Lecture 4 - Localization

Def) R a comm. ring. A subset $U \subseteq R$ is multiplicatively closed if $1 \in U$ & $a, b \in U \Rightarrow ab \in U$.

Example) R is an integral domain ($ab = 0 \Rightarrow a = 0$ or $b = 0$) $\Longleftrightarrow$ $R \setminus \{0\}$ is multiplicatively closed.

Example) If $a \in R$, then $S_a = \{a^n : n \in \mathbb{N}\}$ is mult. closed.

lemma) An ideal $I \subseteq R$ is prime $\Longleftrightarrow$ $R \setminus I$ is multiplicatively closed.

Proof) To say $R \setminus I$ is multiplicatively closed is to say $1 \notin I$ (I is proper), $a \notin I$ & $b \notin I \Rightarrow ab \notin I$ (equiv. $ab \in I \Rightarrow a \in I$ or $b \in I$).

- If R is an integral domain (eg. $\mathbb{Z}$) we can form the field of fractions $\mathrm{Frac}(R)$, whose elts are fractions r/s where $s \neq 0$ (eg. $\mathbb{Q}$)

- Localization generalises this to other multiplicative subsets other than $R \setminus \{0\}$.

Localization at a multiplicative subset

Given $U \subseteq R$ mult. closed, we define an equivalence relation on $R \times U$ by

- $(x,u) \simeq (y,v) \iff \exists\, t \in U$ such that $t(xv - yu) = 0$

Exercise: check this is an equiv. rel.

Remark: If R is an integral domain & U does not cont. 0, the equiv relation reduces to
$$(x,u) \simeq (y,v) \iff xv - yu = 0.$$

We write x/u for the equivalence class $[(x,u)]$ & $U^{-1}R$ for the set of these fractions.

- We define a comm ring structure on $U^{-1}R$ by

- $(x/y)(x'/y') = xx'/yy'$
- $(x/y) + (x'/y') = (xy' + x'y)/yy'$
- Zero $0/1$ & unit $1/1$

Theorem

Let R be a commutative ring & $U \subseteq R$ mult closed.

① With the above operations, $U^{-1}R$ is a commutative ring &
$$p : R \longrightarrow U^{-1}R : r \longmapsto r/1$$
a homomorphism.

② If $u \in U$ then $p(u)$ is invertible (a unit).

Moreover, if $f : R \longrightarrow S$ also inverts elements of U then $\exists !\ \bar{f} : U^{-1}R \longrightarrow S$ such that
$$\begin{tikzcd} R \arrow[r, "p"] \arrow[dr, "f"'] & U^{-1}R \arrow[d, "\bar{f}"] \arrow[dl, phantom, "="] \\ & S \end{tikzcd}.$$

~~Proof~~

The first part is straightforward - one checks that the operations are well defined on equiv. classes (fractions) & the comm. ring structure is easily checked. (Exercise!)

- For the second part, certainly if $u \in U$, then $\rho(u) = u/1$ is invertible, since as $1 \in U$, $1/u \in U^{-1}R$ & then

$$u/1 \cdot 1/u = u/u = 1/1 \quad (\text{as } u.1 - 1.u = 0)$$

- Now suppose $f : R \to S$ has $f(u)$ invertible if $u \in U$.

- If we are to have

$$\begin{tikzcd} R \arrow[r, "\rho"] \arrow[dr, "f"'] & U^{-1}R \arrow[d, "\bar{f}"] \\ & S \end{tikzcd}$$

we must set

$$\bar{f}(r/1) = f(r).$$

At a general element $a/b \in U^{-1}R$, we have $a/b = a/1 \cdot 1/b$ where $b \in U$.
If $\bar{f}$ is to preserve multiplication & unit (& so inverses) we must have
$\bar{f}(a/b) = \bar{f}(a/1) \cdot \bar{f}(1/b) = f(a) \cdot f(b)^{-1}$.
It is straightforward to check that, with this definition, $\bar{f}$ is indeed a homomorphism.

$\square$

Examples

(1) If R is an integral domain, & $U = R \setminus \{0\}$, then $U^{-1}R$ is Field of Fractions of R.

a) E.g. $\mathbb{Z} \longmapsto \mathbb{Q}$

b) If $R = k[x_1, \dots, x_n]$ for k a integral domain, its field of fractions consists of expressions

$$f(x_1, \dots, x_n) / g(x_1, \dots, x_n)$$

with g non-zero, often called rational functions.

(2). If k a Field & $A \subseteq k^n$ a variety, recall the co-ord. ring $k(A)$ whose elts are polynomial functions $A \longrightarrow k$; equiv

$$k(A) = k[x_1, \dots, x_n] / I(A).$$

- If $a \in A$, then

$$I(A) \subseteq I(a) \subseteq k[x_1, \dots, x_n] \text{ maximal}, \text{ so}$$

so image $\bar{I}(a)$ of $I(a)$ maximal in $k(A)$ - contains poly. Functions $A \xrightarrow{f} k$ w/ $f(a) = 0$.

- Since $\bar{I}(a)$ is maximal, it is prime, then $k(A)_{\bar{I}(a)}$ contains fractions

$$f(x) / g(x) \text{ where } g(a) \neq 0 :$$

these are rational functions, which are well defined in a arb small neighborhood of $a \in A$.

(3) Given $a \in R$, recall $S_a = \{a^n : n \in \mathbb{N}\} \subseteq R$.
Write $R_a = S_a^{-1} R$. It consists of fractions
$\{ r/a^n : r \in R \}$.

Note $\mathbb{Z}_p \neq \mathbb{Z}_{(p)}$:(easily confused.

Ideals of localizations

Def^n) A local ring is one with a unique maximal ideal.

Prop^n) If $P \subset R$ is a prime ideal, then R_P is local.

~~Proof~~ Recall $R_P = U^{-1}R$ where $U = R \setminus P$.

- The unique maximal ideal $P^e = \{ a/b : a \in P \}$.
- Lets check an ideal. If $a/b, c/d \in P^e$, then $a/b + c/d = \dfrac{ad+bc}{bd}$ & $ad+bc \in P$ as P an ideal & $a, c \in P$.

Also $ra/b \in P^e$ as $ra \in P$ since P ideal.

- To see P^e max, suppose $P^e \subset I \subset R$. Then I contains element a/b with $a \notin P$.
- But then $a \in U \Rightarrow a/b$ invertible with inv b/a.
- So I contains a unit $\Rightarrow$ contains $1 \Rightarrow$ $I = R$. Hence P^e maximal.

If $I \subset R$ is maximal, each element (a non-unit) a/b.

Hence $a \notin U \Rightarrow a \in P \Rightarrow a/b \in P^e$

so $I \subseteq P^e$. $\square$

Exercise: show that the unique max^l ideal of R_P consists of precisely the non-units.

Localization of modules

- For $U \subseteq R$ mult. closed, we can localize an R-module M in just the same way: we have e-rel on $M \times U$ where

$$(m,u) \sim (n,v) \text{ if } \exists t \in U : t(vm - un) = 0$$

& then $U^{-1}M$ contains e-classes, the fractions m/u where $m \in M$ and $u \in U$.

Addition is as before: $m/u + n/v = \dfrac{vn + un}{uv}$

whilst the action is $r . m/u = rm/u$.

- A more conceptual approach is as follows:

since we have $\rho : R \longrightarrow U^{-1}R : r \mapsto r/1$, $U^{-1}R$ becomes an R-module by restriction:

$$r.\left(s/u\right) = rs/u.$$

- Then we can form the tensor product of R-modules $U^{-1}R \otimes_R M$ (as in Alg 3)

Proposition

We have an isomorphism $U^{-1}M \cong U^{-1}R \otimes_R M$ of R-modules.

Proof/ The assignment

$$\begin{aligned} U^{-1}R \times_R M &\longrightarrow U^{-1}M \\ (r/u, m) &\longmapsto rm/u \end{aligned}$$

is well defined & R-bilinear.

- So it induces a unique R-module map

$$\begin{aligned} U^{-1}R \otimes_R M &\xrightarrow{k} U^{-1}M \\ r/u \otimes m &\longmapsto mr/u \end{aligned}$$

on generators.

It has inverse the assignment

$$\begin{aligned} l : U^{-1}M &\longrightarrow U^{-1}R \otimes M \\ m/u &\longmapsto 1/u \otimes m \end{aligned}$$

which is also an R-module map. (Exercise)

- Clearly $kl = 1$.
- To see $lk : U^{-1}R \otimes M \longrightarrow U^{-1}R \otimes M$ is identity, it suffices to check on generators.

$$r/u \otimes m \overset{l}{\longmapsto} rm/u \overset{k}{\longmapsto} 1/u \otimes rm$$

but $r/u \otimes m = r(1/u \otimes m) = 1/u \otimes rm$ by bilinearity. □

Localization of modules gives a functor

$$\begin{aligned} U^{-1} : R\text{-Mod} &\longrightarrow R\text{-Mod} \\ M &\longmapsto U^{-1}M \quad \& \end{aligned}$$

$$f : M \longrightarrow N \quad \longmapsto \quad U^{-1}f : U^{-1}M \longrightarrow U^{-1}N$$
$$m/u \longmapsto fm/u \quad .$$

We have a natural isomorphism

$$\begin{tikzcd} R\text{-Mod} \arrow[r, bend left, "U^{-1}"] \arrow[r, bend right, "U^{-1}R\otimes -"'] \arrow[r, phantom, "\cong"] & R\text{-Mod} \end{tikzcd} \quad \&$$

since $U^{-1}R \otimes -$ is left adjoint it preserves colims &, in particular, surjections.
Perhaps surprisingly it also preserves injections.

Prop) $U^{-1} : R\text{-Mod} \longrightarrow R\text{-Mod}$ preserves injections.

Proof) Consider $M \xrightarrow{f} N$ mono &
$U^{-1}f : U^{-1}M \longrightarrow U^{-1}N : m/u \longmapsto fm/u$.
If $fm/u = 0$, $\exists\, t \in U$ st $t(fm) = f(tm) = 0$
but f is mono, so $tm = 0$. Hence
$m/u = (1/tu) . (tm/1) = 0$, as required.

$\square$

- It follows (as we will see in homological algebra) that U^{-1} is an exact functor, one preserving kernels, cokernels, short exact sequences etc.
Equiv, it preserves finite lims & colims.
- What is remarkable about localization is that also reflects isomorphisms (& so reflects finite limits & colimits) if we allow U to vary.

~~Prop~~ ① An R-mod $M = 0 \iff M_P = 0$ for each maximal ideal P.

② An R-mod map $f: M \to N$ is surj / inj / iso $\iff f_P : M_P \to N_P$ is surj / inj / iso for each maximal ideal P.

~~Proof~~ ① Will prove that if $M \neq 0$, $\exists$ max ideal P with $M_P \neq 0$.

Choose $m \neq 0 \in M$

- The annihilator $Ann(m) = \{ r \in R : r.m = 0 \}$ is an ideal of R (ie. kernel of $- .m : R \to M$).
- It is a proper ideal as it doesn't contain 1.
- Therefore, by Zorn's lemma, it is contained in a max^l ideal $Ann(m) \subset P \subset R$.

Consider $m/1 \in R_P$.

If $m/1 = 0$ $\exists t \in R - P$ st $tm = 0$. But since $t \notin P$, $t \notin Ann(M)$ so this is impossible.

② Consider $f: M \to N$ & suppose f_P inj all such P, so $Ker(f_P : M_P \to N_P) = 0$. Since $Ker(f_P) = Ker(f)_P$.

So $Ker(f)_P = 0 \Rightarrow Ker(f) = 0$ by first part of prop. Hence f inj.

Argument for surj similar using cokernels

For isos, combine the two. $\square$