can compose them to get $$R \xrightarrow[M \otimes_S N]{} T$$ & we can.

S tensor product of bimodules.

- $M \otimes_S N$ classifies (R,S,T)-linear maps: given a (R,T)-bimodule A, a function $f : M \times N \to A$ is (R,S,T)-linear if
- $f(-,n) : M \to A$ a hom. of left R-mods.
- $f(m,-) : N \to A$ - - - - - right T-mods
- $f(m,sn) = f(ms,n)$ all m,s,n.

- I.e. $\exists\ M \times N \xrightarrow{\theta} M \otimes_S N$ (R,S,T)-linear st given $M \times N \xrightarrow{f} A$ (R,S,T)-linear $\exists!\ M \otimes_S N \xrightarrow{\bar{f}} A \in {}_R\mathrm{Mod}_T$ such that

$$\begin{tikzcd} M \times N \arrow[r, "\theta"] \arrow[dr, "f"'] & M \otimes_S N \arrow[d, dashed, "\bar{f}"] \\ & A \end{tikzcd}$$
$=$.

- As for tensor product of modules over commutative ring in Alg 3, it can be constructed as quotient of <u>free abelian grp</u> $F(M \times N)$ subject to necessary equations (Ex)

Aside

- One might wonder whether

$$\begin{tikzcd} R \arrow[r, "M", "/" marking] \arrow[rr, bend right, "M \otimes_S N"'] & S \arrow[r, "N", "/" marking] & T \end{tikzcd}$$

gives a category structure

but it is only associative <u>up to iso</u> - it is a so-called <u>bicategory</u>, Mod.

Like in any bicat, each hom category $\mathrm{Mod}(R,R) := (R,R)\text{-Bimod}$ is a monoidal category.

Tor functors

- In particular, we obtain a tensor product of right & left R-modules:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "A"] \arrow[rr, bend right, "A \otimes_R B"'] & R \arrow[r, "B"] & \mathbb{Z} \end{tikzcd}$$

which is a $(\mathbb{Z},\mathbb{Z})$-bimodule, ie an abelian group.

For $A \in \mathrm{Mod}_R$ (a right R-mod) obtain

$$R\text{-Mod} \xrightarrow{A \otimes_R -} \mathrm{Ab}$$

left R-mods

& if $B \in R\text{-Mod}$, obtain

$$\mathrm{Mod}\text{-}R \xrightarrow{- \otimes_R B} \mathrm{Ab}$$

- Both functors are rex.

Def^n) $\mathrm{Tor}_n^A(A,B) := L_n(- \otimes_R B)(A)$.

Explicitly, this is given by n'th homology of

$$\cdots \to A_2 \otimes_R B \to A_1 \otimes_R B \to A_0 \otimes_R B$$

where $A_\bullet$ is proj. resolution of $A \in R\text{-Mod}$, so $H_n(A_\bullet \otimes_R B)$.

no proof given

Equivalently, it can be calculated as $L_n(A\otimes_R -)(B)$, so take instead proj. resolution of B, tensor by A:

$$\cdots \longrightarrow A\otimes_R B_2 \longrightarrow A\otimes_R B_1 \longrightarrow A\otimes_R B_0$$

& calculate homology:

so $\underline{H_n(A_\bullet \otimes_R B)} \cong \underline{H_n(A \otimes_R B_\bullet)}$.

Ext functors (again, no proofs)

- Given $A \in R\text{-Mod}$, consider the hom functor

$$\mathrm{Hom}(A,-) = R\text{-Mod}(A,-) : R\text{-Mod} \longrightarrow \mathrm{Ab}$$
$$B \longmapsto R\text{-Mod}(A,B).$$

- It is lex & R-Mod has enough injs, so can form right derived functor $\mathrm{Ext}_n(A,-) := R^n\mathrm{Hom}(A,-) : R\text{-Mod} \longrightarrow \mathrm{Ab}$.

Thus $\mathrm{Ext}_n(A,B) = H^n \mathrm{Hom}(A, B^\bullet)$ where $B^\bullet$ is an injective resolution

$$B^1 \to B^2 \to B^3 \longrightarrow \cdots \quad \text{of } B.$$

- Also have $\mathrm{Hom}(-,B) : R\text{-Mod}^{op} \longrightarrow \mathrm{Ab}$ which is lex, so can calculate its right derived functor, $R^n\mathrm{Hom}(-,B)$, & since injectives in $R\text{-Mod}^{op} \equiv$ projectives in R-Mod, this is calculated as $H^n(\mathrm{Hom}(A_\bullet, B))$ where $A_\bullet$ is a projective resolution of A.

In fact, both coincide:

$$\mathrm{Ext}_n(A,B) = H^n(\mathrm{Hom}(A_\bullet,B)) = H^n(\mathrm{Hom}(A,B^\bullet)).$$

- Ext_n can be understood using extensions:
an extension of A by B is a ses
$$0 \to B \to X \to A \to 0.$$
- Two extensions of A by B are equiv ($\simeq$) if $\exists$ iso of ses of form
$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, "1"'] & X \arrow[r] \arrow[d, "S"] & A \arrow[r] \arrow[d, "1"] & 0 \\ 0 \arrow[r] & B \arrow[r] & Y \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$
- Then $\mathrm{Ext}^1(A,B) \cong \{\text{extensions of } A \text{ by } B\}/\simeq$
- In particular, the rhs is an abelian group too - its unit is $0 \to B \to B \oplus A \to A \to 0$, the split exact sequence.
- Will explore addition (Baer sum) in ex. class.
- Recall A proj. $\Leftrightarrow \hom(A,-)$ exact
$\Leftrightarrow \mathrm{Ext}_1(A,B) = 0$ all B -
this says that each ses with A proj is split (prove directly).
- For higher n, elements of $\mathrm{Ext}_n(A,B)$ are es $0 \to B \to \dots \to \dots \to \dots \to A \to 0$ of length n.

- Special examples of Ext are group cohomology & cohomology of Lie groups - we'll look at these (at least the first) next time, maybe also Hoschild cohomology.