STRUCTURE OF THE COURSE

I. Lie Groups

- Basic Theory
- Representations of Lie groups
- Classification of Lie groups (and Lie algebras)
- Homogeneous spaces, Klein geometries

II. BUNDLES

- Fiber bundles, vector bundles and principal bundles
- Associated vector bundles
- Homogeneous vector bundles

III. CONNECTIONS

- Linear connections on vector bundles (in particular, affine connections)
- Principal connections on principal bundles
- Geometric structures determining (classes of) distinct connections (e.g. Riem. mfds, conformal structures, projective structures, ...)

- Holonomy
- Cartan geometries

I. Lie groups (Čap, Lie groups, lecture notes)

1.1 Basic Theory

For a group G we write:

- $\mu : G \times G \longrightarrow G$ for the multiplication map
- $\nu : G \longrightarrow G$ for the inversion, $\nu(g) = g^{-1}$
- $e \in G$ for the identity / neutral element in G.

Def. 1.1 A topological group is a topological space G equipped with a group structure (μ, ν, e) s.t. μ and ν are continuous.

Remark Any abstract group can be made into a topological group by equipping it with the discret topology.

Def. 1.2 A Lie group is a smooth manifold G equipped with a group (μ, ν, e) s.t. μ and ν are smooth.

Rem. In Def. 1.2 it is enough to require that μ is smooth, since ν is then automatically smooth by applying the Inverse Function/Implicit Function Thm to the equation $\mu(g, \nu(g)) = e$.

Def. 1.3

(1) A homomorphism between topological groups (resp. Lie groups) G and H is a continuous (resp. smooth) map $\varphi : G \to H$ that is also a group homomorphism ($\varphi(g \cdot h) = \varphi(g)\varphi(h)$ $\forall g, h \in G$).

(2) A homomorphism $\varphi: G \to H$ as in (1) is called an isomorphism between top. groups (resp. Lie groups), if φ is a homeomorphism (resp. diffeomorphism). -

Note that in this case, φ^{-1} is also a group homomorphism.

<u>Notation</u> Two Lie groups G and H are called isomorphic, if $\exists$ a Lie group isomorphism between them. We write $G \simeq H$ in this case.

Groups of greatest interest in mathematics and physics consist of bijections $f : M \to M$ of a set M (space) with group multiplication given by composition.

$$\mu(f, \tilde{f}) = f \cdot \tilde{f} = f \circ \tilde{f}$$

$$f, f \in \mathrm{Bij}(M) = \{\text{bijections of } M\}.$$

$$\nu(f) = f^{-1} \qquad e = \mathrm{id}_M .$$

Examples

If M has some extra structure, one can consider subgroups of $(\mathrm{Bij}(M), \circ)$ consisting of bijections preserving that extra structure.

- M topolog. space

$$\mathrm{Homeo}(M) := \{ f \in \mathrm{Bij}(M) : f \text{ is a homeom.} \}$$

- M smooth mfd (resp. smooth oriented mfd)

$$\mathrm{Diff}(M) = \{ f : M \to M \ : \ f \text{ is a diffeom.} \}$$
$$\mathrm{Diff}_+(M) = \{ f : M \to M \ : \ \text{orientation preserv. diffeos} \}$$

- Suppose M is a smooth mfd equipped with a geometric structure like a Riemannian metric g or a symplectic form ω :

$$\mathrm{Isom}(M,g) = \{ f : M \to M \ : f \text{ diffeom. s.t. } f^*g = g \}$$

$$\mathrm{Symp}(M,g) = \{ f \in \mathrm{Diff}(M) : f^*\omega = \omega \}$$

With the exception of $\mathrm{Isom}(M,g)$, these groups are infinite-dimensional and hence not Lie groups in our sense. These groups are all naturally topological groups.

$\mathrm{Isom}(M, g)$ is a Lie group of dim. $\leq \frac{\dim(M)(\dim(M)+1)}{2}$.

Now some examples of actual Lie groups.
Typical examples arise from linear transformations of finite-dim. vector spaces.

Examples

① $\mathbb{R}$, $\mathbb{C}$ with respect to addition $+$ is a Lie group and so is any finite-dim. vector space over $\mathbb{R}$ or $\mathbb{C}$ w.r. to $+$.
$\mathbb{C}$ and any complex vector space is even a complex Lie group

(i.e. a complex mfd with holomorphic group structure).

② $\mathbb{R}\setminus\{0\}$, $\mathbb{C}\setminus\{0\}$ are Lie groups w.r.t. to multiplication.

Also, $U(1) = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a Lie group.

③ The product $G \times H$ of Lie groups is again a Lie group. In particular, the n-dim. torus $T^n := \underbrace{U(1) \times \dots \times U(1)}_{= n \text{ times}}$ is a Lie group.

Also, for m, n natural numbers, $\mathbb{R}^m \times T^n$ is a Lie group. The latter exhaust all connected commutative Lie groups.

④ If G is a Lie group, then a Lie subgroup H of G is a subgroup $H \leq G$ that is also a submfd. Since the multipl. on H is just the restriction of the one from G, it smooth, and so H is a Lie group.

(5) Suppose V is a real or complex finite-dim. vector space.

$$GL(V) = \{\text{linear isomorphisms of } V\} \subseteq \mathrm{End}(V) = \{\text{linear maps } V \to V\}$$

open subset

is a Lie group with respect to composition.

Via choice of basis, we may identify $V \simeq \mathbb{R}^n$ (resp. $\mathbb{C}^n$) and $GL(V)$ with

$$GL(n, \mathbb{K}) = \{A \in M_{n \times n}(\mathbb{K}) : A \text{ invertible}\}$$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

and composition of linear maps becomes matrix multiplication.

Again, $GL(n, \mathbb{C})$ is also a complex Lie group.

It is called the <u>general linear group</u>.

6. Matrix groups (also called linear Lie grps) are Lie subgroups G of $GL(n, \mathbb{R})$ and $GL(n, \mathbb{C})$.

- Special linear group

  $G = SL(n, \mathbb{K}) = \{ A \in GL(n, \mathbb{K}) : \det_{\mathbb{K}}(A) = 1 \}$

  $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ ($SL(n, \mathbb{C})$ is a complex Lie group).

- Orthogonal groups: $I_{p,q} = \begin{pmatrix} Id_p & \\ & -Id_q \end{pmatrix} \in M_{n\times n}(\mathbb{R})$

$$= \begin{pmatrix} \overbrace{1 \ddots 1}^{p} & \\ & \overbrace{-1 \ddots -1}^{q} \end{pmatrix} \qquad n = p+q$$

defines inner product of signature (p,q) on $\mathbb{R}^n$

$$\langle x, y\rangle := x^t I_{p,q}\, y \qquad \forall x, y \in \mathbb{R}^n$$

$$G = O(p,q) = \{ A \in GL(n,\mathbb{R}) : A^t I_{p,q} A = I_{p,q} \}$$
$$= \{ A \in GL(n,\mathbb{R}) : \langle Ax, Ay\rangle = \langle x, y\rangle \ \forall x, y \in \mathbb{R}^n \}$$

linear orthog. group of signature (p,q)

$O(n,0) =: O(n) = \{ A \in GL(n,\mathbb{R}) : A^t = A^{-1} \}$

$O(n-1,1)$ Lorentzian group.

- linear symplectic group: $J_n = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix} \in M_{2n\times 2n}(\mathbb{R})$

defines symplectic form on $\mathbb{R}^{2n}$: $\omega(x,y) := x^t J_n y$

$$G = Sp(2n,\mathbb{R}) = \{ A \in GL(2n,\mathbb{R}) : A^t J_n A = J_n \}$$
$$= \{ A \in GL(2n,\mathbb{R}) : \omega(Ax,Ay) = \omega(x,y) \ \forall x,y \in \mathbb{R}^{2n} \}$$

More examples in the seminar.

Suppose (G, μ, ν, e) is a Lie group.

Then we denote by

- $\lambda_g : G \to G$ left-multipl. by $g \in G$

  $\lambda_g(h) := \mu(g,h) = g \cdot h = gh \quad \forall h \in G$

- $\rho^g : G \to G$ right-multipl. by $g \in G$

  $\rho^g(h) := \mu(h,g) = hg \quad \forall h \in G$

Lemma 1.4

For every $g \in G$, λ_g (resp. ρ^g) is a diffeomorphism with inverse $\lambda_{g^{-1}}$ (resp. $\rho^{g^{-1}}$). In particular, if $U \subseteq G$ is an open neighbhd of $g \in G$, then $\lambda_h(U)$ (resp. $\rho^h(U)$) is an open neighbhd of hg (resp. gh).

Moreover, for $g, h \in G$, $\lambda_{gh} = \lambda_g \circ \lambda_h$ and $\rho^{hg} = \rho^g \circ \rho^h$.

Proof

$$\begin{tikzcd} \underset{G}{h} \arrow[r, mapsto] \arrow[rr, bend left, "\lambda_g"] & \underset{G \times G}{(g,h)} \arrow[r, "\mu"] & G \end{tikzcd}$$

is smooth as composition of two smooth maps.

Lemma 1.5

① For $g, h \in G$, $\xi \in T_g G$ and $\eta \in T_h G$ one has

$$\underset{(g,h)}{T}\mu(\xi, \eta) = \underline{T_h \lambda_g \eta} + \underline{T_g \rho^h \xi}$$

$$\left(\underline{T(G \times G)} \simeq TG \times TG \right)$$

② For any $g \in G$,

$$T_g \nu \,\underline{=}\, -T_e \rho^{g^{-1}} \circ T_g \lambda_{g^{-1}} = -T_e \lambda_{g^{-1}} \circ T_g \rho^{g^{-1}}$$

In particular, $T_e \nu = -\mathrm{Id}_{T_e G}$

<u>Proof</u> $T_{(g,h)}\mu : T_gG \times T_hG \longrightarrow T_{gh}G$ is linear

$\simeq T_{(g,h)}(G\times G)$

$$\Longrightarrow T_{(g,h)}\mu(\xi,\eta) = \underline{\underline{T_{(g,h)}\mu(\xi,0)}} + \underline{T_{(g,h)}\mu(0,\eta)}$$

Let $c : (-\varepsilon,\varepsilon) \to G$ C^∞-curve representing ξ :

$c(0) = g \qquad c'(0) = \xi$.

Then $t \mapsto (c(t),h)$ represents $(\xi,0) \in T_{(g,h)}(G\times G)$

$\mu(c(t),h) = \rho^h(c(t))$

$$\Longrightarrow T_{(g,h)}\mu(\xi,0) = \frac{d}{dt}\Big|_{t=0} \mu(c(t),h) = \frac{d}{dt}\Big|_{t=0} \rho^h(c(t)) = \underline{\underline{T_g\rho^h\xi}}$$

Similarly, $T_{(g,h)}\mu(0,\eta) = \frac{d}{dt}\Big|_{t=0} \mu(g, c(t)) = \frac{d}{dt}\Big|_{t=0} \lambda_g(c(t))$

$= T_h \lambda_g \eta$

for a curve $c : (-\varepsilon, \varepsilon) \to G$

with $c(0) = h$, $c'(0) = \eta$.

② $e = \mu(g, \nu(g))$

$\Longrightarrow$ differentiate, $0 = T_{(g,g^{-1})}\mu(\zeta, T_g \nu \zeta) \overset{①}{=}$

$= T_g \rho^{g^{-1}} \zeta + \underline{T_{g^{-1}} \lambda_g} \, \underline{T_g \nu \zeta} \qquad \forall \zeta \in T_g G$

$$\Longrightarrow \quad \underline{\underline{T_{g^{-1}}\lambda_g}}\, T_g \nu\, \xi = -T_g \rho^{g^{-1}} \xi \iff T_g \nu\, \xi = -T_e \lambda_{g^{-1}} \circ T_g \rho^{g^{-1}} \xi$$

Second formula follows similarly from diff. $e = \mu(\nu(g), g)$.

$\square$

Recall that for any (local) diffeom. $f : M \to M$ on a mfd, $f^* : \Gamma(TM) \to \Gamma(TM)$, given by

$$f^*\xi = (Tf)^{-1} \circ \xi \circ f, \quad \text{is linear and } f^*[\xi, \eta] = [f^*\xi, f^*\eta].$$

Def. 1.6 Suppose G is a Lie group. Then a vector field $\xi \in \underbrace{\Gamma(TG)}_{= \mathfrak{X}(G)}$ is called left (resp. right) invariant, if $\lambda_g^* \xi = \xi \ \forall g \in G$ (resp. $(\rho^g)^* \xi = \xi \ \forall g \in G$).

Denote by $\mathfrak{X}_L(G)$ (resp. $\mathfrak{X}_R(G)$) the set of left- (resp. right) invariant vectorfields on G.

By linearity of the pull-back and its compatibility with $[\,,\,]$, it is a subalgebra of the infinite-dim. Lie algebra $(\mathfrak{X}(G), [\,,\,])$.