L11 2024 - Representations of groups

- Let k be a field & Vect be the category of vector spaces over k (or k-modules).
- For V a vector space, let $Gl(V)$ be the group of invertible linear transformations $V \longrightarrow V$.
- Let G be a group.

Def^n) A G-module / representation of G is a group homomorphism
$$\rho : G \longrightarrow Gl(V).$$

- Equivalently, a G-module is specified by:
  - a vector space V
  - For $g \in G$, $v \in V$, an element $\rho(g).v \in V$

(normally written as $g.v$ or just gv)

such that

- $g(v+w) = gv + gw$ & $g(\lambda v) = \lambda g v$
- $gh(v) = g(hv)$ & $ev = v$.

unit

Examples

1) If V is a vector space, it has a trivial G-module structure : $g.v = v$ all $v \in V$.

2) The regular G-module is the vector space kG with basis G: its elements are k-linear sums $\lambda_1 g_1 + \dots + \lambda_n g_n$ where $\lambda_i \in k$, $g_i \in G$.

It is a G-module with action

$$g.(\lambda_1 g_1 + \dots + \lambda_n g_n) = \lambda_1 g g_1 + \dots + \lambda_n g g_n .$$

3) • More generally, if G acts on a set X, then if we form the vector space kX with basis X, then it has G-module structure

$$g.(\lambda_1 x_1 + \dots + \lambda_n x_n) = \lambda_1 g x_1 + \dots + \lambda_n g x_n .$$

• Such examples called permutation repres.

• Preceding example takes $X = G$.

• Eg. Symmetric group S_n acts on $\{1, \dots, n\}$ so obtain S_n-module on $k\{1, \dots, n\}$.

Matrix representations

- If V is a finite dim. vector space, of dim. n, then choosing a basis $B = \{v_1, \dots, v_n\}$ of V induces an iso of groups

$$Gl(V) \cong Gl_n(k)$$
$$X \longmapsto [X]_B$$

group of invertible $(n \times n)$-matrices with values in k.

- Therefore, composing with this iso $G \xrightarrow{\sigma} Gl(V) \cong Gl_n(k)$ gives a bij^n between representations of G on V & homomorphisms $G \xrightarrow{\sigma} Gl_n(k)$, sometimes called matrix representations

- In other words, for each $g \in G$, we have a matrix $\rho(g)$ such that $\rho(e) = 1$, $\rho(gh) = \rho(g)\rho(h)$

- Often, when giving examples of G-representations (on f.d. vect. spaces) we give them as matrix reps.

Examples

① (Trivial rep) For any d, define

$$\rho : G \to Gl_d(V)$$
$$g \longmapsto 1 \quad \text{all } g \in G.$$

② Let $C_n = \langle g : g^n = 1 \rangle$ be cyclic group of order n.

A 1-d complex rep is a homomorphism $C_n \xrightarrow{f} \mathbb{C}$.

- Then $\rho(g)^n = \rho(g^n) = \rho(g) = 1$ $\rho(g)$ is a n'th complex root of unity : of form $\cos(2k\pi/n) + i\sin(2k\pi/n)$ for $k = 0, \ldots, n-1$.
- There are n such representations.

- Eg. for C_4, there are 4, corresp. to roots $\{1, i, -1, -i\}$.

③ Let D_8 be the dihedral group

$$\langle a, b : a^4 = b^2 = 1,\ b^{-1}ab = a^{-1} \rangle.$$

It has 8 elements and describes symmetries of the square, which are generated by a rotation of order 4 & a reflection.

Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ & $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the matrices for such a rotation & reflection & define $\rho : G \to Gl_2(K) : a, b \longmapsto A, B$.

As $A^4 = B^2 = 1$ & $B^{-1}AB = A^{-1}$, this is a 2-d representation.

It is easy to see ρ is injective - it is a faithful representation.

Def) Matrix reps $\sigma, \tau : G \longrightarrow GL_n(k)$ are **equivalent** $\Longleftrightarrow$ $\exists\, T \in GL_n(k)$ such that

- $\sigma(g) = T^{-1}\tau(g)T$ all $g \in G$.

G-modules as modules over a ring kG

Def^n) The group algebra kG is a k-algebra whose vector space is kG as above: its elements are k-linear sums

$$\lambda_1 g_1 + \dots + \lambda_n g_n \quad \text{where } \lambda_i \in k, g_i \in G.$$

Multiplication on kG linearly extends the multiplic on G: $(\sum_i \alpha_i g_i)(\sum_j \alpha_j g_j) = \sum_{i,j} \alpha_i \alpha_j (g_i g_j)$.

& unit $e \in G$.

- Of course, kG is a ring. In fact, kG-modules over the ring kG $\equiv$ G-modules.

- Indeed, given a kG-mod A abelian group with action $\cdot$,
- It has k-vect space str $(\lambda, v) \longmapsto \lambda e . v$ & then G-module str: $(g, v) \longmapsto 1g . v$.
- Conversely, if V is a G-module with action $\cdot$, we define an action of kG on V by linear extension

$$(\sum_i \lambda_i g_i).v = \sum_i \lambda_i (g_i . v).$$

- The two processes are inverse.

- Henceforth, we will identify G-modules & kG-modules.

In particular, we have a category

$$G\text{-Mod} := kG\text{-Mod}$$

a morphism $f: V \longrightarrow W$ is a linear transformation such that $f(g \cdot v) = g \cdot f(v)$ $\forall g \in G, v \in V$.

These are called G-equivariant maps.

Exercise

Two matrix reps $\sigma, \tau : G \to Gl_n(k)$ are equiv $\Leftrightarrow$ the corresponding G-modules are isomorphic.

Other perspectives

- Let ΣG be the category with one object $\bullet$ & morphisms $\bullet \xrightarrow{g} \bullet$ the elements of G.
- A G-module is just a functor

$$\begin{array}{ccc} \Sigma G & \longrightarrow & \mathrm{Vect} \\ \bullet & & V \\ g\downarrow & \longmapsto & \downarrow g\cdot - \\ \bullet & & V \end{array}$$

- One doesn't need the assumption that k is a field - one can consider representations $\Sigma G \longrightarrow R\text{-Mod}$ over any ring R, & when $R = \mathbb{Z}$, we obtain functors $\Sigma G \longrightarrow \mathrm{Ab}$, which are the G-modules we considered last week.

Note that functors $\Sigma G \longrightarrow \mathrm{Set}$ are precisely G-sets - really, a rep of G is a functor from ΣG to some nice category of our choosing.

- However there is a nice structure theory for G-modules over a field k, & we will henceforth assume that k is a field.

Basic constructions

① As for modules over any ring, given $f: V \to W \in G\text{-Mod}$, we can form the submodules

$$\ker(f) \hookrightarrow V \quad \& \quad \mathrm{im}(f) \hookrightarrow W.$$

② We also have the direct sum (biprod.) $V \oplus W$ as usual:

action $g(v \oplus w) = gv \oplus gw$.

③ If $U, V \subseteq W$ are submodules of W, then we obtain the induced map

$$\begin{tikzcd} U \arrow[d, hook] \arrow[dr, bend left, hook] & \\ U \oplus V \arrow[r, "p"] & W \\ V \arrow[u, hook] \arrow[ur, bend right, hook] & \end{tikzcd} : u \oplus v \longmapsto u + v$$

The p is an iso $\Leftrightarrow$

- $\forall w \in W$, $\exists!\, u \in U, v \in V$ such that $u + v = w$

  In this case, we say $W = U \oplus V$ is an internal direct sum.

- Equivalently, $W = U + V$ & $U \cap V = \{0\}$.

Projections & direct sums

Def) $p : V \to V \in G\text{-Mod}$ is a projection if $p^2 = p$.

Lemma

If $p : V \to V$ is a projection, then $V = \ker(p) \oplus \operatorname{im}(p)$ – each direct sum of G-submodules arises in this way.

Proof

- Write $v = pv + (pv - v)$.
  - This is unique as if $v \in \ker(p) \cap \operatorname{im}(p)$, then $pv = 0$, but as $v = px$, we have $pv = p^2x = px = v$ so $v = 0$.
- Conversely, if $V = U \oplus W$ for submodules, define $p : V \longrightarrow V$ by $p(u+w) = u$. $\square$

Decomposing G-modules

Def) Let $V \neq 0$ be a G-module.

- It is said to be irreducible if its only submodules are $0 = \{0\}$ & V itself.
- Else, it is reducible.

- A matrix rep $\rho : G \longrightarrow GL_n(K)$ is said to be reducible if the corresponding G-module (K^n with action $g.v = \rho(g)v$) is reducible.

What does this mean in el. terms?

- It means K^n has G-submodule $U < K^n$.
- The subspace $U = \langle u_1, \dots, u_m \rangle$ extends to a basis $B = \langle u_1, \dots, u_m, v_1, \dots, v_\ell \rangle$.
- Let T be base-change matrix from standard basis to B.
- In this basis, the matrix rep $[g]_B = T^{-1}\rho(g)T$ has form

$$[g]_B = \left[\begin{array}{c|c} A(g) & B(g) \\ \hline 0 & C(g) \end{array}\right]$$

where each $A(g)$ has dim $m < n$.

- In this way, we see ρ is reducible $\Leftrightarrow$ it is equiv. to matrix rep. ad. block decomp of above form.

Theorem (Maschke)

Let G be a finite group & suppose char(K) does not divide order of G, $|G|$. (eg. if $K = \mathbb{R}$ or $\mathbb{C}$)

If V is a G-module & $U \leq V$ a proper submodule, then $\exists$ G-submodule W st $V = U \oplus W$.

Proof

- Firstly, as U subspace of V, can find linearly independent vectors giving **vector subspace W_0** st $V = U \oplus W_0$ as a vector space.
- This gives a projection of vector spaces
  $p : V \to V : u + w \mapsto u$ with image U & kernel W_0, but p need not be a G-module map.

- We will modify p to a G-module map
  $q : V \to V$ st $q^2 = q$ & $\operatorname{im} q = U$;
  then $V = \operatorname{im} q \oplus \ker q = U \oplus \ker q$, a dir. sum of G-submodules, as required.

- For $g \in G$, let $p_g : V \to V : v \mapsto g^{-1}(p(gv))$.
  As a composite of 3 linear maps, p_g is linear map.

- Define $q = \frac{1}{|G|} \sum_{g \in G} p_g$ as the "average" of these maps, which is again
- linear as it is a linear comb. of linear maps.
- To check q a G-module map:

$q(hv) = \frac{1}{|G|} \sum_{g \in G} g^{-1} p(g.hv)$.

Since each elt of G is uniquely of form gh^{-1} for some $g \in G$,

$$\frac{1}{|G|} \sum_{g \in G} g^{-1} p(g.hv) =$$

$$\frac{1}{|G|} \sum_{g \in G} (gh^{-1})^{-1} p(gh^{-1}.hv) \quad \text{using } V \text{ a } G\text{-mod.}$$

$$= \frac{1}{|G|} \sum_{g \in G} hg^{-1} p(gv)$$

$$= h. \frac{1}{|G|} \sum_{g \in G} g^{-1} p(gv) = h.q(v).$$

- Remains to show $\underline{\operatorname{im} q = U}$.

Let $u \in U$. Then

$$q(u) = \frac{1}{|G|} \sum_g g^{-1} p(gu) \quad \left(\begin{array}{l}\text{as } gu \in U \text{ so} \\ p(gu) = gu\end{array} \right)$$

$$= \frac{1}{|G|} \sum_g g^{-1} gu$$

$$= \frac{1}{|G|} \sum_{g \in G} u = \frac{|G|}{|G|} u = u.$$

- Hence $U \leq \operatorname{im} q$. To see $\operatorname{im} q \leq U$, observe that since p takes it's image in U, each $g^{-1} p(gv) \in U$; hence $q(v) \in U$ all $v \in V$.

$\underline{\text{Therefore } \operatorname{im} q = U}$.

Since $qv \in U$ & $qu = u$ all $u \in U$, we get $qqv = qv$ all $v \in V$, as required. $\square$

Theorem

Let G be a finite group & suppose char(K) does not divide order of G, $|G|$. (eg. if $K = \mathbb{R}$ or $\mathbb{C}$).

Then each non-zero finite dim. G-module V admits a decomposition

$$V = V_1 \oplus \dots \oplus V_n$$ as direct sum of irreducible G-submodules.

Proof) - By ind. on dimension of V.

- If $\dim V = 1$, trivial as each 1-d G-module is irreducible.
- Else, suppose it is true for all W st. $\dim(W) < \dim(V)$.
- Suppose $U \leq V$ is a proper G-submodule. Then by Mashke's Theorem,

$$V = U \oplus W \quad \text{for } U, W \text{ proper submodules}$$

Then $\dim(U), \dim(W) < \dim(V)$ so

$$V = U \oplus W = (u_1 \oplus \dots \oplus u_m) \oplus (v_1 \oplus \dots \oplus v_e)$$

where all the u_i & v_j are irreducible.

$\square$