Homological algebra

- Study of invariants of structures using chain complexes & their cohomology, originally homology of spaces but later groups, ...

Exact sequences

- R a ring (not necessarily commutative)
- Mod_R the category of left R-modules.
- Consider $A \xrightarrow{f} B \xrightarrow{g} C \in \mathrm{Mod}_R$ in which $g \circ f = 0$, the zero homomorphism.
- Then $g(f(a)) = 0$ for all $a \in A$ so that $\mathrm{im}(f) \leq \mathrm{Ker}(g)$.

Def) - The sequence is said to be exact at B if $\mathrm{im}(f) = \ker(g)$.

Examples

- $\{0\} = 0 \longrightarrow A \xrightarrow{f} B$ is exact $\Leftrightarrow \ker f = 0$ $\Leftrightarrow$ f is injective (mono).
- $A \xrightarrow{f} B \longrightarrow 0$ is exact $\Leftrightarrow \mathrm{im} f = B$ $\Leftrightarrow$ f is surjective (or epi, as we will see later)

Def^n) A short exact sequence is a sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

which is exact in each position:

- ex @ A : f is injective
- ex @ C : g is surjective
- ex @ B : $\mathrm{im} f = \ker g$, or equivalently,

$$B/\mathrm{im} f = B/\ker g,$$

but since g is surjective, $B/\ker g \cong C$,

so exactness at B says that $B/\mathrm{im} f \cong C$.

Exercise: Using this,

show that each ses is of the form

$$0 \hookrightarrow A \hookrightarrow B \longrightarrow B/A \longrightarrow 0$$

for A a submodule of B,

(up to isomorphism of sequences)

$\boxed{\text{Def}^n}$ • A chain complex **A** is a sequence

$$\cdots A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \cdots$$

of R-modules (for $n \in \mathbb{Z}$)

where $d_n \circ d_{n+1} = 0 \quad \forall n$.

• The elts of $Z_n := \ker d_n \subseteq A_n$ are called n-cycles.

• Elts of $B_n := \operatorname{im} d_{n+1} \leq \ker d_n$ are called n-boundaries.

Def^n) The n-th homology of A is the R-module $H_n(A) = \dfrac{\ker d_n}{\operatorname{im} d_{n+1}} = \dfrac{Z_n}{B_n}$.

Remark) A is exact @ $A_n \iff H_n(A) = 0$.

Thus the n-th homology measures the failure of A to be exact at A_n.

• If A is exact at all n, it is called a long exact sequence.

The category of chain complexes

- A chain map $f: A \longrightarrow B$ of chain complexes consists of maps $f_n: A_n \longrightarrow B_n$ $\forall n \in \mathbb{Z}$

such that
$$\begin{tikzcd} A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, "="] & A_n \arrow[d, "f_n"] \\ B_{n+1} \arrow[r, "d_{n+1}"'] & B_n \end{tikzcd}$$
$\forall n$.

Notation: One often just writes $d: B_{n+1} \to B_n$ when the context is clear.

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Chain complexes and chain maps form a category $Ch(Mod_R)$.

Proposition

The n-th homology determines a functor
$$H_n : Ch(Mod_R) \longrightarrow Mod_R .$$

Proof) It sends $A \longmapsto H_n(A)$.

At $f : A \longrightarrow B$, then given $x \in Z_n(A) \subseteq A$, then $dfx = fdx = 0$ so $fx \in Z_n(B)$;

similarly if $x \in B_n(A)$ then $f(x) \in B_n(B)$;

hence $Z_nA \xrightarrow{f} Z_nB \twoheadrightarrow Z_nB/B_nB$ sends B_nA to 0, so we obtain

$$H_n(A) = \frac{Z_nA}{B_nA} \xrightarrow{H_n(f)} \frac{Z_nB}{B_nB} = H_n(B)$$
$$[x] \longmapsto [fx]$$

This is clearly functorial. $\square$

Homotopy & homology

Def^n) let $f, g : A \rightrightarrows B$ be chain maps. A chain htpy s from f to g (written $s : f \sim g$) is a sequence of homomorphisms $s_n : A_n \longrightarrow B_{n+1}$ as in the picture below :

$$\begin{tikzcd}
\cdots A_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}-g_{n+1}"'] & A_n \arrow[r, "d_n"] \arrow[dl, "s_n"'] \arrow[d, "f_n-g_n"] & A_{n-1} \cdots \arrow[dl, "s_{n-1}"'] \arrow[d, "f_{n-1}-g_{n-1}"] \\
\cdots B_{n+1} \arrow[r, "d_{n+1}"'] & B_n \arrow[r, "d_n"'] & B_{n-1} \cdots
\end{tikzcd}$$

such that

$$d_{n+1} s_n + s_{n-1} d_n = f_n - g_n \quad \forall n \in \mathbb{Z}.$$

- A chain map $f : A \to B$ is null homotopic if $f \sim 0$.
- It is a htpy equivalence if $\exists g : B \to A$ such that $fg \sim 1_B$ & $gf \sim 1_A$.

Proposition

Homotopy is an e-rel, compatible with composition.

$\boxed{\text{Proof}}$ - Taking $s_n = 0$ shows $f \sim f$.

If $s : f \sim g$ & $t : g \sim h$ take $s+t : f \sim h$;

indeed

$$f - h = (f-g)+(g-h) = (ds+sd)+(dt+td)$$
$$= d(s+t) + (s+t)d$$

- If $s : f \sim g$ then $-s : g \sim f$ where $(-s)_n = -s_n$. (so have e-rel)

- If

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "s"] & B \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "s'"] & C \end{tikzcd}$$

must show $f'f \sim g'g$.

- Suffices to show $f'f \sim f'g \sim g'g$ by transitivity of $\sim$.
- Consider $f's$ with $(f's)_n : A_n \xrightarrow{s_n} B_{n+1} \xrightarrow{f'_{n+1}} C_{n+1}$

- Then $d(f's) + (f's)d = f'ds + f'sd$

$$= f'(ds + sd)$$

$$= f'(f - g)$$

$$= f'f - f'g.$$

And let $(s'g)_n = A_n \xrightarrow{g_n} B_n \xrightarrow{s'_n} C_{n+1}$.

Then $d(s'g) + (s'g)d = ds'g + s'dg$

$$= (ds' + s'd)g$$

$$= (f' - g')g$$

$$= f'g - g'g. \quad \square$$

Proposition

If $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "s"] & B \end{tikzcd}$ then $H_n f = H_n g$.

$\boxed{\text{Proof}}$ Consider $x \in Z_n(A) = \ker(d : A_n \to A_{n-1})$

- We must show $f(x), g(x) \in Z_n(B)$ coincide modulo

$$B_n(B) = \operatorname{im}(d_{n+1} : B_{n+1} \to B_n)$$

But $f(x) - g(x) = sdx + dsx$
$= dsx \in B_n(B)$

as $sdx = s0 = 0$.

- Hence $H_n(f) = H_n(g)$.

Corollary

If $f : A \longrightarrow B$ is **homotopy equivalence**, then $H_n f : H_n A \longrightarrow H_n B$ is invertible.

~~Proof~~ **Suppose** $\exists\, g : B \longrightarrow A$ **such that** $gf \sim 1_A$ & $fh \sim 1_B$.

Then $H_n(g)H_n(f)$ = by functoriality

$H_n(gf)$ = by previous prop

$H_n(1_A)$ = by functoriality

$1_{H_n(A)}$

Sim $H_n(f)H_n(g) = 1_{H_n(B)}$. $\square$

Homology of spaces

- We now take a look at the motivating context of topological spaces.

- We will do homology of simplicial complexes first, then singular homology.

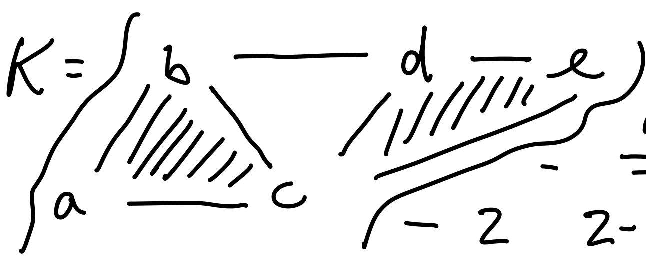
Simp. complexes give an abstract description of spaces that can be built from points, lines, triangles & higher dimensional triangles, or simplices.

Def A <u>simplicial complex</u> K consists of:

- a set K_0 of vertices +
- a set $S(K)$ of non-empty finite subsets of $U(K)$ called <u>simplices</u> such that:
- if X is a simplex & $Y \subseteq X$ is a non-empty subset then Y is a simplex.

Notation) The simplices with $n+1$-elements are called <u>n-simplices</u>.

Example) Triangulated spaces can be viewed as simplicial complexes.



is a simp. complex with
- 5 vertices $a, \ldots, e$
- 7 1-simplices $\{a,b\}, \{b,c\}, \ldots$
- 2 2-simplices $\{a,b,c\}, \{c,d,e\}$.

- This is a 2-dimensional simplicial complex.

- A <u>1-dimensional</u> simplicial complex is a <u>graph</u>.

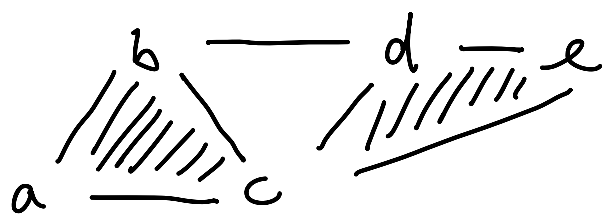
Remark) Spaces "describable" by simp. complexes include most of interest: spheres, tori ...

- Given K a simplicial complex, with an ordering on its set of vertices we have a function

$$d_i : K_n \longrightarrow K_{n-1} \text{ for } 0 \le i \le n$$

which omits the i'th elt. of a n-simplex.

Example)



- Ordering $a<b<c<d<e$

Write $ab = \{a,b\}$ etc

Then $K_1 \xrightarrow{d_1} K_0$

sends $ab \longmapsto a$

$cd \longmapsto c$

Then $K_2 \xrightarrow{d_0} K_1$

sends $abc \longmapsto bc$

$cde \longmapsto de$

- Let $C(K)_n$ = free abelian group on K_n - elements are sums $\sum \underbrace{n_j}_{\text{integer}} \underbrace{x_j}_{n\text{-simplex}}$.

- **Obtain**

$$C(K)_n \xrightarrow{d_i} C(K)_{n-1} \text{ by extension}$$

$$\sum n_j x_j \longmapsto \sum n_j d_i(x_j)$$

& take the **alternating sum**

$$C(K)_n \xrightarrow{d_n = \sum_{i=0}^{n} (-1)^i d_i} C(K)_{n-1}$$

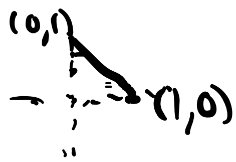
& this is a chain complex

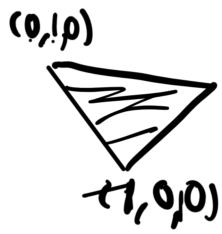
& the n'th simplicial homology is $H_n(C(K))$.

Singular homology

- X a top space.
- $\Delta_{n-1} = \{ (x_0, x_1, \dots, x_n) \in \mathbb{R}^n : \sum x_i = 1, x_i \geq 0 \}$

so $\Delta_0 =$ 

$\Delta_1 =$ 

, $\Delta_2 =$ 

etc

- Δ_n is called <u>standard n-simplex</u>.
- Define $S_n X = \mathbb{F}(\mathrm{Top}(\Delta_n, X))$, the <u>free abelian group</u> on the set of continuous maps $\Delta_n \longrightarrow X$.

- As before, we obtain face maps
$$S_n(X) \xrightarrow{d_i} S_{n-1}(X) \text{ for } 0 \leq i \leq n$$
& taking alternating sum
$$\cdots S_n(X) \xrightarrow{\sum_{i=0}^{n} (-1)^i d_i} S_{n-1}(X) \cdots$$
obtain <u>singular chain complex $S(X)$</u>.

- Its homology $H_n(X)$ is the n'th singular homology of X.
- When X is a space with a triangulation, this coincides with simplicial homology.

Cohomology

- So far we have talked about chain complexes & homology.
- Dually we have cochain complexes & cohomology.

Def) A cochain complex is a diagram

$$\cdots X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \cdots \text{ s.t.}$$

$$d^n \circ d^{n-1} = 0 \text{ all } n \in \mathbb{Z}.$$

Remark) Equivalently a chain complex in the opposite abelian cat $(R\text{-Mod})^{op}$ - see next week.

- Everything about chain complexes has a dual version for cochain complexes.
- The nth cohomology of a cochain complex X is defined as

$$H^n X := \ker d^n / \operatorname{im} d^{n-1}.$$

Examples

① If X is a ch. complex of ab. groups & A an **abelian group**

then $R\text{-Mod}(X^n, A) \xrightarrow{R\text{-Mod}(d^{n+1}, A)} R\text{-Mod}(X^{n+1}, A) \cdots$

$$a \longmapsto a \circ d^{n+1}.$$

is a cochain complex $R\text{-Mod}(X, A)$ in Ab.

② Recall if X a top space, can form $SX \in Ch(Ab)$, whose homology is the singular homology of X.

If A is an abelian group, then $H_n(Ab(SX, A))$ is the so-called singular cohomology of X of X with coefficients in A.