POSLOUPNOSTI – Domácí práce

① Limity

a) $\lim\limits_{n\to\infty} \sqrt[3]{n^3+n^2} - n \cdot \dfrac{\sqrt[3]{(n^3+n^2)^2} + n\cdot\sqrt[3]{n^3+n^2} + n^2}{-\,{}''\,-} =$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{n^3} + n^2 - \cancel{n^3}}{\sqrt[3]{n^6+2n^5+n^4} + \sqrt[3]{n^6+n^5} + n^2} =$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{n^2}}{\cancel{n^2}\left(\sqrt[3]{1+\frac{2}{n}+\frac{1}{n^2}} + \sqrt[3]{1+\frac{1}{n}} + 1\right)} = \dfrac{1}{1+1+1} = \underline{\underline{\dfrac{1}{3}}}$

b) $\lim\limits_{n\to\infty} \left(\sqrt{(n+a)(n+b)} - n\right) \cdot \dfrac{\sqrt{(n+a)(n+b)}+n}{\sqrt{(n+a)(n+b)}+n} =$

$= \lim\limits_{n\to\infty} \dfrac{\cancel{n^2} + an + bn + ab - \cancel{n^2}}{\sqrt{n^2+an+bn+ab}+n} = \lim\limits_{n\to\infty} \dfrac{n\left(a+b+\overbrace{\frac{ab}{n}}^{\to 0}\right)}{n\left(\sqrt{1+\underbrace{\frac{a}{n}+\frac{b}{n}+\frac{ab}{n^2}}_{\to 0}}+1\right)} =$

$= \underline{\underline{\dfrac{a+b}{2}}}$

c) $\lim\limits_{n\to\infty} \sqrt[n]{2^n+3^n} = \lim\limits_{n\to\infty} \sqrt[n]{3^n\left(\left(\frac{2}{3}\right)^n+1\right)} = \lim\limits_{n\to\infty} 3\sqrt[n]{\underbrace{\left(\tfrac{2}{3}\right)^n}_{\to 0}+1} = \underline{\underline{3}}$

② Limity

a) $a_n = \dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \dots + \dfrac{1}{n(n+1)}$

$a_n = \left(1-\cancel{\tfrac{1}{2}}\right) + \left(\cancel{\tfrac{1}{2}}-\cancel{\tfrac{1}{3}}\right) + \left(\cancel{\tfrac{1}{3}}-\cancel{\tfrac{1}{4}}\right) + \dots + \left(\cancel{\tfrac{1}{n}} - \tfrac{1}{n+1}\right) =$

$= 1 - \dfrac{1}{n+1}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = \underline{\underline{1}}$

b) $a_n = \dfrac{1}{3} + \dfrac{1}{8} + \dots + \dfrac{1}{n^2-1}$

$a_n = \left(\tfrac{1}{2} - \cancel{\tfrac{1}{6}}\right) + \left(\tfrac{1}{4} - \cancel{\tfrac{1}{8}}\right) + \left(\cancel{\tfrac{1}{6}} - \tfrac{1}{10}\right) + \left(\cancel{\tfrac{1}{8}} - \tfrac{1}{12}\right) + \dots + \left(\cancel{\tfrac{1}{2(n-1)}} - \tfrac{1}{2(n+1)}\right.$

vše se odečte, zbyde $\frac{1}{2}$ a $\frac{1}{4}$

$\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{1}{2} + \frac{1}{4}\right) = \underline{\underline{\dfrac{3}{4}}}$

c) $\lim\limits_{n\to\infty} \frac{\sin(n^2)+n}{n+1} = \lim\limits_{n\to\infty} \frac{\sin(n^2)}{n+1} + \lim\limits_{n\to\infty} \frac{n}{n+1} =$

$= \lim\limits_{n\to\infty} \underbrace{\sin(n^2)}_{\text{ohraničená}} \cdot \underbrace{\frac{1}{n+1}}_{\searrow 0} + \lim\limits_{n\to\infty} \frac{1}{1+\underbrace{\frac{1}{n}}_{\searrow 0}} = 0 + 1 = \underline{\underline{1}}$

③ Konvergence

a) $a_n = \frac{n-1}{n}$

$a_{n+1} - a_n = \frac{n}{n+1} - \frac{n-1}{n} = \frac{n^2-(n+1)(n-1)}{n(n+1)} = \frac{\cancel{n^2}-\cancel{n^2}+1}{n(n+1)} =$

$= \frac{1}{n(n+1)} > 0 \quad \forall n \in \mathbb{N} \Rightarrow a_{n+1} > a_n$

rostoucí

$0 < a_n < 1$ omezená

$\Rightarrow \{a_n\}$ je konvergentní

b) $a_n = \frac{(n!)^2}{(2n)!}$

$a_{n+1} = \frac{((n+1)!)^2}{(2n+2)!} = \frac{((n+1)\cdot n!)^2}{(2n+2)(2n+1)(2n)!} = \underbrace{\frac{(n+1)^2}{(2n+2)(2n+1)}}_{<1} \cdot a_n < a_n$

$\Rightarrow$ klesající

$0 < a_n < \frac{1}{2}$ ohraničená

$\Rightarrow \{a_n\}$ je konvergentní

④ Hromadné body

a) $\{a_n\} = \{ \frac{1}{2}; \frac{1}{2}; \frac{1}{4}; \frac{3}{4}; \frac{1}{8}; \frac{7}{8}; \dots; \frac{1}{2^n}; \frac{2^n-1}{2^n}; \dots \}$

$k_n = 2n-1: \quad \{a_{k_n}\} = \{ \frac{1}{2}; \frac{1}{4}; \dots; \frac{1}{2^n}; \dots \}$

$\lim\limits_{n\to\infty} a_{k_n} = 0$

$k_n = 2n: \quad \{a_{k_n}\} = \{ \frac{1}{2}; \frac{3}{4}; \frac{7}{8}; \dots; \frac{2^n-1}{2^n}, \dots \}$

$\lim\limits_{n\to\infty} a_{k_n} = 1$

$H_{a_n} = \{0, 1\}$

b) $a_n = \frac{n}{n+1} \cdot \underbrace{\sin \frac{n\pi}{6}}_{\text{nekonverguje}}$

$\frac{n}{n+1} \searrow 1$

$\sin \frac{n\pi}{6} \in \{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, \dots\}$

$H_{a_n} = \{0, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1\}$

c) $a_n = \sqrt[n]{1 + (-1)^n}$

$k_n = 2n-1: \quad a_{k_n} = \sqrt[n]{1 + (-1)^n} = \sqrt[n]{1-1} = 0$

$\lim_{n \to \infty} a_{k_n} = 0$

$k_n = 2n: \quad a_{k_n} = \sqrt[n]{1 + (-1)^n} = \sqrt[n]{1+1} = \sqrt[n]{2}$

$\lim_{n \to \infty} a_{k_n} = \lim_{n \to \infty} \sqrt[n]{2} = \lim_{n \to \infty} (2)^{\frac{1}{n}} = 1$

$H_{a_n} = \{0, 1\}$

⑤ Limsup ; Liminf

a) ~~...~~ $a_n = (-1)^{n-1} (2 + \frac{1}{n})$

~~...~~ $a_n = \left\langle \begin{array}{l} 2 + \frac{1}{n} \\ -(2 + \frac{1}{n}) \end{array} \right.$

$H(a_n) = \{-2, 2\}$

$\limsup_{n \to \infty} a_n = \underline{\underline{2}}$; $\liminf_{n \to \infty} a_n = \underline{\underline{-2}}$

b) $a_n = 1 + \frac{n}{n+1} \cos \frac{n\pi}{2}$

$\frac{n}{n+1} \searrow 1$

$\cos \frac{n\pi}{2} \in \{0, -1, 0, 1, \dots\}$

$H_{a_n} = \{0, 1, 2\} \Rightarrow \limsup_{n \to \infty} a_n = \underline{\underline{2}}$; $\liminf_{n \to \infty} a_n = \underline{\underline{0}}$

c) $a_n = (-1)^n \left(1 + \frac{1}{n}\right)^n + \sin \frac{n\pi}{4}$

$\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$; $(-1)^n \in \{-1, 1\}$; $\sin \frac{n\pi}{4} \in \{ \frac{\sqrt{2}}{2}, 1, 0, -1, -\frac{\sqrt{2}}{2} \}$

$H_{a_n} = \{ -e + \frac{\sqrt{2}}{2} ; \cancel{-e+1} ; \cancel{-e} ; \cancel{-e-1} ; -e - \frac{\sqrt{2}}{2} ; \cancel{e + \frac{\sqrt{2}}{2}} ; e+1 ;$

$e ; e-1 ; \cancel{e - \frac{\sqrt{2}}{2}} \}$

$\limsup\limits_{n \to \infty} a_n = \underline{\underline{e+1}}$; $\liminf\limits_{n \to \infty} a_n = \underline{\underline{-e - \frac{\sqrt{2}}{2}}}$