Algebra 4 - Marked assignment 1

① Let R be a comm. ring which is Noetherian. Using results from the course or otherwise, show that an R-module M is Noetherian $\iff$ it is finitely generated. (2 pts)

② A comm ring R is Artinian if each descending chain of ideals

$$\ldots \subseteq I_{n+1} \subseteq I_n \subseteq \ldots I_1 \subseteq I_0 \subseteq R$$

eventually stabilises

Which of the following rings are Artinian and why?

(a) $\mathbb{Z}$

(b) $\mathbb{Z}/n\mathbb{Z}$

(c) $\mathbb{R}[x]$ (3 pts)

3 (Generalised Chinese remainder theorem)

Let R be a comm. ring & $Q_1, \dots, Q_n$ be ideals such that $Q_i + Q_j = R$ for $i \neq j$.* Show that we have an iso of rings $R / \bigcap_{i=1,\dots,n} Q_i \cong \prod_{i=1,\dots,n} (R/Q_i)$ as follows.

(a) Consider the ring homomorphism
$$R \xrightarrow{\varphi} \prod_{i=1,\dots,n} (R/Q_i)$$
$$r \longmapsto (r \bmod Q_1, \dots, r \bmod Q_n)$$
& show it has kernel $\bigcap_{i=1,\dots,n} Q_i$. (1pt)

(b) Let m be a maximal ideal of R. Show that at most one of the Q_i is contained in m. (1pt)

(c) Show if Q_i is not contained in m, then $(Q_i)_m = R_m$. (1pt)

(d) Now localizing the map φ at m, & using the properties of localization in the course, conclude that φ_m & hence φ is surjective to obtain the result. (2pts)

Note*: $Q_i + Q_j$ is the ideal $\{a+b : a \in Q_i, b \in Q_j\}$.