1. KONTROLNÍ TEST – SKUPINA A

1. Řešte nerovnici

$$2-3x > |x-1| + |x+1|$$

2. $$f(x) = \frac{x}{\sqrt{1+x^2}}$$

a) Rozhodněte, zda je funkce $f(x)$ sudá nebo lichá.

b) Vypočtěte $(f \circ f)(x)$.

3. Vypočtěte limitu posloupnosti

$$\lim_{n\to\infty} \sqrt{n^2+n+1} - \sqrt{n^2+2n+3}$$

4. Rozhodněte, zda je zadaná posloupnost konvergentní

$$a_n = \frac{3^n}{n!}$$

5. Určete hromadné body, limsup a liminf posloupnosti

$$a_n = \left(\sin \frac{n\pi}{6}\right)^n$$

6. Určete definiční obor funkce

$$f(x) = \arcsin(3 - \sqrt{4-x})$$

7. Rozložte funkci $f(x)$ na parciální zlomky

$$f(x) = \frac{x}{x^3+1}$$

8. Vypočtěte limitu funkce

$$\lim_{x\to 3} \frac{x^2+x-12}{x^2-5x+6}$$

1. KONTROLNÍ TEST – SKUPINA B

1. Řešte nerovnici

$$|x+3| - |x| < 1$$

2. $$f(x) = \frac{x}{\sqrt[3]{1+x^3}}$$

a) Rozhodněte, zda je funkce $f(x)$ sudá nebo lichá.

b) Vypočtěte $(f \circ f)(x)$.

3. Vypočtěte limitu posloupnosti

$$\lim_{n\to\infty} \sqrt{n+3} - \sqrt{n-1}$$

4. Rozhodněte, zda je zadaná posloupnost konvergentní

$$a_n = \frac{n!}{n^n}$$

5. Určete hromadné body, limsup a liminf posloupnosti

$$a_n = \frac{n}{\sqrt{2n^2+1}} \cdot \cos \frac{n\pi}{2}$$

6. Určete definiční obor funkce

$$f(x) = \arccos \frac{2x}{1+x^2}$$

7. Rozložte funkci $f(x)$ na parciální zlomky

$$f(x) = \frac{-5x+2}{x^4-x^3+2x^2}$$

8. Vypočtěte limitu funkce

$$\lim_{x\to 4} \frac{x^2-2x-8}{x^2-5x+4}$$

(1A) $2-3x > |x-1| + |x+1|$

1b.

$$\begin{array}{c|ccc} & (-\infty,-1) & (-1,1) & (1,\infty) \\ |x-1| & - & - & + \\ |x+1| & - & + & + \end{array}$$

I. $x \in (-\infty, -1)$

$2-3x > -x+1-x-1$

$2 > x$

$\Rightarrow \underline{x \in (-\infty, -1)}$

II. $x \in \langle -1, 1 \rangle$

$2-3x > -x+1+x+1$

$-3x > 0$

$x < 0$

$\Rightarrow \underline{x \in \langle -1, 0)}$

III. $x \in (1, \infty)$

$2-3x > x-1+x+1$

$x < \frac{2}{5}$

$\Rightarrow x \in \emptyset$

Ř: $\underline{\underline{x \in (-\infty, 0)}}$

(1B) $|x+3| - |x| < 1$

1b.

$$\begin{array}{c|ccc} & (-\infty,-3) & (-3,0) & (0,\infty) \\ |x-3| & - & + & + \\ |x| & - & - & + \end{array}$$

I. $x \in (-\infty, -3)$

$-x-3+x < 1$

$-3 < 1$ ✓

$\Rightarrow \underline{x \in (-\infty, -3)}$

II. $x \in \langle -3, 0 \rangle$

$x+3+x < 1$

$x < -1$

$\Rightarrow \underline{x \in \langle -3, -1)}$

III. $x \in (0, \infty)$

$x+3-x < 1$

$3 < 1$ *nelze*

$\Rightarrow \underline{x \in \emptyset}$

Ř: $\underline{\underline{x \in (-\infty, -1)}}$

(2A) $f(x) = \dfrac{x}{\sqrt{1+x^2}}$

1b.

a) $f(-x) = \dfrac{-x}{\sqrt{1+(-x)^2}} = -\dfrac{x}{\sqrt{1+x^2}} = -f(x) \Rightarrow$ <u>lichá</u>

b) $(f \circ f)(x) = f(f(x)) = \dfrac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\left(\frac{x}{\sqrt{1+x^2}}\right)^2}} = \dfrac{\frac{x}{\sqrt{1+x^2}}}{\sqrt{1+\frac{x^2}{1+x^2}}} =$

$$= \frac{\frac{x}{\cancel{\sqrt{1+x^2}}}}{\frac{\sqrt{1+2x^2}}{\cancel{\sqrt{1+x^2}}}} = \underline{\underline{\frac{x}{\sqrt{1+2x^2}}}}$$

(2B)
1b.

$$f(x) = \frac{x}{\sqrt[3]{1+x^3}}$$

a) $f(-x) = \frac{-x}{\sqrt{1+(-x)^3}} = -\frac{x}{\sqrt{1-x^3}}$ ani sudá ani lichá

př. $f(\frac{1}{2}) \neq f(-\frac{1}{2})$; $-f(\frac{1}{2}) \neq f(\frac{1}{2})$

b) stejný postup jako v př. 1A : $(f \circ f)(x) = \underline{\underline{\frac{x}{\sqrt[3]{1+2x^3}}}}$

(3A)
1b.

$$\lim_{n\to\infty} \left(\sqrt{n^2+n+1} - \sqrt{n^2+2n+3}\right) \cdot \frac{\sqrt{n^2+n+1} + \sqrt{n^2+2n+3}}{\sqrt{n^2+n+1} + \sqrt{n^2+2n+3}} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}+n+1-\cancel{n^2}-2n-3}{\sqrt{n^2+n+1}+\sqrt{n^2+2n+3}} = \lim_{n\to\infty} \frac{-n-2}{\sqrt{n^2+n+1}+\sqrt{n^2+2n+3}} =$$

$$= \lim_{n\to\infty} \frac{n(-1-\frac{2}{n})}{n\left(\sqrt{1+\frac{1}{n}+\frac{1}{n^2}} + \sqrt{1+\frac{2}{n}+\frac{3}{n^2}}\right)} = \underline{\underline{-\frac{1}{2}}}$$

(3B)
1b.

$$\lim_{n\to\infty} \left(\sqrt{n+3} - \sqrt{n-1}\right) \cdot \frac{\sqrt{n+3}+\sqrt{n-1}}{\sqrt{n+3}+\sqrt{n-1}} = \lim_{n\to\infty} \frac{\cancel{n}+3-\cancel{n}+1}{\sqrt{n+3}+\sqrt{n+1}} =$$

$$= \lim_{n\to\infty} \frac{4}{\sqrt{n+3}+\sqrt{n+1}} = \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}\left(\frac{4}{\sqrt{n}}\right)}{\cancel{\sqrt{n}}\left(\sqrt{1+\frac{3}{n}}+\sqrt{1+\frac{1}{n}}\right)} = \frac{4\cdot 0}{2} = \underline{\underline{0}}$$

(4A)
2b.

$$a_n = \frac{3^n}{n!}$$

omezená zdola : $\frac{3^n}{n!} \geq 0 \quad \forall n \in \mathbb{N}$

shora : $\frac{3^n}{n!} = \underbrace{\frac{3}{n}}_{\leq 1} \underbrace{\frac{3}{(n-1)} \cdots}_{\leq 1} \underbrace{\frac{3}{3}}_{\leq 1} \underbrace{\frac{3}{2}}_{\leq 2} \underbrace{\frac{3}{1}}_{\leq 3} \leq 3 \cdot 2 = 6$

monotonie :

$$a_{n+1} = \frac{3^{n+1}}{(n+1)!} = \frac{3\cdot 3^n}{(n+1)n!} = \frac{3}{n+1} \cdot \frac{3^n}{n!} = \frac{3}{n+1} \cdot a_n$$

$a_{n+1} \nless a_n$; $a_{n+1} \ngtr a_n$ ⟹ není monotónní

⟹ posloupnost není konvergentní

4B

$a_n = \frac{n!}{n^n}$

2b.

omezená zdola: $\frac{n!}{n^n} \geq 0$

shora: $\frac{n!}{n^n} = \frac{n(n-1)\cdots\cdots 2 \cdot 1}{n \cdot n \cdots\cdots n \cdot n} \leq 1$

$\underset{\leq 1}{\frac{n}{n}} \quad \underset{\leq 1}{\frac{n-1}{n}} \quad \cdots \quad \underset{\leq 1}{\frac{2}{n}} \quad \underset{\leq 1}{\frac{1}{n}}$

monotonie:

$$a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}} = \frac{\cancel{(n+1)}(n!)}{\cancel{(n+1)}(n+1)^n} = \frac{n!}{(n+1)^n}$$

$$\frac{a_n}{a_{n+1}} = \frac{\frac{\cancel{n!}}{n^n}}{\frac{\cancel{n!}}{(n+1)^n}} = \frac{(n+1)^n}{n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n > 1$$

$\Rightarrow a_n > a_{n+1} \Rightarrow$ klesající

$\Rightarrow$ posloupnost je konvergentní

5A

$a_n = \left(\sin \frac{n\pi}{6}\right)^n$

2b.

$H_{a_n} = \{0, 1\}$

$\limsup_{n\to\infty} a_n = 1 \qquad \liminf_{n\to\infty} a_n = 0$

5B

$a_n = \frac{n}{\sqrt{2n^2+1}} \cos \frac{n\pi}{2}$

2b.

$H_{a_n} = \left\{ -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0 \right\}$

$\limsup_{n\to\infty} a_n = \frac{\sqrt{2}}{2} \qquad \liminf_{n\to\infty} a_n = -\frac{\sqrt{2}}{2}$

6A

$f(x) = \arcsin(3 - \sqrt{4-x})$

1b.

1. $4 - x \geq 0 \Rightarrow x \leq 4$
2. $3 - \sqrt{4-x} \in \langle -1, 1 \rangle$

$3 - \sqrt{4-x} \geq -1$

$4 \geq \sqrt{4-x} \quad /^2$

$16 \geq 4 - x$

$x \geq -12$

$3 - \sqrt{4-x} \leq 1$

$2 \leq \sqrt{4-x} \quad /^2$

$4 \leq 4 - x$

$0 \leq -x$

$x \leq 0$

$\Rightarrow D(f) = \langle -12, 0 \rangle$

6B

1b.

$$f(x) = \arccos \frac{2x}{1+x^2}$$

1. $1+x^2 \neq 0$ ✓

2. $\frac{2x}{1+x^2} \in \langle -1, 1 \rangle$

$$\frac{2x}{1+x^2} \geq -1$$

$$\frac{2x+1+x^2}{1+x^2} \geq 0$$

$$2x+1+x^2 \geq 0$$

$$(x+1)^2 \geq 0 \ ✓$$

$$\frac{2x}{1+x^2} \leq 1$$

$$\frac{2x-1-x^2}{1+x^2} \leq 0 \quad / \cdot (-1)$$

$$\frac{x^2-2x+1}{1+x^2} \geq 0$$

$$(x-1)^2 \geq 0 \ ✓$$

$$\underline{\underline{D(f) = \mathbb{R}}}$$

7A

1b.

$$f(x) = \frac{x}{x^3+1} = \frac{x}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$x = A(x^2-x+1) + (Bx+C)(x+1)$$

$$x = Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

x^0: $0 = A + C$

x^1: $1 = -A + B + C$ $\Rightarrow$

x^2: $0 = A + B$

$A = -\frac{1}{3}$

$B = \frac{1}{3}$

$C = \frac{1}{3}$

$$\underline{\underline{f(x) = -\frac{1}{3(x+1)} + \frac{x+1}{3(x^2-x+1)}}}$$

7B

1b.

$$f(x) = \frac{-5x+2}{x^4-x^3+2x^2} = \frac{-5x+2}{x^2(x^2-x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2-x+2}$$

$$-5x+2 = Ax(x^2-x+2) + B(x^2-x+2) + (Cx+D)x^2$$

$$-5x+2 = Ax^3 - Ax^2 + 2Ax + Bx^2 - Bx + 2B + Cx^3 + Dx^2$$

x^0: $2 = 2B$

x^1: $-5 = 2A - B$ $\Rightarrow$

x^2: $0 = -A + B + D$

x^3: $0 = A + C$

$A = -2$

$B = 1$

$C = 2$

$D = -3$

$$\underline{\underline{f(x) = -\frac{2}{x} + \frac{1}{x^2} + \frac{2x-3}{x^2-x+1}}}$$

8A
1b.

$$\lim_{x \to 3} \frac{x^2 + x - 12}{x^2 - 5x + 6} = \lim_{x \to 3} \frac{\cancel{(x-3)}(x+4)}{\cancel{(x-3)}(x+2)} = \lim_{x \to 3} \frac{x+4}{x-2} = \frac{7}{1} = \underline{\underline{7}}$$

8B
1b.

$$\lim_{x \to 4} \frac{x^2 - 2x - 8}{x^2 - 5x + 4} = \lim_{x \to 4} \frac{(x-4)(x+2)}{(x-4)(x-1)} = \lim_{x \to 4} \frac{x+2}{x-1} = \frac{6}{3} = \underline{\underline{2}}$$