Marked assignment 2 - Algebra 4

(Due **Sunday 28'th April**)

① Let K be a simplicial complex (as defined in the notes) with an ordering on it's set of vertices, & let K_n be the set of n-simplices in K. Then we have the maps $d_i : K_n \longrightarrow K_{n-1}$ for $0 \leq i \leq n$ which remove the i'th element.

a) Show that these satisfy the equations $d_i d_j = d_{j-1} d_i$ for $0 \leq i < j \leq n$.

b) Taking free abelian groups $C(K_n)$ we can extend these to homomorphisms $C(K_n) \xrightarrow{d_i} C(K_{n-1})$ for $0 \leq i \leq n$ & take alternating sum $d^n = \sum_{i=0}^{n} (-1)^i d_i : C(K_n) \longrightarrow C(K_{n-1})$.
Using the equations in ⓐ, show that
$C(K) := \cdots C(K_{n+1}) \xrightarrow{d^{n+1}} C(K_n) \xrightarrow{d^n} C(K_{n-1}) \cdots$
is a chain complex.
3

② Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a right exact functor between abelian categories. Prove that each left derived functor $L_i F : \mathcal{A} \longrightarrow \mathcal{B}$ is additive.
2

- In the course we introduced $\mathrm{Ext}^n(A,-)$ as the n'th right derived functor of $\mathrm{Hom}(A,-) : \mathrm{Mod}_R \longrightarrow \mathrm{Ab}$, but then mentioned that elements of $\mathrm{Ext}^n(A,B)$ (for $n>0$) are equiv. classes of exact sequences

$$0 \rightarrow B \rightarrow X_1 \rightarrow \cdots \rightarrow X_n \rightarrow A \rightarrow 0.$$

- Here we explore some of the algebra of such exact sequences.

③ (Splicing of exact sequences)

Consider exact sequences

$$0 \to B \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \to X_n \xrightarrow{f_n} A \to 0 \quad \&$$

$$0 \to C \xrightarrow{g} Y \xrightarrow{h} B \to 0.$$

- Show that

$$0 \to C \xrightarrow{g} Y \xrightarrow{f_0 h} X_1 \xrightarrow{f_1} X_2 \to \cdots \to X_n \xrightarrow{f_n} A \to 0$$

is exact. 2

④ Can you see a relationship between ③ & the les of cohomology for Ext? (I am not expecting a rigorous proof here)

- Can you generalise splicing as in Q3 to longer sequences? 1

⑤ (Decomposing exact sequences)

The first question shows how to "splice" exact sequences into longer ones. 2

Show that every exact sequence of length n can be obtained by splicing n short exact sequences.