Chapter VI

LIMITING DISTRIBUTIONS OF TEST STATISTICS UNDER THE ALTERNATIVES

1. Contiguity

1.1. Asymptotic methods. Contiguity. The asymptotic approach consists in regarding a given testing problem as a member of a sequence $\{H_\nu, K_\nu\}$, $\nu \geqq 1$, of similar testing problems. In this sequence the ν-th testing problem concerns N_ν observations $X_1, \ldots, X_{N_\nu}$ with $N_\nu \to \infty$ as $\nu \to \infty$. As a rule, H_ν depends on ν through N_ν only, i.e. $H_\nu = H(N_\nu)$, whereas K_ν depends on some parameters $d_{\nu i}$, $0 \leqq i \leqq N_\nu$, in addition. For example, we might assume that $H_\nu = H_0$, H_0 being applied to $N = N_\nu$ observations, and that K_ν consists of a single density q_ν,

$$q_\nu = \prod_{i=1}^{N_\nu} f_0(x_i - d_{\nu i}) .$$

Of course, there are infinitely many such sequences, and we try to choose one which resembles the given testing problem as much as possible. First of all it would be desirable to keep the envelope power function $\beta(\alpha, H_\nu, K_\nu)$ independent of ν. Since this is usually difficult or even impossible, we shall be satisfied with the existence of a limit $\beta(\alpha)$:

$$\lim_{\nu \to \infty} \beta(\alpha, H_\nu, K_\nu) = \beta(\alpha) , \quad 0 \leqq \alpha \leqq 1 . \tag{1}$$

As in the previous chapter, we shall also consider indexed sets of testing problems $\{H_d, K_d, d \in D\}$, where the convergence will be equivalent to the convergence to a fixed limit for all sequences selected from the set and satisfying certain requirements. As a rule, K_d will be simple, consisting of a density q_d.

The limiting relation (1) entails that $\beta(\alpha, H_\nu, K_\nu)$ will approximately equal $\beta(\alpha)$ for $\nu \geqq \nu_0$. The usefulness of the asymptotic results will depend on whether the problems "H_ν against K_ν" with $\nu \geqq \nu_0$, may occur in practice or not. The value of ν_0 is usually guessed on the basis of numerical calculations for selected ν's and the assumption that the convergence is more or less monotone.

In this book we shall not investigate the somewhat degenerate cases in which

$$\beta(\alpha) = 1 \quad \text{for all} \quad \alpha > 0 . \tag{2}$$

We shall even exclude the cases in which

$$\beta(\alpha) \nrightarrow 0 \quad \text{for} \quad \alpha \to 0\,. \tag{3}$$

However, it may be shown, that in problems dealt with in the sequel, (3) implies (2).

The requirement that (3) should not take place finds its theoretical expression in the notion of contiguity, which is due to LeCam (1960). The notion of contiguity is basic for the asymptotic methods of the theory of hypothesis testing.

Consider a sequence $\{p_\nu, q_\nu\}$ of simple hypotheses p_ν and simple alternatives q_ν defined on measure spaces $(\boldsymbol{X}_\nu, \mathscr{A}_\nu, \mu_\nu)$, $\nu \geqq 1$, respectively.

Definition. If for any sequence of events $\{A_\nu\}$, $A_\nu \in \mathscr{A}_\nu$,

$$[P_\nu(A_\nu) \to 0] \Rightarrow [Q_\nu(A_\nu) \to 0] \tag{4}$$

holds, we say that the densities q_ν are *contiguous* to the densities p_ν, where $\mathrm{d}P_\nu = p_\nu\,\mathrm{d}\mu_\nu$, $\mathrm{d}Q_\nu = q_\nu\,\mathrm{d}\mu_\nu$, $\nu \geqq 1$.

If H_ν is composite, we say that q_ν is contiguous to H_ν if for each ν the convex hull $\bar{H}_\nu$ of H_ν contains a density p_ν such that (4) holds.

If both H_ν and K_ν are composite, we say that K_ν is contiguous to H_ν if (4) holds for some $p_\nu \in \bar{H}_\nu$ and $q_\nu \in \bar{K}_\nu$.

Contiguity implies that any sequence of random variables converging to zero in P_ν-probability converges to zero in Q_ν-probability, $\nu \to \infty$.

1.2. LeCam's first lemma. According to the Neyman-Pearson lemma, for any event A_ν there exists a critical function Φ_ν such that

$$\begin{aligned} \Phi_\nu &= 0\,, \quad \text{if} \quad q_\nu < k_\nu p_\nu\,, \\ &= \xi\,, \quad \text{if} \quad q_\nu = k_\nu p_\nu\,, \\ &= 1\,, \quad \text{if} \quad q_\nu > k_\nu p_\nu\,, \end{aligned} \tag{1}$$

where $0 \leqq \xi \leqq 1$, and that

$$P_\nu(A_\nu) = \int \Phi_\nu\,\mathrm{d}P_\nu\,,$$

$$Q_\nu(A_\nu) \leqq \int \Phi_\nu\,\mathrm{d}Q_\nu\,.$$

Thus contiguity will follow if we show that

$$\left[\int \Phi_\nu\,\mathrm{d}P_\nu \to 0\right] \Rightarrow \left[\int \Phi_\nu\,\mathrm{d}Q_\nu \to 0\right] \tag{2}$$

for critical functions of the type (1).

Introduce the likelihood ratio $L_\nu = q_\nu/p_\nu$, or more precisely,

$$
\begin{aligned}
L_\nu(x_\nu) &= q_\nu(x_\nu)/p_\nu(x_\nu)\,, && \text{if} \quad p_\nu(x_\nu) > 0\,,\\
&= 1\,, && \text{if} \quad p_\nu(x_\nu) = q_\nu(x_\nu) = 0\,,\\
&= \infty\,, && \text{if} \quad p_\nu(x_\nu) = 0 < q_\nu(x_\nu)\,,
\end{aligned} \tag{3}
$$

where x_ν denotes the typical point of the space $\mathbf{X}_\nu$, $\nu \geqq 1$.

Let F_ν be the distribution function of L_ν under P_ν:

$$F_\nu(x) = P_\nu(L_\nu \leqq x)\,, \tag{4}$$

where $L_\nu = L_\nu(X_\nu)$, $\nu \geqq 1$.

Lemma. *Assume that F_ν given by (4) converges weakly (at continuity points) to a distribution function F such that*

$$\int_0^\infty x \, \mathrm{d}F(x) = 1\,. \tag{5}$$

Then the densities q_ν are contiguous to the densities p_ν, $\nu \geqq 1$.

Proof. Take a sequence of critical functions Φ_ν of the type (1) and such that

$$\int \Phi_\nu \, \mathrm{d}P_\nu \to 0\,. \tag{6}$$

Then note that

$$
\begin{aligned}
\int \Phi_\nu \, \mathrm{d}Q_\nu &= \int_{\{L_\nu \leqq y\}} \Phi_\nu \, \mathrm{d}Q_\nu + \int_{\{L_\nu > y\}} \Phi_\nu \, \mathrm{d}Q_\nu \leqq \\
&\leqq y \int \Phi_\nu \, \mathrm{d}P_\nu + \int_{\{L_\nu > y\}} \mathrm{d}Q_\nu = \\
&= y \int \Phi_\nu \, \mathrm{d}P_\nu + 1 - \int_{\{L_\nu \leqq y\}} \mathrm{d}Q_\nu = \\
&= y \int \Phi_\nu \, \mathrm{d}P_\nu + 1 - \int_{\{L_\nu \leqq y\}} L_\nu \, \mathrm{d}P_\nu = \\
&= y \int \Phi_\nu \, \mathrm{d}P_\nu + 1 - \int_0^y x \, \mathrm{d}F_\nu\,.
\end{aligned} \tag{7}
$$

Now for any $\varepsilon > 0$ we can find a continuity point y of F such that, in view of (5),

$$1 - \int_0^y x \, \mathrm{d}F < \tfrac{1}{2}\varepsilon\,.$$

Since $F_\nu \to F$ entails

$$\int_0^y x\, \mathrm{d}F_\nu \to \int_0^y x\, \mathrm{d}F$$

we shall have for some ν_0

$$(8) \qquad 1 - \int_0^y x\, \mathrm{d}F_\nu < \tfrac{1}{2}\varepsilon\,, \quad \nu \geqq \nu_0\,.$$

Furthermore, (6) ensures the existence of ν_1 such that

$$(9) \qquad y \int \Phi_\nu\, \mathrm{d}P_\nu < \tfrac{1}{2}\varepsilon\,, \quad \nu \geqq \nu_1\,.$$

Finally, from (7) through (9) it follows that

$$\int \Phi_\nu\, \mathrm{d}Q_\nu < \varepsilon \quad \text{for} \quad \nu \geqq \max(\nu_0, \nu_1)\,.$$

Thus $\int \Phi_\nu\, \mathrm{d}Q_\nu \to 0$, which concludes the proof.

Remark. Note that contiguity does not entail that the Q_ν are absolutely continuous with respect to the P_ν. The singular part of Q_ν, however, must tend to zero,

$$Q_\nu(p_\nu = 0) \to 0$$

as a consequence of $P_\nu(p_\nu = 0) = 0 \to 0$.

The asymptotic distribution of the likelihoods L_ν will regularly be log-normal. We shall say that a random variable Y is log-normal (μ, σ^2), if $\log Y$ is normal (μ, σ^2). Now let us establish the condition under which we obtain

$$\mathrm{E}Y = \int_0^\infty x\, \mathrm{d}F = 1$$

for a log-normal random variable Y. We obviously have

$$\mathrm{E}Y = \mathrm{E}\exp(\log Y) = (2\pi)^{-\frac{1}{2}}\sigma^{-1}\int_{-\infty}^{\infty} \exp\left[x - \tfrac{1}{2}(x-\mu)^2\sigma^{-2}\right]\mathrm{d}x = e^{\mu + \frac{1}{2}\sigma^2}\,.$$

This equals 1 for

$$(10) \qquad \mu = -\tfrac{1}{2}\sigma^2\,.$$

Thus we have the following

Corollary. *If L_ν is asymptotically log-normal $(-\frac{1}{2}\sigma^2, \sigma^2)$, then the densities q_ν are contiguous to the densities p_ν.*

1.3. LeCam's second lemma. Assume that $x_\nu = (x_1, \ldots, x_{N_\nu})$ and

$$p_\nu(x_\nu) = \prod_{i=1}^{N_\nu} f_{\nu i}(x_i) \tag{1}$$

and

$$q_\nu(x_\nu) = \prod_{i=1}^{N_\nu} g_{\nu i}(x_i)\,. \tag{2}$$

From (1) and (2) we have

$$\log L_\nu = \sum_{i=1}^{N_\nu} \log\left[g_{\nu i}(x_i)/f_{\nu i}(x_i)\right] \tag{3}$$

which makes sense even if $\log L_\nu = \pm\infty$, since on the right side the summands are $<\infty$ with P_ν-probability 1 and are $> -\infty$ with Q_ν-probability 1.

Thus we may regard $\log L_\nu$ as an extended random variable allowed to attain $-\infty$ with positive probability under P_ν. However, asymptotic normality of $\log L_\nu$ is defined in the same way as for an ordinary random variable, i.e. as convergence of $P_\nu(\log L_\nu < x)$ to a normal distribution function in every real point x. Thus asymptotic normality entails $P_\nu(\log L_\nu = -\infty) \to 0$.

In what follows we restrict ourselves to cases in which the summands in (3) are uniformly asymptotically negligible, i.e.

$$\lim_{\nu\to\infty} \max_{1 \leq i \leq N_\nu} P_\nu\left(\left|\frac{g_{\nu i}(X_i)}{f_{\nu i}(X_i)} - 1\right| > \varepsilon\right) = 0\,. \tag{4}$$

Under this condition necessary and sufficient conditions of asymptotic normality are well-known. These conditions are considerably simpler if the summands have finite variance. However, this fails sometimes to be satisfied in (3) within the class of problems considered below. For this reason we instead consider the statistic

$$W_\nu = 2\sum_{i=1}^{N_\nu} \{[(g_{\nu i}(X_i)/f_{\nu i}(X_i)]^{\frac{1}{2}} - 1\} \tag{5}$$

which always consists of summands with finite variances, as may be easily seen, and has additional advantages. The following lemma, due to LeCam, shows that asymptotic normality of $\log L_\nu$ may be established by proving asymptotic normality of W_ν.

Lemma. *Assume that* (4) *holds and that the statistics* W_ν, $\nu \geqq 1$, *are asymptotically normal* $(-\frac{1}{4}\sigma^2, \sigma^2)$ *under* P_ν.

Then the statistics $\log L_\nu$ *satisfy*

$$\lim_{\nu\to\infty} P_\nu(|\log L_\nu - W_\nu + \tfrac{1}{4}\sigma^2| > \varepsilon) = 0\,, \quad \varepsilon > 0\,, \tag{6}$$

and are asymptotically normal $(-\frac{1}{2}\sigma^2, \sigma^2)$ *under* P_ν.

Proof. If a function $h(x)$ has a second derivative $h''(x)$, then

$$(7)\qquad h(x) = h(x_0) + (x - x_0)\, h'(x_0) + + \tfrac{1}{2}(x - x_0)^2 \int_0^1 2(1 - \lambda)\, h''[x_0 + \lambda(x - x_0)]\, d\lambda\,,$$

as may be easily seen by integration by parts. Thus, putting

$$(8)\qquad T_{\nu i} = 2[g_{\nu i}(X_i)/f_{\nu i}(X_i)]^{\frac{1}{2}} - 2\,,$$

we obtain

$h(x) = 2 \log x$
$x = \frac{1}{2} T_{\nu i} + 1$
$x - x_0 = \frac{1}{2} T_{\nu i}$

$$\log (g_{\nu i}/f_{\nu i}) = 2 \log (1 + \tfrac{1}{2}T_{\nu i}) = = T_{\nu i} - \tfrac{1}{4}T_{\nu i}^2 \int_0^1 [2(1 - \lambda)/(1 + \tfrac{1}{2}\lambda T_{\nu i})^2]\, d\lambda\,.$$

Consequently,

$$(9)\qquad \log L_\nu = W_\nu - \frac{1}{4} \sum_{i=1}^{N_\nu} T_{\nu i}^2 \int_0^1 [2(1 - \lambda)/(1 + \tfrac{1}{2}\lambda T_{\nu i})^2]\, d\lambda\,.$$

This holds even for $\log L_\nu = -\infty$.

Introduce

$$T_{\nu i}^\delta = T_{\nu i}\,, \quad \text{if} \quad |T_{\nu i}| \leqq \delta\,,$$
$$= 0\,, \quad \text{otherwise}\,.$$

As is well known (Loève (1955), p. 316), asymptotic normality $(-\frac{1}{4}\sigma^2, \sigma^2)$ of W_ν implies under (4) that for every $\delta > 0$

$$(10)\qquad \sum_{i=1}^{N_\nu} P_\nu(|T_{\nu i}| > \delta) \to 0\,,$$

$$(11)\qquad \sum_{i=1}^{N_\nu} \mathsf{E} T_{\nu i}^\delta \to -\tfrac{1}{4}\sigma^2\,,$$

$$(12)\qquad \sum_{i=1}^{N_\nu} \operatorname{var} T_{\nu i}^\delta \to \sigma^2\,.$$

Now (10), holding for *every* $\delta > 0$, entails

$$(13)\qquad \sum_{i=1}^{N_\nu} T_{\nu i}^2 \int_0^1 [2(1 - \lambda)/(1 + \tfrac{1}{2}\lambda T_{\nu i})^2]\, d\lambda \sim \sum_{i=1}^{N_\nu} (T_{\nu i}^\delta)^2$$

where $\sim$ denotes that the ratio of both sides tends to 1 in P_ν-probability. Thus, in order to prove (6), it remains to show that

$$\sum_{i=1}^{N_\nu} (T_{\nu i}^\delta)^2 \to \sigma^2 \tag{14}$$

in P_ν-probability. For this purpose it suffices to prove

$$\sum_{i=1}^{N_\nu} \mathsf{E}(T_{\nu i}^\delta)^2 \to \sigma^2 \tag{15}$$

and

$$\lim_{\delta \to 0} \limsup_{\nu \to \infty} \sum_{i=1}^{N_\nu} \operatorname{var} (T_{\nu i}^\delta)^2 = 0 , \tag{16}$$

since then (14) will follow by the Chebyshev inequality. Further, in view of (12), (15) is equivalent to

$$\sum_{i=1}^{N_\nu} (\mathsf{E}T_{\nu i}^\delta)^2 \to 0 . \tag{17}$$

We first prove (17). If $\delta > 2$, then $T_{\nu i}^\delta \leqq T_{\nu i}$, since $T_{\nu i} \geqq -2$, in view of (8). Consequently,

$$\begin{aligned} \mathsf{E}T_{\nu i}^\delta &\leqq \mathsf{E}T_{\nu i} = 2\mathsf{E}\{g_{\nu i}(X_i)/f_{\nu i}(X_i)\}^{\frac{1}{2}} - 2 \leqq \\ &\leqq 2\{\mathsf{E}[g_{\nu i}(X_i)/f_{\nu i}(X_i)]\}^{\frac{1}{2}} - 2 = \\ &= 2\left\{\int_{\{f_{\nu i}>0\}} g_{\nu i}(x)\, \mathrm{d}x\right\}^{\frac{1}{2}} - 2 \leqq 0 . \end{aligned} \tag{18}$$

Thus, for $\delta > 2$,

$$\sum_{i=1}^{N_\nu} (\mathsf{E}T_{\nu i}^\delta)^2 \leqq \min_{1 \leqq i \leqq N_\nu} \mathsf{E}T_{\nu i}^\delta \sum_{i=1}^{N_\nu} \mathsf{E}T_{\nu i}^\delta ,$$

and (17) follows from (11) and from the fact that

$$\min_{1 \leqq i \leqq N} \mathsf{E}T_{\nu i}^\delta \to 0 ,$$

which is an easy consequence of (4). Now it remains to note that the validity of (17) for any $\delta > 2$ entails its validity for any $\delta > 0$, because of (12) and of

$$\sum_{i=1}^{N_\nu} \mathsf{E}(T_{\nu i}^{\delta_1})^2 \leqq \sum_{i=1}^{N_\nu} \mathsf{E}(T_{\nu i}^{\delta_2})^2 , \quad \delta_1 < \delta_2 .$$

As for (16), first note that

$$\sum_{i=1}^{N_\nu} \operatorname{var} [(T_{\nu i}^\delta)^2] \leqq \sum_{i=1}^{N_\nu} \mathsf{E}(T_{\nu i}^\delta)^4 \leqq \delta^2 \sum_{i=1}^{N_\nu} \mathsf{E}(T_{\nu i}^\delta)^2 .$$

Thus, on account of (15),

$$\limsup_{\nu\to\infty} \sum_{i=1}^{N_\nu} \operatorname{var}(T_{\nu i}^{\delta})^2 \leq \delta^2\sigma^2 . \tag{19}$$

Consequently, (16) holds.

Finally, asymptotic normality $(-\frac{1}{2}\sigma^2, \sigma^2)$ is an immediate consequence of (6). Q.E.D.

The reader, having finished the above proof, can observe that we did not evade niceties connected with the truncation of summands. However, they are concentrated in the above proof and will not trouble us any more.

1.4. LeCam's third lemma. Limiting distributions of test statistics S_ν *under the alternative* are important from the point of view of the power properties of the respective tests. Unfortunately, their derivations are considerably more difficult than the proofs of limiting distributions under the hypothesis. Nonetheless, in the contiguity case, the difficulties are essentially diminished by the following lemma due to LeCam.

We say that the pair $(S_\nu, \log L_\nu)$ is asymptotically jointly normal $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_{12})$ if it converges in distribution to a normal vector (Z_1, Z_2) such that $\mathsf{E}Z_i = \mu_i$, $\operatorname{var} Z_i = \sigma_i^2$, $i = 1, 2$, and $\operatorname{cov}(Z_1, Z_2) = \sigma_{12}$. (For convergence in distribution in $k \geq 2$ dimensions see the definitions of Section V.2.1.)

Lemma. *Assume that the pair $(S_\nu, \log L_\nu)$ is under P_ν asymptotically jointly normal $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_{12})$ with $\mu_2 = -\frac{1}{2}\sigma_2^2$.*

Then S_ν is under Q_ν asymptotically normal $(\mu_1 + \sigma_{12}, \sigma_1^2)$.

Proof. Obviously,

$$\begin{aligned} Q_\nu(S_\nu \leq x) &= \int_{\{S_\nu \leq x\}} \mathrm{d}Q_\nu = \\ &= \int_{\{S_\nu \leq x\}} L_\nu \,\mathrm{d}P_\nu + Q_\nu(p_\nu = 0, S_\nu \leq x) = \\ &= \int_{-\infty}^{x} \int_{-\infty}^{\infty} e^v \,\mathrm{d}F_\nu(u, v) + Q_\nu(p_\nu = 0, S_\nu \leq x) , \end{aligned} \tag{1}$$

where $F_\nu(u, v)$ denotes the distribution function of $(S_\nu, \log L_\nu)$. Now $\mu_2 = -\frac{1}{2}\sigma_2^2$ implies contiguity (Corollary 1.2), and hence

$$Q_\nu(p_\nu = 0, S_\nu \leq x) \to 0 , \tag{2}$$

since $P_\nu(p_\nu = 0, S_\nu \leqq x) = 0 \to 0$. Furthermore, for any $c > 0$

$$(3) \qquad \int_{-\infty}^{x} \int_{-c}^{c} e^v \, \mathrm{d}F_\nu(u, v) \to \int_{-\infty}^{x} \int_{-c}^{c} e^v \, \mathrm{d}\Phi(u, v),$$

where $\Phi(u, v)$ denotes the two-dimensional normal distribution function with parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_{12})$. Actually, $F_\nu \to \Phi$ according to our assumption, and the function

$$h(u, v) = e^v, \quad -\infty < u < x, \quad -c \leqq v \leqq c$$
$$= 0, \quad \text{otherwise},$$

is uniformly bounded and continuous except on the set $\{(u, v) : v = -c \text{ or } v = c \text{ or } u = x\}$, which obviously has Φ-probability 0. Thus we may apply D1 of Section V.2.1.

Now (1), (2) and (3) will imply

$$(4) \qquad Q_\nu(S_\nu \leqq x) \to \int_{-\infty}^{x} \int_{-\infty}^{\infty} e^v \, \mathrm{d}\Phi(u, v)$$

if we show that for every ε there exist c_0 and ν_0 such that

$$(5) \qquad \int_{-\infty}^{x} \int_{-\infty}^{-c_0} e^v \, \mathrm{d}F_\nu + \int_{-\infty}^{x} \int_{c_0}^{\infty} e^v \, \mathrm{d}F_\nu < \varepsilon, \quad \nu \geqq \nu_0.$$

In other words we must show that the truncated parts of the integral are uniformly small if c_0 is sufficiently large. However, (5) is an easy consequence of contiguity. Actually, if (5) were not true for some $\varepsilon > 0$, we would have a sequence of pairs (c_j, ν_j) such that

$$(6) \qquad \lim_{j \to \infty} c_j = \infty, \quad \lim_{j \to \infty} \nu_j = \infty,$$

and

$$Q_{\nu_j}(\log L_{\nu_j} < -c_j \text{ or } \log L_{\nu_j} > c_j) =$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{-c_j} e^v \, \mathrm{d}F_{\nu_j} + \int_{-\infty}^{\infty} \int_{c_j}^{\infty} e^v \, \mathrm{d}F_{\nu_j} \geqq$$
$$\geqq \int_{-\infty}^{x} \int_{-\infty}^{-c_j} e^v \, \mathrm{d}F_{\nu_j} + \int_{-\infty}^{x} \int_{c_j}^{\infty} e^v \, \mathrm{d}F_{\nu_j} \geqq \varepsilon.$$

On the other hand, since $\log L_\nu$ is asymptotically normal under P_ν,

$$P_{\nu_j}(\log L_{\nu_j} < -c_j \text{ or } \log L_{\nu_j} > c_j) \to 0,$$

because of (6). This contradicts contiguity, and thereby (4) is proved.

since $P_\nu(p_\nu = 0, S_\nu \leqq x) = 0 \to 0$. Furthermore, for any $c > 0$

$$\int_{-\infty}^{x}\int_{-c}^{c} e^{v}\,\mathrm{d}F_\nu(u, v) \to \int_{-\infty}^{x}\int_{-c}^{c} e^{v}\,\mathrm{d}\Phi(u, v)\,, \tag{3}$$

where $\Phi(u, v)$ denotes the two-dimensional normal distribution function with parameters $(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \sigma_{12})$. Actually, $F_\nu \to \Phi$ according to our assumption, and the function

$$\begin{aligned} h(u, v) &= e^{v}\,, \quad -\infty < u < x\,, \quad -c \leqq v \leqq c \\ &= 0\,, \quad \text{otherwise,} \end{aligned}$$

is uniformly bounded and continuous except on the set $\{(u, v) : v = -c \text{ or } v = c \text{ or } u = x\}$, which obviously has Φ-probability 0. Thus we may apply D1 of Section V.2.1.

Now (1), (2) and (3) will imply

$$Q_\nu(S_\nu \leqq x) \to \int_{-\infty}^{x}\int_{-\infty}^{\infty} e^{v}\,\mathrm{d}\Phi(u, v) \tag{4}$$

if we show that for every ε there exist c_0 and ν_0 such that

$$\int_{-\infty}^{x}\int_{-\infty}^{-c_0} e^{v}\,\mathrm{d}F_\nu + \int_{-\infty}^{x}\int_{c_0}^{\infty} e^{v}\,\mathrm{d}F_\nu < \varepsilon\,, \quad \nu \geqq \nu_0\,. \tag{5}$$

In other words we must show that the truncated parts of the integral are uniformly small if c_0 is sufficiently large. However, (5) is an easy consequence of contiguity. Actually, if (5) were not true for some $\varepsilon > 0$, we would have a sequence of pairs (c_j, ν_j) such that

$$\lim_{j\to\infty} c_j = \infty\,, \quad \lim_{j\to\infty} \nu_j = \infty\,, \tag{6}$$

and

$$\begin{aligned} &Q_{\nu_j}(\log L_{\nu_j} < -c_j \text{ or } \log L_{\nu_j} > c_j) = \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{-c_j} e^{v}\,\mathrm{d}F_{\nu_j} + \int_{-\infty}^{\infty}\int_{c_j}^{\infty} e^{v}\,\mathrm{d}F_{\nu_j} \geqq \\ &\geqq \int_{-\infty}^{x}\int_{-\infty}^{-c_j} e^{v}\,\mathrm{d}F_{\nu_j} + \int_{-\infty}^{x}\int_{c_j}^{\infty} e^{v}\,\mathrm{d}F_{\nu_j} \geqq \varepsilon\,. \end{aligned}$$

On the other hand, since $\log L_\nu$ is asymptotically normal under P_ν,

$$P_{\nu_j}(\log L_{\nu_j} < -c_j \text{ or } \log L_{\nu_j} > c_j) \to 0\,,$$

because of (6). This contradicts contiguity, and thereby (4) is proved.

Now, by easy computations, we derive that

$$\text{(7)} \quad \int_{-\infty}^{x}\int_{-\infty}^{\infty} e^{v}\,\mathrm{d}\Phi = \int_{-\infty}^{x}\int_{-\infty}^{\infty} (\sigma_1\sigma_2 \,.\, 2\pi)^{-1}\,[(1-\varrho^2)]^{-\frac{1}{2}}\,.$$

$$.\exp\{v - [2(1-\varrho^2)]^{-1}\,[(u-\mu_1)^2\,\sigma_1^{-2} -$$

$$- 2\varrho(u-\mu_1)(v+\tfrac{1}{2}\sigma_2^2)(\sigma_1\sigma_2)^{-1} + (v+\tfrac{1}{2}\sigma_2^2)^2\,\sigma_2^{-2}]\}\,\mathrm{d}u\,\mathrm{d}v =$$

$$= \sigma_1^{-1}(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} \exp\left[-\tfrac{1}{2}(u-\mu_1-\sigma_{12})^2\,\sigma_1^{-2}\right]\mathrm{d}u\,,$$

where $\varrho = \sigma_{12}(\sigma_1\sigma_2)^{-1}$. Combining (4) and (7), we easily conclude the proof.

Remark. The above Lemma holds even if $\sigma_2^2 = 0$, i.e. if $\log L_\nu$ converges to 0 in probability.

2. Simple linear rank statistics

2.1. Location alternatives for H_0. We shall consider alternatives

$$\text{(1)} \quad q_d = \prod_{i=1}^{N} f_0(x_i - d_i)\,,$$

where f_0 is a known density with $I(f_0) < \infty$, and $d = (d_1, \ldots, d_N)$ is an arbitrary vector. Recall that the vector d runs through the set of all real vectors of all finite dimensions, and that the asymptotic statements concern sequences $\{d_\nu = (d_{\nu 1}, \ldots \ldots, d_{\nu N_\nu})\}$ selected from this set. However, to simplify the notation, we shall drop the index ν. First of all, we shall establish conditions under which sequences of such alternatives are contiguous with respect to corresponding sequences of the hypotheses $H_0 = H_{0N}$. For this purpose we associate with each q_d the following density

$$\text{(2)} \quad p_d = \prod_{i=1}^{N} f_0(x_i - \bar{d})\,.$$

Obviously $p_d \in H_{0N}$, where N is the dimension of d, and it depends on d only through $\bar{d} = N^{-1}\sum d_i$. If we show that under certain conditions the densities q_d are contiguous with respect to the densities p_d, then, a fortiori, they will be contiguous with respect to the hypotheses H_{0N}. The densities p_d have been chosen so as to be least favourable for H_{0N} against q_d, in an asymptotic sense.

According to LeCam's first lemma, the densities q_d are contiguous to the densities p_d, if $\log L_d$, where $L_d = q_d/p_d$, is asymptotically normal $(-\frac{1}{2}\sigma^2, \sigma^2)$. Moreover, on

Application 2. Approximate scores corresponding to some density f such that $\varphi(u, f)$ is a sum of monotone and square integrable functions. If, in addition, $\varphi(u, f)$ is skew symmetric, then

$$S_{mn} = \sum_{i=1}^{m} \varphi\left(\frac{R_i}{m+n+1}, f\right), \tag{10}$$

with (8) and (9) still applicable. If $\varphi(u, f)$ is not symmetric, we have to subtract the expectation under H_0, i.e. to put

$$S_{mn} = \sum_{i=1}^{m} \varphi\left(\frac{R_i}{m+n+1}, f\right) - \frac{m}{N} \sum_{i=1}^{N} \varphi\left(\frac{i}{m+n+1}, f\right).$$

However, since $\int_0^1 \varphi(u, f)\,\mathrm{d}u = 0$, the correction is asymptotically negligible, as may be shown.

2.4. Rank statistics for H_0 against regression. Now we shall investigate the limiting distribution under q_d of the statistics

$$S_c = \sum_{i=1}^{N} (c_i - \bar{c})\, a_N(R_i)\,. \tag{1}$$

Theorem. *Let q_d be given by* (2.1.1) *and assume that* (2.1.4) *and* (2.1.5) *hold. Then, under q_d the statistics S_c given by* (1), *where the scores satisfy* (2.3.1), *are for* $\sum_{i=1}^{N} (c_i - \bar{c})^2 / \max_{1 \leq i \leq N} (c_i - \bar{c})^2 \to \infty$ *asymptotically normal* (μ_{dc}, σ_c^2) *with*

$$\mu_{dc} = \left[\sum_{i=1}^{N} (c_i - \bar{c})(d_i - \bar{d})\right] \int_0^1 \varphi(u)\, \varphi(u, f_0)\,\mathrm{d}u \tag{2}$$

and

$$\sigma_c^2 = \left[\sum_{i=1}^{N} (c_i - \bar{c})^2\right] \int_0^1 [\varphi(u) - \bar{\varphi}]^2\,\mathrm{d}u\,. \tag{3}$$

The assertions remain true if we replace (2.1.1), (2.1.5) *and* $\varphi(u, f_0)$ *by* (2.2.1), (2.2.3) *and* $\varphi_1(u, f)$, *respectively.*

Proof. Without loss of generality, we may assume that

$$\sum_{i=1}^{N} (c_i - \bar{c})^2 = 1 \tag{4}$$

and

$$\sum_{i=1}^{N} (c_i - \bar{c})(d_i - \bar{d}) \to b_{12}\,. \tag{5}$$

Note that under (4) $\sum(c_i - \bar{c})^2/\max(c_i - \bar{c})^2 \to \infty$ is equivalent to

$$\max_{1 \leq i \leq N} (c_i - \bar{c})^2 \to 0 \,. \tag{6}$$

Furthermore, if S_c^φ is given by

$$S_c^\varphi = \sum_{i=1}^{N} (c_i - \bar{c})\, a_N^\varphi(R_i)\,, \tag{7}$$

where the scores are related to φ by (V.1.4.12), then $(S_c - S_c^\varphi)(\text{var } S_c^\varphi)^{-\frac{1}{2}} \to 0$ in probability under H_0 (see the proof of Theorem V.I.6.a). We shall denote this briefly by $S_c \sim S_c^\varphi$. Furthermore, inspecting the proof of Theorem V.1.5.a, we see that $S_c^\varphi \sim T_c$, T_c given by

$$T_c = \sum_{i=1}^{N} (c_i - \bar{c})\, \varphi(U_i)\,, \tag{8}$$

where $U_i = F_d(X_i)$, $F_d(x) = P_d(X_i \leqq x)$ with P_d given by (2.1.2). Thus $S_c \sim T_c$, and S_c may be replaced by T_c in considerations concerning the limiting distribution.

On the other hand, we know from Theorem 2.1 that $\log L_d \sim (T_d - \frac{1}{2}b^2)$, T_d given by (2.1.13), or equivalently, by

$$T_d = \sum_{i=1}^{N} (d_i - \bar{d})\, \varphi(U_i, f_0)\,. \tag{9}$$

Thus

$$(S_c, \log L_d) \sim (T_c, T_d - \tfrac{1}{2}b^2)\,, \tag{10}$$

where it should be noted that T_c and T_d differ not only in their regression constants but also in their φ-functions. Consequently, if we show that (T_c, T_d) is under P_d asymptotically jointly normal with $\mu_1 = \mu_2 = 0$, variances $\sigma_1^2 = \int_0^1 [\varphi(u) - \bar{\varphi}]^2 \, du$ and $\sigma_2^2 = b^2$, and covariance $\sigma_{12} = b_{12} \int_0^1 \varphi(u)\, \varphi(u, f_0)\, du$, we can conclude that $(S_c, \log L_d)$ is asymptotically jointly normal with the same parameters except for $\mu_2 = -\frac{1}{2}b^2$, and the theorem will follow immediately from LeCam's third lemma.

Now, since the U_i's are independent and uniformly distributed under P_d, we have $\mathsf{E}T_c = \mathsf{E}T_d = 0$, and in view of (2.1.5) and (5),

$$\text{var } T_c = \sum_{i=1}^{N} (c_i - \bar{c})^2 \int_0^1 [\varphi(u) - \bar{\varphi}]^2 \, du\,,$$

$$\text{var } T_d = \sum_{i=1}^{N} (d_i - \bar{d})^2 \int_0^1 \varphi^2(u, f_0)\, du \to b^2\,,$$

$$\text{cov}(T_c, T_d) = \Big[\sum_{i=1}^{N} (c_i - \bar{c})(d_i - \bar{d})\Big] \int_0^1 \varphi(u)\, \varphi(u, f_0)\, du \to$$

$$\to b_{12} \int_0^1 \varphi(u)\, \varphi(u, f_0)\, du\,.$$

Thus the limiting parameters have the required values. In view of D3 of Section V.2.1, it remains to show, for all real λ_1 and λ_2, that $\lambda_1 T_c + \lambda_2 T_d$ is either asymptotically normal $(0, \sigma_{cd}^2)$ with $\sigma_{cd}^2 = \text{var}(\lambda_1 T_c + \lambda_2 T_d)$, or $\text{var}(\lambda_1 T_c + \lambda_2 T_d) \to 0$. Write

$$\lambda_1 T_c + \lambda_2 T_d = \sum_{i=1}^{N} [\lambda_1(c_i - \bar{c})\, \varphi(U_i) + \lambda_2(d_i - \bar{d})\, \varphi(U_i, f_0)] ,$$

and assume that

(11) $$\text{var}(\lambda_1 T_c + \lambda_2 T_d) \to v^2 > 0 .$$

Now put

$$\begin{aligned} Z_{1i} &= \lambda_1(c_i - \bar{c})\, [\varphi(U_i) - \bar{\varphi}] , \\ Z_{2i} &= \lambda_2(d_i - \bar{d})\, \varphi(U_i, f_0) , \\ Z_i &= Z_{1i} + Z_{2i} , \\ Z_i(\delta) &= Z_i , \quad \text{if} \quad |Z_i| > \delta , \\ &= 0 , \quad \text{if} \quad |Z_i| \leqq \delta , \end{aligned}$$

and define $Z_{1i}(\delta)$ and $Z_{2i}(\delta)$ similarly. In view of (11) the Lindeberg condition for $\lambda_1 T_c + \lambda_2 T_d$ may be expressed as follows:

(12) $$\sum_{i=1}^{N} \mathsf{E}[Z_i(\delta)]^2 \to 0 , \quad \delta > 0 .$$

However, we obviously have

$$[Z_i(\delta)]^2 \leqq 4[Z_{1i}(\tfrac{1}{2}\delta)]^2 + 4[Z_{2i}(\tfrac{1}{2}\delta)]^2 ,$$

so that (12) follows from

(13) $$\sum_{i=1}^{N} \mathsf{E}[Z_{1i}(\delta)]^2 \to 0 , \quad \delta > 0 ,$$

and

(14) $$\sum_{i=1}^{N} \mathsf{E}[Z_{2i}(\delta)]^2 \to 0 , \quad \delta > 0 .$$

Finally, observe that (13) and (14) are equivalent to the Lindeberg condition for $\lambda_1 T_c$ and $\lambda_2 T_d$, respectively, in view of (4) and (2.1.5). Moreover, from (6) and (2.1.4) it follows that this condition is satisfied in both cases (see the proof of Theorem V.1.2). This concludes the proof for q_d given by (2.1.1). If q_d were given by (2.2.1), we would proceed quite similarly. Q.E.D.

The density q_Δ will be associated with the density

$$p = \prod_{i=1}^{N} f_0(x_i)\,, \tag{3}$$

which obviously belongs to H_1. Our aim is to establish the limiting distribution of

$$S_N^+ = \sum_{i=1}^{N} a_N(R_i^+) \operatorname{sign} X_i \tag{4}$$

under q_Δ.

Theorem. *Let $q_{N\Delta}$ be given by (1), where f_0 is symmetric about zero, $I(f_0) < \infty$, and (N, Δ) satisfy (2). Further assume that the functions $a_N(1 + [uN])$, $0 < u < 1$, converge in quadratic mean to a square integrable function $\varphi^+(u)$. Then the statistics (4) are under $q_{N\Delta}$ asymptotically normal (μ_N, σ_N^2) with*

$$\mu_N = \Delta N \int_0^1 \varphi^+(u)\, \varphi^+(u, f_0)\, \mathrm{d}u \tag{5}$$

and

$$\sigma_N^2 = \sum_{i=1}^{N} [a_N(i)]^2 \sim N \int_0^1 [\varphi^+(u)]^2\, \mathrm{d}u\,. \tag{6}$$

Remark. Recall that $\varphi^+(u, f_0) = \varphi(\frac{1}{2} + \frac{1}{2}u, f_0)$.

Proof. Following the pattern of the proof of Theorem 2.1, we can show that

$$\log L_{N\Delta} \sim \Delta \sum_{i=1}^{N} \varphi^+(U_i^+, f_0) \operatorname{sign} X_i - \tfrac{1}{2}b^2 \tag{7}$$

where $U_i^+ = F^+(|X_i|)$. The rest follows from $S_N^+ \sim T_N$, T_N given by (V.1.7.5).

Application 1. Put $a_N(i) = a_N^+(i, f)$, f symmetric about zero, $I(f) < \infty$. Then

$$S_N^+ = \sum_{i=1}^{N} a_N^+(i, f) \operatorname{sign} X_i \tag{8}$$

is under $q_{N\Delta}$ asymptotically normal (μ_N, σ_N^2) with

$$\mu_N = N\Delta \int_0^1 \varphi^+(u, f)\, \varphi^+(u, f_0)\, \mathrm{d}u\,, \tag{9}$$

$$\sigma_N^2 = N \int_0^1 [\varphi^+(u, f)]^2\, \mathrm{d}u\,. \tag{10}$$

Note that

$$\int_0^1 \varphi^+(u,f)\,\varphi^+(u,f_0)\,du = \int_0^1 \varphi(u,f)\,\varphi(u,f_0)\,du\,.$$

Application 2. Put $a_N(i) = \varphi(\frac{1}{2} + \frac{1}{2}i/(N+1), f)$, where f is symmetric about zero, $I(f) < \infty$, and $\varphi(u,f)$ is a finite sum of monotone and square integrable functions. Then the statistics

$$S_N^+ = \sum_{i=1}^{N} \varphi\left(\frac{1}{2} + \frac{R_i^+}{N+1}, f\right) \operatorname{sign} X_i \tag{11}$$

are asymptotically normal (μ_N, σ_N^2) with μ_N and σ_N^2 given by (9) and (10) respectively.

2.6. Rank statistics for H_2. Unfortunately, we do not have at our disposal satisfactory methods for treating the distribution of these statistics under the alternatives. Heuristic considerations suggest, however, that we should obtain, possibly under some additional assumptions, the following result: Consider

$$q_{N\Delta} = \prod_{i=1}^{N} h_\Delta(x_i, y_i)\,, \tag{1}$$

with

$$h_\Delta(x, y) = \int_{-\infty}^{\infty} f_0(x - \Delta z)\, g_0(y - \Delta z)\, dM(z)\,, \tag{2}$$

(see Section II.4.11). Assume that the variance corresponding to $M(z)$ is finite and positive, and that

$$N\Delta^4 I(f_0)\, I(g_0) \to b^2\,, \quad 0 < b^2 < \infty\,. \tag{3}$$

Further, introduce the statistic

$$S_N = \sum_{i=1}^{N} a_N(R_{Ni})\, b_N(Q_{Ni}) \tag{4}$$

and suppose that

$$\lim_{N\to\infty} \int_0^1 [a_N(1 + [uN]) - \varphi(u)]^2\, du = 0 \tag{5}$$

and

$$\lim_{N\to\infty} \int_0^1 [b_N(1 + [uN]) - \psi(u)]^2\, du = 0\,.$$