1.2. Representations of Lie groups and Lie algebras

Def. 1.24 Suppose G is a Lie group.

A representation of G on a finite-dim. real vector space V is a Lie group homomorphism $\varphi : G \to GL(V)$.

Equivalently, it is a smooth map $\varphi : G \times V \to V$ s.t.

- $\varphi(g, -) : V \to V$ linear $\forall g \in G$
- $\varphi(e, v) = v \quad \forall v \in V$
- $\varphi(g, \varphi(h, v)) = \varphi(gh, v) \quad \forall g, h \in G, v \in V$.

<u>Remark</u> One often just refers to V as a representation of G in Def. 1.23, if it is understood what the map $\varphi: G \to GL(V)$.

<u>Examples</u>

1. $GL(V) = G$ $\qquad \dim(V) = n$

Defining / standard repres on V: $\varphi: GL(V) \times V \to V$

$$(A, v) \mapsto Av = \varphi(A, v)$$

Via choice of basis, one can identify

$$GL(V) \simeq GL(n, \mathbb{R})$$

and φ with $GL(n,\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n$

$(A,v) \mapsto Av$ (matrix multipl. of $A \in GL(n,\mathbb{R})$ with a vector in $\mathbb{R}^n$)

Similarly, any matrix group $H \subseteq GL(V)$ has a standard representation, namely V.

② Adjoint representation of a Lie group G on its Lie algebra $\mathfrak{g}$

Denote by $\mathrm{con}_g : G \to G$ conjugation in G :

$$\mathrm{con}_g(h) := g h g^{-1} \quad \forall\, h \in G .$$

It is a Lie group homomorphism.

$$Ad : G \to GL(\mathfrak{g})$$

$$Ad(g) := T_e con_g : \mathfrak{g} \to \mathfrak{g}$$

is called the adjoint representation of G on its Lie alg. $\mathfrak{g}$.

Let us check this is really a representation:

$$con_g = \lambda_g \circ \rho^{g^{-1}} = \rho^{g^{-1}} \circ \lambda_g \Longrightarrow T_e con_g = \underline{T_{g^{-1}} \lambda_{g^{-1}} \circ T_e \rho^{g^{-1}}}$$

$$= T_g \rho^{g^{-1}} \circ T_e \lambda_g$$

$\mathrm{con}_{gh} = \mathrm{con}_g \circ \mathrm{con}_h \Longrightarrow \mathrm{Ad}(gh) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h)$

$\mathrm{con}_{g^{-1}} = (\mathrm{con}_g)^{-1} \Longrightarrow \mathrm{Ad}(g^{-1}) = \mathrm{Ad}(g)^{-1}$

$\Longrightarrow$ $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is a group homomorphism.

To see that Ad is smooth, we can equiv. show that $(g, X) \mapsto \mathrm{Ad}(g)(X)$ is smooth. Setting $F: G \times \mathfrak{g} \to TG \times TG \times TG$, $F(g, X) = (0_g, X, 0_{g^{-1}})$, we have

$\underline{(T\mu \circ (\mathrm{id}_{TG} \times T\mu) \circ F)(g, X)} = T_{g^{-1}}\lambda_g \circ T_e \rho^{g^{-1}} X = \mathrm{Ad}(g) X$

which is smooth as a composition of smooth maps.

If $G = GL(n,\mathbb{R})$, then

$$\mathrm{con}_A(B) = ABA^{-1}$$

is linear as a map $\mathrm{con}_A : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$

$$\Rightarrow \mathrm{Ad}(A)(X) = T_{\mathrm{Id}}\,\mathrm{con}_A(X) = \mathrm{con}_A(X) = AXA^{-1}$$

$\forall X \in \mathfrak{gl}(n,\mathbb{R})$
$\forall A \in GL(n,\mathbb{R})$.

Def. 1.25 Suppose $\mathfrak{g}$ is a real (or complex) Lie algebra over $\mathbb{K} = \mathbb{R}$ (or $= \mathbb{C}$).

A representation of $\mathfrak{g}$ on a finite-dim. vector space V over $\mathbb{K}$ is a Lie algebra homomorphism

$$\psi : \mathfrak{g} \to \mathfrak{gl}(V)$$

i.e. a linear map s.t. $\psi([X,Y]) = [\psi(X), \psi(Y)]$

$$= \psi(X) \circ \psi(Y) - \psi(Y) \circ \psi(X)$$

$\forall X, Y \in \mathfrak{g}$.

Equivalently, a bilinear map $\psi : \mathfrak{g} \times V \to V$ s.t.

$$\psi([X,Y],v) = \psi(X,\psi(Y,v)) - \psi(Y,\psi(X,v)) \quad \forall X,Y \in \mathfrak{g}, \ \forall v \in V.$$

By Prop. 1.12, any representation $\varphi : G \to GL(V)$ of a Lie group G induces a representation $\varphi' = T_e\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ of its Lie algebra $\mathfrak{g}$.

For $G = GL(n, \mathbb{R})$, the standard representation φ of $GL(n, \mathbb{R})$ gives rise to standard representation of $\mathfrak{gl}(n, \mathbb{R})$

$$\varphi' : \mathfrak{gl}(n, \mathbb{R}) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$(X, v) \longmapsto Xv$$

Similarly, for any matrix group and its standard representation.

For the adjoint representation of a Lie group G,

$Ad : G \longrightarrow GL(\mathfrak{g})$, the induced representation

of $\mathfrak{g}$, called the adjoint representation of $\mathfrak{g}$, is

given by

$$ad : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$$

$$ad(X)(Y) = [X,Y] \qquad \forall\, X, Y \in \mathfrak{g}.$$

as the following proposition shows.

Prop. 1.26 G Lie group with Lie alg. $(\mathfrak{g}, [\,,\,])$.

① For $X \in \mathfrak{g}$ and $g \in G$, $L_X(g) = R_{Ad(g)(X)}(g)$.

② For $X, Y \in \mathfrak{g}$, $ad(X)(Y) = [X, Y]$

③ For $X \in \mathfrak{g}$, $g \in G$ one has

$$\exp(t \underbrace{Ad(g)(X)}) = g \exp(tX) g^{-1}$$

④ For $X, Y \in \mathfrak{g}$ one has

$$Ad(\exp(X))(Y) = e^{ad(X)}(Y) = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{ad(X)^k Y}_{[X,[X,\ldots[X,Y]\ldots]]} = Y + [X, Y] + \frac{1}{2}[X,[X,Y]] + \ldots$$

with $\overbrace{[X,[X,\ldots[X}^{k}, Y]\ldots]]$

Proof

① $\lambda_g = \rho^g \circ \mathrm{con}_g$

$$\Longrightarrow \underset{\underset{L_X(g)}{\shortparallel}}{T_e\lambda_g X} = \underline{T_e}\rho^g \underbrace{T_e \mathrm{con}_g X}_{\underline{\underline{Ad(g)(X)}}} = R^{(g)}_{Ad(g)(X)} \quad \forall X \in \mathfrak{g}.$$

② Choose a basis $X_1, \dots, X_n$ of $\mathfrak{g}$, then $Ad(g) : \mathfrak{g} \to \mathfrak{g}$ corresponds to an $n \times n$ matrix $(a_{ij}(g))$ for any $g \in G$ and $a_{ij} : G \to \mathbb{R}$ are smooth.

Matrix representation of $\operatorname{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ equals

$$X \cdot a_{ij} = T_e a_{ij} X = (L_X \cdot a_{ij})(e) \qquad \forall X \in \mathfrak{g} .$$

Any $Y \in \mathfrak{g}$ can be written as $\underline{Y = \sum_{i=1}^{n} y_i X_i}$.

$$\Longrightarrow \quad L_Y(g) \overset{(1)}{=} R_{\underbrace{\mathrm{Ad}(g)(Y)}_{\sum_{ij} a_{ij}(g) y_j \underline{X_i}}}(g) = \sum_{ij} y_j a_{ij}(g) R_{X_i}(g)$$

$$\Longrightarrow \underline{[L_X, L_Y]} = \sum_{ij} y_j \underbrace{[L_X, a_{ij} R_{X_i}]}_{\overset{\text{Prop. 1.14}}{=} (L_X \cdot a_{ij}) R_{X_i}} = \sum_{ij} y_j (L_X \cdot a_{ij}) R_{X_i}$$

Evaluting at e yields : $\underline{[X,Y]} = \sum_{i,j} y_j \underline{(X, a_{ij})} X_i$

$= \mathrm{ad}(X)(Y)$

③ Since $\mathrm{Ad}(g) = T_e \,\mathrm{con}_g$, the result follows from ① of Thm. 1.23,

$$\underline{\mathrm{con}_g}(\exp(tX)) = \exp(\mathrm{Ad}(g)(tX)) = \exp(t\,\mathrm{Ad}(g)(X))$$

$\forall g \in G, X \in \mathfrak{g}$.

④ $\mathrm{Ad} : G \to GL(\mathfrak{g})$ Lie group hom.

$T_e \mathrm{Ad} = \mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$

By ① of Thm. 1.23:

$$\mathrm{Ad}(\exp(X))(Y) = \exp(\mathrm{ad}(X))(Y)$$
$$= e^{\mathrm{ad}(X)}(Y)$$
$$= \sum_{k=0}^{\infty} \frac{\mathrm{ad}(X)^k}{k!} Y$$

$\square$

Prop. 1.27 G Lie group with Lie algebra $\mathfrak{g}$.
Let $\varphi: G \to GL(V)$ be a represent. of G with induced represent. $\varphi': \mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$.

① $\varphi(\exp(tX))(v) = \exp(t\varphi'(X))(v)$ $\forall t \in \mathbb{R}$, $\forall X \in \mathfrak{g}$, $\forall v \in V$

② $\varphi'(X)v = \frac{d}{dt}\Big|_{t=0} \varphi(\exp(tX))(v)$

Proof

① follow from ① of Thm. 1.23 and ② is an immediate consequ. of ①.

1.3 Lie subgroups and virtual Lie subgroups

<u>Prop. 1.28</u> Suppose H is a Lie subgroup of a Lie group G. Then H is closed as subset of the topolog. space G.

<u>Proof</u>

Any submfd N of a mfd M is locally closed, i.e. open in its closure $\overline{N}$ ($\Leftrightarrow$ every point $x \in N$ has a neighbhd U in M s.t. $U \cap N$ is closed in U)

For any subgroup H of a topolog. group G, $\overline{H}$ is also a subgroup of G $\left(\overset{\in H}{h_n} \xrightarrow[n\to\infty]{} h \quad \overset{\in H}{g_n} \xrightarrow[n\to\infty]{} g \Rightarrow \underset{\in H}{h_n g_n} \to h\cdot g \in \overline{H} \right)$

If H is a Lie subgroup of G, H is open and dense in $\overline{H}$. Hence, for $g \in \overline{H}$, $\lambda_g(H) \subseteq \overline{H}$ is open in $\overline{H}$.

Since H is dense in $\overline{H}$, $\lambda_g(H) \cap H \neq \emptyset$, which implies $g \in H$.

$\square$

Conversely, one has:

<u>Thm. 1.29</u> Suppose H is a subgroup of a Lie group G that is closed as a subset of the topolog. space G. Then H is a Lie subgroup.

<u>Proof</u> We write $\mathfrak{g}$ for the Lie alg. of G and set

$$\mathfrak{h} := \left\{ c'(0) \;:\; \begin{array}{l} c: \mathbb{R} \to G \text{ is smooth, } c(0) = e \\ \text{and } c \text{ has values in } H \end{array} \right\}$$

$$\subseteq \mathfrak{g}$$

Claim 1 $\quad \mathfrak{h}$ is a linear subspace of $\mathfrak{g}$.

If $c_1, c_2 : \mathbb{R} \to H \subseteq G$ $\quad C^\infty$-curves, $c_1(0) = c_2(0) = e$.

Then $\quad c(t) := c_1(t)\, c_2(at) \quad a \in \mathbb{R}$

is C^∞-curve with values in H $\quad c(0) = e$.

$$\Rightarrow \quad \underset{\displaystyle \underset{\displaystyle T_e\mu\left(c_1'(0),\, a c_2'(0)\right) = c_1'(0) + a c_2'(0)}{\parallel}}{c'(0)} \in \underline{\mathfrak{h}}$$

Claim 2: Suppose $(X_n)_{n\in\mathbb{N}}$ is sequence in $\mathfrak{g}$ s.t.

$\lim_{n\to\infty} X_n = X \in \mathfrak{g}$ and let $(t_n)_{n\in\mathbb{N}}$ be a sequence in

$\mathbb{R}_{>0}$ with $\lim_{n\to\infty} t_n = 0$.

Then, if $\exp(t_n X_n) \in H \ \forall n\in\mathbb{N}$, then $\exp(tX) \in H \ \forall t\in\mathbb{R}$.

Fix $t\in\mathbb{R}$. For $n\in\mathbb{N}$ let a_n be the largest integer $\leq \frac{t}{t_n}$.

$\Longrightarrow$ $a_n t_n \leq t$ and $t - a_n t_n < t_n$, so

$$\lim_{n\to\infty} a_n t_n = t$$

$$\underset{\text{continuity of exp}}{\Longrightarrow} \quad \lim_{n\to\infty}\Big(\underbrace{\exp(t_n X_n)}_{\in H}^{a_n}\Big) = \lim_{n\to\infty} \exp(a_n t_n X_n) = \exp(tX) \in H$$

Since H is closed.

<u>Claim 3</u> $\quad \mathfrak{h} = \{X \in \mathfrak{g} : \exp(tX) \in H \ \forall t \in \mathbb{R}\}$

RHS $\subseteq \mathfrak{h}$ by Def. of $\mathfrak{h}$.

To show $\underset{\smile}{\mathfrak{h}} \subseteq \overset{\frown}{\text{RHS}}$, let $c : \mathbb{R} \to H \subseteq G$ be a smooth curve with $c(0) = e$. Then $c'(0) \in \mathfrak{h}$.

Then $\exists\, \varepsilon > 0$ and a C^∞-curve $v : (-\varepsilon, \varepsilon) \to \mathfrak{g}$

s.t. $c(t) = \exp(v(t)) \quad \forall t \in (-\varepsilon, \varepsilon) \quad (v(0) = 0 \in \mathfrak{g})$.

$$\Longrightarrow \quad c'(0) = \frac{d}{dt}\Big|_{t=0} \exp(v(t)) = \underbrace{T_0 \exp}_{= \mathrm{Id}_{\mathfrak{g}}} v'(0) = v'(0)$$

$$= \lim_{n \to \infty} n\, v\left(\frac{1}{n}\right)$$

Set $t_n = \frac{1}{n}$ and $X_n = n \underbrace{v\left(\frac{1}{n}\right)}$ for suff. large n.

$$\Longrightarrow \quad \exp(t_n X_n) = \exp\left(v\left(\frac{1}{n}\right)\right) = c\left(\frac{1}{n}\right) \in H$$

for suff. large n

By claim 2, $\exp(t\, c'(0)) \in H \quad \forall t \in \mathbb{R}$

<u>Claim</u>: Write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$ as a vector space, where $\mathfrak{k}$ is a linear compliment of $\mathfrak{g}$ in $\mathfrak{g}$.

Then $\exists$ an open neighbhd W of $0 \in \mathfrak{k}$ in $\mathfrak{k}$ s.t. $\exp(W) \cap H = \{e\}$.

Conversely, assume that's not the case. Then $\exists$ a sequence of elements $\underset{\neq 0}{Y_n} \in \mathfrak{k}$ s.t. $\lim\limits_{n \to \infty} Y_n = 0$ and $\exp(Y_n) \in H$.

For a norm $\|\cdot\|$ on $\mathfrak{k}$, put $X_n = \frac{1}{\|Y_n\|} Y_n$. By passing to a subsequence if necessary, we can assume that

$\lim_{n\to\infty} X_n =: X \in \mathfrak{k}$. Then $\|X\| = 1$, in particular, $X \neq 0$.

Set $t_n = \|Y_n\|$. Then $\exp(t_n X_n) = \exp(Y_n) \in H$

and Claim 2 and 3 show that $X \in \mathfrak{g}$, which is a contradiction to $X \neq 0$ and $X \in \mathfrak{k}$.

We define the following smooth map

$$F: \mathfrak{g} \times \mathfrak{k} \to G$$
$$F(X,Y) = \exp(X)\exp(Y)$$

Since T_0F is a linear isomorphism, $\exists$ open neighbhds

V and W of $0 \in \mathfrak{g}$ and $0 \in \mathfrak{k}$ s.t.

$$F|_{V \times W} : V \times W \to F(V, W) =: U$$

is a diffeomorphism onto an open neighbhd U of e in G.

By shrinking W, we may assume that $\exp(W) \cap H = \{e\}$ by Claim 4.

F restricted to $\underline{V \times \{0\}}$ is a bijection onto $\underline{U \cap H}$.

Indeed, $\exp(V) \subseteq U \cap H$, since $V \subseteq \mathfrak{g}$. Moreover, any $x \in U \cap H$ can be written uniquely as $x = \exp(X) \exp(Y)$ for $X \in V$, $Y \in W$.

$$\Longrightarrow \quad \exp(Y) = \underbrace{\exp(-X)}_{\in H} \cdot \underset{\substack{\cap \\ H}}{x} \in H$$

By construction, this implies $\exp(Y) = e$, hence $Y = 0$.

Therefore, $\left(U, u := F|_{V \times W}^{-1}\right)$ is a submfd. chart for H defined around $e \in G$ and $\left(\lambda_h(U), u \circ \lambda_{h^{-1}}\right)$ for $h \in H$ is a submfd. chart around $h \in H$.

$\square$