Exercises 3

(1) Hilbert's Nullstellensatz fails over $\mathbb{R}$

- The Nullstellensatz says that if k is algebraically closed, & if $S \subseteq k[x_1, \ldots, x_n]$ a set of polynomials, then

$$I\,V(S) = \mathrm{Rad}\langle S \rangle .$$

- Show this is false over $\mathbb{R}$, by considering the polynomial $x^2+1 \in \mathbb{R}[x]$.

(2)

let $\bar{a} \in K^n$ for K a Field.

The goal of the First ex. is to show $I(\bar{a}) = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$ & that this is maximal.

a) Show $K[x_1, \ldots, x_n] \xrightarrow{ev_{\bar{a}}} K$ is a surjective algebra homomorph with kernel $I(\bar{a})$ & conclude this is maximal.

b) Show $I(\bar{0}) = \langle x_1, \ldots, x_n \rangle$.

c) Describe an algebra iso

$$\begin{aligned} K[x_1, \ldots, x_n] &\longrightarrow K[x_1, \ldots, x_n] \\ x_i &\longmapsto x_i - a_i \end{aligned}$$

& use it to prove the claim.

③ The Zariski Topology

For k a field, the set k^n has a topology with closed sets the varieties $U(I)$ where I is a set of polynomials.

Recall that the axioms for a topology on X (expressed using closed sets) say:

- ϕ, X are closed
- closed sets are closed under inf. intersection & finite union.

Verify these for k^n by showing

ⓐ $U(\phi) = k^n$, $U(k[x_1,\dots,x_n]) = \phi$,

ⓑ $\bigcap_{i\in X} V(A_i) = V(\bigcup_{i\in I} A_i)$

ⓒ $U(A) \cup U(B) = U(AB)$ where

$AB = \{fg : f\in A, g\in B\}$.