GEJZA WIMMER

BIOŠTATISTIKA

MODEL S NÁHODNÝMI EFEKTMI

1 ÚVOD

Semestrálna prednáška „Bioštatistika“ obsahuje základy teórie odhadu v modeli s náhodnými efektmi a v zmiešanom lineárnom modeli. Oba modely vznikajú aj pri riešení problémov v biológii, medicíne, genetike, poľnohospodárstve atď. Cieľom tejto prednášky je ukázať na príkladoch genézu modelu s náhodnými efektmi ako aj zmiešaného lineárneho modelu, odhady parametrov, testy o parametroch, súvis s modelom analýzy rozptylu a kovariančnou analýzou. Jej súčasťou sú aj dôležité vety týkajúce sa kvadratických štatistík.

2 MODEL ANALÝZY ROZPTYLU – MODEL S PEVNÝMI EFEKTMI

Pri mnohých pokusoch (vyšetreniach) sa meria (sleduje, vyšetruje, určuje) hodnota kvantitatívnej náhodnej premennej (znaku) na štatistických jednotkách, na ktoré pôsobia rôzne faktory $A, B, C, \ldots$, ktorých účinky nás zaujímajú. Každý faktor vystupuje na rôznych kvalitatívnych alebo kvantitatívnych úrovniach $A_1, A_2, \ldots, A_I, B_1, B_2, \ldots, B_J$ atď. Podmienky pri ktorých sa merania uskutočňujú, a ktorých vplyv na výsledok pokusu nás zaujíma, môžu byť vyjadrené v termínoch úrovní určitých faktorov. Faktormi môžu byť aj javy, ktoré z experimentu nemôžeme vylúčiť, ale môžeme od nich očakávať vplyv na výsledok experimentu, hoci tento vplyv na výsledok nie je predmetom nášho výskumu.

Predpokladáme, že každá úroveň faktora spôsobuje na výsledok pokusu pevný (aj keď nie známy) efekt. Pokusy robíme pri rôznych známych úrovniach jednotlivých faktorov.

Vznikajú otázky:

Vplývajú faktory na výsledok pokusu?
Ako ohodnotíme (odhadneme) vplyv jednotlivých faktorov na výsledok pokusu?

Zaujímajú nás všetky úrovne faktorov, ktoré s meraním nejakým spôsobom súvisia. Jednotlivé merania sú nezávislé. Ďalej predpokladáme, že výsledky merania sú normálne rozdelené, pričom vplyv určitej úrovne niektorého faktora je vyjadrený pevným efektom – (pevným) posunom strednej hodnoty merania.

Príklad 1 ([7], str. 316)
Skúmajme vplyv rôznych druhov penicilínu na bacil $\mathcal{B}$ (substilis). Vyšetroval sa vplyv štyroch druhov penicilínu na rast bacilu $\mathcal{B}$. Faktor A (druh penicilínu) má 4

úrovne. Na každej úrovni sa päťkrát nezávisle merala veľkosť bacila pod vplyvom úrovne faktora A. Výsledky merania sú v tabuľke 1.

Tabuľka 1

číslo pokusu j	druh penicilínu – i			
	1	2	3	4
1	10,6	7,3	8,2	7,5
2	8,5	9,1	7,7	6,6
3	9,8	8,4	8,0	5,1
4	8,3	8,8	7,2	7,1
5	8,1	7,6	6,4	6,7

Pokus z príkladu 1 modelujeme nasledujúcim spôsobom. Výsledok j-teho pokusu ($j = 1, 2, \ldots, 5$) pri pôsobení i-tej úrovne faktora A (i-teho druhu penicilínu) ($i = 1, 2, 3, 4$) je y_{ij}. Je to realizácia náhodnej premennej Y_{ij}, pričom

$$(1) \qquad Y_{ij} = \mu_{ij} + \varepsilon_{ij}\,, \qquad \begin{array}{l} i = 1, 2, 3, 4\,, \\ j = 1, 2, 3, 4, 5\,, \end{array}$$

kde $\varepsilon_{ij} \sim N(0, \sigma^2)$; μ_{ij} a σ^2 sú neznáme konštanty.

Pretože predpokladáme, že každý druh penicilínu (úroveň faktora A) má na výsledok rovnaký (pevný) efekt, je pre $i = 1, 2, 3, 4$

$$\mu_{ij} = \mu_i, \qquad j = 1, 2, 3, 4, 5,$$

teda

$$Y_{ij} = \mu_i + \varepsilon_{ij}, \qquad \begin{array}{l} i = 1, 2, 3, 4\,, \\ j = 1, 2, 3, 4, 5\,. \end{array}$$

Ak označíme

$$\frac{1}{4}\sum_{i=1}^{4} \mu_i = \mu; \quad \mu_i - \mu = \alpha_i, \quad i = 1, 2, 3, 4,$$

môžeme (1) písať ako

$$(2) \qquad Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \qquad \begin{array}{l} i = 1, 2, 3, 4\,, \\ j = 1, 2, 3, 4, 5\,, \end{array}$$

kde $\varepsilon_{ij} \sim N(0, \sigma^2)$ sú nezávislé a modelujú náhodnú chybu i, j-teho výsledku (merania), α_i je (pevný) efekt i-tej úrovne A_i faktora A, μ je celková stredná hodnota.

Model (2) nazývame jednofaktorový vyvážený model analýzy rozptylu (model s pevnými efektmi). Vyváženosť znamená, že pre každú úroveň faktora A bol vykonaný rovnaký počet meraní. Analýzou rozptylu testujeme hypotézu

$$H_0\colon \qquad \mu_1 = \mu_2 = \mu_3 = \mu_4$$

týkajúcu sa zhody jednotlivých úrovní faktora A, alebo (čo je to isté)

$$H_0\colon \qquad \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$$

týkajúcu sa zhody efektov, resp. významnosti efektov (t.j. či je niektorý z efektov štatisticky významne rôzny od nuly).

V prípade zamietnutia nulovej hypotézy, metódami analýzy rozptylu odpovieme aj na otázku experimentátora: „Ktoré skupiny sa od seba líšia?“ (V príklade 1 „Ktoré druhy penicilínu sa od seba líšia vzhľadom na rast bacilu $\mathcal{B}$?“). Všeobecné riešenie pozri napr. v [2] alebo [14].

Príklad 2 ([1], str. 49)

Analyzujeme problém, či vplýva u detí vek a nezávisle od toho telesná výška na vitálnu kapacitu pľúc. Zároveň sa vyskytuje aj otázka, či vplyv výšky v spojitosti s vekom je vždy rovnaký, alebo sa mení. Označme y_{ijk} vitálnu kapacitu pľúc nameranú na k-tom náhodnom dieťati s telesnou výškou na i-tej úrovni (A_i) a vekom na j-tej úrovni (B_j). V tabuľke 2 sú v okienku patriacom A_i, B_j $(i, j = 1, 2, 3, 4)$ vždy dve čísla. Dolné je počet detí n_{ij} meraných pri daných úrovniach faktorov A_i a B_j a horné je vitálna kapacita $\sum_{k=1}^{n_{ij}} y_{ijk}$ v dm^3. Ďalej vieme, že $\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{n_{ij}} y_{ijk}^2 = 2561,4759$.

Tabuľka 2

telesná výška v cm	vek v rokoch			
	12–13	13–14	14–15	15–16
140–149	23,2 11	6,8 3	12,6 5	2,7 1
150–159	55,98 23	102,2 42	112,75 46	61,15 23
160–169	28,3 10	100,3 36	158,35 56	160,45 53
170–179	3,1 1	8,8 3	24,3 8	50,5 15

Nameraný výsledok y_{ijk} považujeme za realizáciu náhodnej premennej Y_{ijk}, pričom

$$(3) \qquad Y_{ijk} = \mu_{ij} + \varepsilon_{ijk}, \qquad \begin{aligned} i &= 1,2,3,4\,,\\ j &= 1,2,3,4\,,\\ k &= 1,2,\ldots,n_{ij}\,, \end{aligned}$$

$\varepsilon_{ijk} \sim N(0, \sigma^2)$ sú nezávislé a μ_{ij} sú konštanty. Model (3) píšeme vo forme

$$(4) \qquad Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}, \qquad \begin{aligned} i &= 1,2,3,4\,,\\ j &= 1,2,3,4\,,\\ k &= 1,2,\ldots,n_{ij}\,, \end{aligned}$$

ktorý je preparametrizovaný, α_i je efekt i-tej úrovne A_i faktora A, β_j je efekt j-tej úrovne B_j faktora B a γ_{ij} je tzv. interakcia t.j. v tomto prípade efekt spojitosti i-tej úrovne výšky s j-tou úrovňou veku. Model (4) je dvojfaktorový nevyvážený model analýzy rozptylu s interakciami.

Metódami analýzy rozptylu testujeme v modeli (4) hypotézy o nulovosti interakcií, nulovosti α_i pre všetky i (vplyv faktora A na výsledok), nulovosti β_j pre všetky j (vplyv faktora B). Taktiež sa dá odpovedať na otázku, ktoré skupiny sa od seba líšia vzhľadom na faktor A, resp. vzhľadom na faktor B.

3 MODEL S NÁHODNÝMI EFEKTMI

V medicíne, ale aj v iných vedeckých, ekonomických a technických disciplínach vzniká aj iný typ problému. Mnohé (napr. fyziologické) veličiny sú premenlivé (nie sú konštantné). Túto premenlivosť treba vziať do úvahy, keď porovnávame hodnoty takejto veličiny namerané na rôznych indivíduách (inter-individuálne kolísanie), alebo medzi skupinami (intra-individuálne kolísanie). Aby sme mohli vyjadriť toto kolísanie, opakujeme meranie na tom istom indivíduu, preto sú tieto merania závislé. Nejde nám o odhad pevných efektov, ale o zistenie ich variability.

Príklad 3 ([1], str. 80)

U desiatich potkaních samcov sa spočítal počet kapilár na 1000 svalových vláknach v troch rôznych rezoch na svale gastrocnemius. Zaujíma nás variabilita jednotlivých meraní u toho istého zvieraťa a variabilita hodnôt medzi zvieratami. Výsledky meraní sú v tabuľke 3.

TABUĽKA 3

číslo rezu j	číslo pokusného zvieraťa – i									
	1	2	3	4	5	6	7	8	9	10
1	1049	1152	1166	1534	1368	1300	1589	1223	1254	
2	955	1208	1068	1252	1295	1443	1314		1377	1277
3	1242		1127	1532	1507	1424	1328	1231	1298	1588

Ak označíme y_{ij} výsledok merania počtu kapilár na 1000 svalových vláknach u i-teho potkana ($i = 1, 2, \ldots, 10$) v j-tom reze ($j = 1, 2, 3$), tak túto hodnotu považujeme za realizáciu náhodnej premennej

(5) $$Y_{ij} = \mu + A_i + \varepsilon_{ij}, \qquad \begin{aligned} i &= 1, 2, \ldots, 10\,, \\ j &= 1, 2, 3\,, \end{aligned}$$

kde μ je (konštantná) stredná hodnota (pre všetky merania), A_i je náhodná premenná vyjadrujúca premenlivosť hodnôt medzi zvieratami a ε_{ij} vyjadruje variabilitu jednotlivých hodnôt u toho istého zvieraťa. Predpokladáme, že stredná hodnota

(6.a) $$\mathcal{E}(A_i) = 0, \qquad i = 1, 2, \ldots, 10,$$

ďalej

$$\mathcal{E}(\varepsilon_{ij}) = 0, \qquad i = 1, 2, \ldots, 10, \ j = 1, 2, 3, \tag{6.b}$$

disperzia

$$\mathcal{D}(A_i) = \sigma_A^2, \qquad i = 1, 2, \ldots, 10, \tag{6.c}$$

$$\mathcal{D}(\varepsilon_{ij}) = \sigma_0^2 \qquad i = 1, 2, \ldots, 10, \ j = 1, 2, 3, \tag{6.d}$$

kovariancia

$$\operatorname{cov}(A_k, A_l) = 0 \quad \text{pre všetky } k \neq l, \tag{6.e}$$

$$\operatorname{cov}(A_i, \varepsilon_{jk}) = 0 \quad \text{pre všetky } i, j, k, \tag{6.f}$$

$$\operatorname{cov}(\varepsilon_{ij}, \varepsilon_{kl}) = 0 \quad \text{pre všetky } i \neq k, \tag{6.g}$$

$$\operatorname{cov}(\varepsilon_{ij}, \varepsilon_{il}) = 0 \quad \text{pre všetky } j \neq l. \tag{6.h}$$

Platí

$$\mathcal{D}(Y_{ij}) = \sigma_A^2 + \sigma_0^2. \tag{7}$$

Úlohou je odhadnúť σ_A^2 a σ_0^2, ďalej testovať, či $\sigma_A^2 = 0$.

Model (5) s podmienkami (6.a) až (6.h) sa nazýva model s jedným náhodným efektom. σ_A^2 a σ_0^2 sú variančné komponenty jednotlivého merania (vidieť to z (7)).

Príklad 4 ([1], str. 83)

Na detskej klinike v Halle skúmali okamžitú adaptáciu detí nyktometrom (zisťuje sa počet identifikovaných plôšok pod určitou krivkou). Vykonalo sa niekoľko meraní v 4 rôznych obdobiach roka u 5 náhodne vybratých detí navštevujúcich druhú triedu. Výsledky sú v tabuľke 4. Zaujímajú nás variančné komponenty.

Ak označme y_{ijk} hodnotu registrovanú nyktometrom u i-teho žiaka ($i = 1, 2, 3, 4, 5$) v j-tom termíne pri k-tom vyšetrení ($k = 1, 2, \ldots, n_{ij}$), tak v tabuľke 4 v (i, j)-tom okienku sú dve čísla. Horné je počet bodov, ktoré spolu získal i-ty žiak v j-tom termíne a dolné je n_{ij} (počet vyšetrení i-teho žiaka v j-tom termíne). Ďalej $\sum_{i=1}^{5} \sum_{j=1}^{4} \sum_{k=1}^{n_{ij}} y_{ijk}^2 = 143033,5$.

Tabuľka 4

číslo žiaka i	číslo vyšetrenia – j			
	1	2	3	4
1	177 4	131 4	255,5 5	246 5
2	119,5 4	109 4	173,5 5	167,5 5
3	167 4	127 4	215 5	166 5
4	142,5 4	153 4	256 6	171,5 4
5	155,5 4	131 4	269,5 6	178 4

Hodnotu y_{ijk} považujeme za realizáciu náhodnej premennej Y_{ijk}, pričom predpokladáme, že

(8) $$Y_{ijk} = \mu + A_i + B_j + C_{ij} + \varepsilon_{ijk}, \qquad \begin{array}{l} i = 1,2,3,4,5\,, \\ j = 1,2,3,4\,, \\ k = 1,2,\dots,n_{ij}\,, \end{array}$$

μ je konštanta a

(9.a) $$\mathcal{E}(A_i) = \mathcal{E}(B_j) = \mathcal{E}(C_{ij}) = \mathcal{E}(\varepsilon_{ijk}) = 0$$

pre všetky i, j, k,

(9.b) $$\mathcal{D}(A_i) = \sigma_A^2, \qquad i = 1,2,3,4,5,$$

(9.c) $$\mathcal{D}(B_j) = \sigma_B^2, \qquad j = 1,2,3,4,$$

(9.d) $$\mathcal{D}(C_{ij}) = \sigma_{AB}^2, \quad \text{pre všetky } i, j,$$

(9.e) $$\mathcal{D}(\varepsilon_{ijk}) = \sigma_0^2, \quad \text{pre všetky } i, j, k.$$

Náhodné premenné $A_i, B_j, C_{ij}, \varepsilon_{ijk}$ sú medzi sebou nekorelované (pre všetky i, j, k).

A_i predstavujú (náhodný) efekt žiaka,

B_j predstavujú (náhodný) efekt termínu a C_{ij} predstavujú (náhodný) efekt interakcie žiaka s termínom.

Platí

$$\mathcal{D}(Y_{ijk}) = \sigma_A^2 + \sigma_B^2 + \sigma_{AB}^2 + \sigma_0^2,$$

kde σ_A^2 je miera variability medzi žiakmi, σ_B^2 je miera variability medzi termínmi, σ_{AB}^2 je miera variability interakcie (vedľajšieho účinku žiaka s termínom), σ_0^2 je miera variability nameraných hodnôt jednotlivých vyšetrení. Model (8) s podmienkami (9.a) až (9.e) nazývame modelom s dvoma náhodnými efektmi a interakciou.

Poďme si ešte raz analyzovať rozdiel medzi modelmi s pevnými (analýza rozptylu) a náhodnými efektmi.

Príklad 5 ([7], str. 307)

Presnosť chromatografického určenia dietylénglykolu. V laboratóriu pracujú traja laboranti na troch chromatografoch a rutinne určujú obsah (v %) dietylénglykolu v etylénglykole. Chceme zistiť, či na výsledok analýzy má vplyv prístroj, či laborant a či vplyv súčinnosti laboranta a prístroja je rovnaká. Na každom prístroji B_1, B_2, B_3 vykonal každý laborant A_1, A_2, A_3 dve opakované merania (meral sa ten istý roztok). Namerané hodnoty percentuálneho obsahu dietylénglykolu sú v tabuľke 5.

Tabuľka 5

laborant	prístroj		
	B_1	B_2	B_3
A_1	0,110 0,116	0,101 0,102	0,108 0,109
A_2	0,112 0,111	0,115 0,106	0,111 0,109
A_3	0,114 0,112	0,107 0,109	0,113 0,110

Ak označíme y_{ijk} výsledok i-teho laboranta na j-tom prístroji pri k-tom určení $(i, j = 1, 2, 3,\ k = 1, 2)$, tak y_{ijk} je realizácia náhodnej premennej Y_{ijk}, pričom

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}, \qquad \begin{array}{r} i, j = 1, 2, 3\,, \\ k = 1, 2\,, \end{array}$$

α_i je (pevný) efekt i-teho laboranta, β_j je (pevný) efekt j-teho prístroja a γ_{ij} je (pevný) efekt súčinnosti i-teho laboranta s j-tym prístrojom, $\varepsilon_{ijk} \sim N(0, \sigma_0^2)$ považujeme za nezávislé (v tomto kroku sa vedome dopúšťame „deformácie“ modelu oproti skutočnosti). Prezentovaný model je modelom dvojfaktorovej analýzy rozptylu s interakciami (vyvážený model s pevnými efektmi).

Príklad 6 ([7], str. 311)

Vplyvy pôsobiace na výsledok chromatografického určenia dietylénglykolu .

V laboratóriu pracuje veľa laborantov a majú veľa prístrojov. Náhodne sa vybrali traja laboranti A_1, A_2, A_3 a tri prístroje B_1, B_2, B_3. Výsledky uvažujeme rovnaké ako v tabuľke 5. Máme určiť variabilitu laborantov, prístrojov a interakcií na výslednú variabilitu výsledkov.

Opäť označíme y_{ijk} výsledok i-teho náhodne vybratého laboranta na j-tom náhodne vybratom prístroji pri k-tom určení ($i, j = 1, 2, 3,\ k = 1, 2$). Potom y_{ijk} považujeme za realizáciu náhodnej premennej Y_{ijk}, pričom

$$Y_{ijk} = \mu + A_i + B_j + C_{ij} + \varepsilon_{ijk}, \qquad \begin{array}{l} i, j = 1, 2, 3\,, \\ k = 1, 2\,, \end{array}$$

μ je konštanta, $A_i, B_j, C_{ij}, \varepsilon_{ijl}$ sú nekorelované náhodné premenné s nulovými strednými hodnotami,

$$\mathcal{D}(A_i) = \sigma_A^2,\ \mathcal{D}(B_j) = \sigma_B^2,\ \mathcal{D}(C_{ij}) = \sigma_{AB}^2,\ \mathcal{D}(\varepsilon_{ijk}) = \sigma_0^2,$$

pre všetky i, j, k. Úlohu modelujeme dvojfaktorovým modelom s náhodnými efektmi a interakciou.

Ešte na jednej dvojici príkladov si ukážme rozdiel medzi modelom analýzy rozptylu (model s pevnými efektmi) a modelom s náhodnými efektmi.

Príklad 7 ([7], str. 313)

Máme 5 laboratórií a 2 metódy stanovenia arzénika v potrave (metóda A–reakcia s molybdénovým roztokom, metóda B–reakcia s dietylditiokarbamátom strieborným podľa Vašáka a Šedivca). K vzorke potravy pridáme 15 g arzénika a vzorku nezávisle analyzujeme oboma metódami vo všetkých piatich laboratóriach. Vedú obe metódy vo všetkých laboratóriach k rovnakým výsledkom? Údaje sú v tabuľke 6.

Tabuľka 6

laboratórium	metóda	obsah arzénika v g					
1	A	12,9	13,2	12,9	12,9	13,1	13,0
	B	13,4	13,0	13,0	17,0	16,6	
2	A	14,6	16,2	14,0	15,0	15,5	13,7
	B	14,8	15,2	14,6	15,0		
3	A	13,4	13,0	13,2	13,2	13,1	13,2
	B	14,8	14,8	15,0	14,9	14,8	
4	A	13,3	13,8	12,5	13,5	13,6	12,8
	B	14,8	14,8	15,0	14,5	15,4	15,2
5	A	15,9	14,8	15,3	15,6	14,9	15,2
	B	13,8	14,1	13,8	13,9	14,0	

Úlohu riešime modelom dvojfaktorovej analýzy rozptylu s interakciami

$$Y_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \varepsilon_{ijk}, \qquad \begin{aligned} i &= 1,2,3,4,5\,, \\ j &= 1,2\,, \\ k &= 1,2,\dots,n_{ij}\,. \end{aligned}$$

Hodnota y_{ijk} je nameraná hodnota v i-tom laboratóriu pri j-tej metóde k-ty krát. Je to realizácia náhodnej premennej Y_{ijk}; α_i je (pevný) efekt i-teho laboratória, β_j je (pevný) efekt j-tej metódy, γ_{ij} je (pevná) interakcia j-tej metódy v i-tom laboratóriu; $\varepsilon_{ijk} \sim N(0, \sigma_0^2)$ sú nezávislé náhodné premenné (pre všetky i, j, k).

Príklad 8 ([7], str. 313, modifikovaný)

Máme veľa laboratórií a veľa metód na určenie arzénika v potrave. Náhodne vyberieme 5 laboratórií a 2 metódy. Meranie opakujeme niekoľkokrát. Výsledky sú v tabuľke 6. Zaujíma nás variabilita jednotlivých meraní (v tom istom laboratóriu), variabilita medzi laboratóriami, variabilita medzi metódami, variabilita náhodného efektu interakcie laboratória a metódy.

Úlohu riešime dvojfaktorovým modelom s náhodnými efektmi

$$Y_{ijk} = \mu + A_i + B_j + C_{ij} + \varepsilon_{ijk}, \qquad \begin{aligned} i &= 1,2,3,4,5\,, \\ j &= 1,2\,, \\ k &= 1,2,\dots,n_{ij}\,. \end{aligned}$$

A_i predstavuje (náhodný) efekt laboratória, B_j predstavuje (náhodný) efekt metódy a C_{ij} predstavuje (náhodný) efekt interakcie laboratória a metódy. Platia vzťahy ako v modeli (8) s podmienkami (9.a) až (9.e).

4 MODEL SO ZMIEŠANÝMI EFEKTMI

Príklad 9 ([7], str. 311, modifikovaný)

Uvažujme rovnaké zadanie ako v príklade 5 s tým rozdielom, že z laborantov náhodne vyberieme troch. Faktor A (laboranti) je faktor s náhodnými efektmi a faktor B (prístroja) je faktor s pevnými efektmi.

Príklad riešime modelom

$$Y_{ijk} = \mu + \beta_j + A_i + C_{ij} + \varepsilon_{ijk}, \qquad \begin{aligned} i &= 1,2,3\,, \\ j,k &= 1,2\,, \end{aligned} \tag{10}$$

kde μ je (neznáma) konštanta, β_j je (pevný, neznámy) efekt j-teho prístroja, A_i je (náhodný) efekt i-teho laboranta, C_{ij} je (náhodný) efekt interakcie (súčinnosti) i-teho laboranta s j-tym prístrojom, $\varepsilon_{ijk} \sim N(0, \sigma_0^2)$ sú nezávislé náhodné premenné. Ďalej predpokladáme, že $A_i \sim N(0, \sigma_A^2)$ sú medzi sebou navzájom nezávislé aj nezávislé so všetkými ε_{ijk}. $C_{ij} \sim N(0, \sigma_{AB}^2)$ sú medzi sebou nezávislé aj nezávislé s A_i, ε_{ijk} pre všetky i, j, k. Maticovo môžeme model (10) zapísať ako

$$\boldsymbol{Y}_{18,1} = \begin{pmatrix} Y_{111} \\ Y_{112} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{221} \\ Y_{222} \\ Y_{231} \\ Y_{232} \\ Y_{311} \\ Y_{312} \\ Y_{321} \\ Y_{322} \\ Y_{331} \\ Y_{332} \end{pmatrix} = \begin{pmatrix} 1&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0 \\ 1&1&0&0&1&0&0&1&0&0&0&0&0&0&0&0 \\ 1&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0 \\ 1&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0 \\ 1&0&0&1&1&0&0&0&0&1&0&0&0&0&0&0 \\ 1&0&0&1&1&0&0&0&0&1&0&0&0&0&0&0 \\ 1&1&0&0&0&1&0&0&0&0&1&0&0&0&0&0 \\ 1&1&0&0&0&1&0&0&0&0&1&0&0&0&0&0 \\ 1&0&1&0&0&1&0&0&0&0&0&1&0&0&0&0 \\ 1&0&1&0&0&1&0&0&0&0&0&1&0&0&0&0 \\ 1&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0 \\ 1&0&0&1&0&1&0&0&0&0&0&0&1&0&0&0 \\ 1&1&0&0&0&0&1&0&0&0&0&0&0&1&0&0 \\ 1&1&0&0&0&0&1&0&0&0&0&0&0&1&0&0 \\ 1&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0 \\ 1&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0 \\ 1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&1 \\ 1&0&0&1&0&0&1&0&0&0&0&0&0&0&0&1 \end{pmatrix} \begin{pmatrix} \mu \\ \beta_1 \\ \beta_2 \\ \beta_3 \\ A_1 \\ A_2 \\ A_3 \\ C_{11} \\ C_{12} \\ C_{13} \\ C_{21} \\ C_{22} \\ C_{23} \\ C_{31} \\ C_{32} \\ C_{33} \end{pmatrix} + \boldsymbol{\varepsilon}_{18,1},$$

čiže

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\gamma} + \boldsymbol{Z}\boldsymbol{\delta} + \boldsymbol{\varepsilon} = (\boldsymbol{X} \vdots \boldsymbol{Z}\ ) \begin{pmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\delta} \end{pmatrix} + \boldsymbol{\varepsilon}, \tag{11}$$

kde $\boldsymbol{\gamma} = (\mu, \beta_1, \beta_2, \beta_3)'$ je vektor pevných (neznámych) parametrov, $\boldsymbol{\delta} = (A_1, A_2, A_3, C_{11}, C_{12}, \ldots, C_{33})'$ je vektor náhodných parametrov (efektov). Matica plánu $(\boldsymbol{X} \vdots \boldsymbol{Z}\ )$ je známa. Niekedy je užitočné rozdeliť maticu plánu na submatice podľa jednotlivých „skupín“ efektov. V našom prípade dostávame model

$$\boldsymbol{Y} = (\boldsymbol{X}_1 \vdots \boldsymbol{X}_2)\boldsymbol{\gamma} + (\boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2)\boldsymbol{\delta} + \boldsymbol{\varepsilon}, \tag{12}$$

kde matica $\boldsymbol{X}_1 = \mathbf{1} = (1, 1, \ldots 1)'$ je rozmeru 18×1 a prislúcha konštantnému (pevnému) parametru μ, matica

$$\boldsymbol{X}_2 = \begin{pmatrix} 1&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&1&0 \\ 0&0&1 \\ 0&0&1 \\ 1&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&1&0 \\ 0&0&1 \\ 0&0&1 \\ 1&0&0 \\ 1&0&0 \\ 0&1&0 \\ 0&1&0 \\ 0&0&1 \\ 0&0&1 \end{pmatrix}$$

prislúcha pevnému vektoru $(\beta_1, \beta_2, \beta_3)'$ (efekt prístroja), matica

$$
\boldsymbol{Z}_1 = \begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
$$

prislúcha vektoru náhodných efektov (A_1, A_2, A_3) (efekt laboranta) a matica

$$
\boldsymbol{Z}_2 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

prislúcha vektoru náhodných efektov $(C_{11}, C_{12}, \ldots, C_{33})'$ (efekt interakcie).

Vektor $\boldsymbol{Y}$ má strednú hodnotu

$$
\mathcal{E}(\boldsymbol{Y}) = \boldsymbol{X}\boldsymbol{\gamma} = (\boldsymbol{X}_1 \vdots \boldsymbol{X}_2) \begin{pmatrix} \mu \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} \tag{13}
$$

a kovariančnú maticu

$$
\operatorname{cov}(\boldsymbol{Y}) = \sigma_A^2 \boldsymbol{Z}_1 \boldsymbol{Z}_1' + \sigma_{AB}^2 \boldsymbol{Z}_2 \boldsymbol{Z}_2' + \sigma_0^2 \boldsymbol{I}. \tag{14}
$$

Model (10) (resp. (11) alebo (12)), (pričom observovaný vektor má strednú hodnotu (13) a kovariančnú maticu (14)) patrí medzi modely so zmiešanými efektmi (alebo medzi zmiešané lineárne modely).

5 ODHADY V ZMIEŠANOM LINEÁRNOM MODELI PODĽA HENDERSONA

V mnohých biologických, ale aj priemyselných, ekonomických a iných situáciach sa dostávame k modelu

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{Z}_1\boldsymbol{b}_1 + \boldsymbol{Z}_2\boldsymbol{b}_2 + \ldots + \boldsymbol{Z}_s\boldsymbol{b}_s + \boldsymbol{\varepsilon} = (\boldsymbol{X}\vdots\boldsymbol{Z}\ ) \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{b} \end{pmatrix} + \boldsymbol{\varepsilon}, \tag{15}$$

kde náhodný observačný vektor $\boldsymbol{Y} \in \mathbb{R}^n$, známa matica plánu $(\boldsymbol{X}\vdots\boldsymbol{Z}\ ) = (\boldsymbol{X}\vdots\boldsymbol{Z}_1\vdots\boldsymbol{Z}_2\vdots\ldots\vdots\boldsymbol{Z}_s)$ je rozmerov $n\times k$. Matica $\boldsymbol{X}$ rozmerov $n\times q$ „prislúcha“ pevným efektom (neznámym parametrom) $\boldsymbol{\beta} \in \mathbb{R}^q$. Predpokladáme, že model (15) má s náhodných efektov, teda $\boldsymbol{b} = (\boldsymbol{b}_1', \boldsymbol{b}_2', \ldots, \boldsymbol{b}_s')'$ je vektor náhodných efektov (náhodných premenných), pričom $\boldsymbol{b}_i \in \mathbb{R}^{m_i}$, $i = 1, 2, \ldots, s$,

$$\mathcal{E}(\boldsymbol{b}_i) = \boldsymbol{0}, \qquad i = 1, 2, \ldots, s,$$

$$\mathrm{cov}(\boldsymbol{b}) = \mathrm{cov}(\begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_s \end{pmatrix}) = \begin{pmatrix} \sigma_1^2\boldsymbol{I}_{m_1,m_1} & 0 & \ldots & 0 \\ 0 & \sigma_2^2\boldsymbol{I}_{m_2,m_2} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & \sigma_s^2\boldsymbol{I}_{m_s,m_s} \end{pmatrix}.$$

Matice $\boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots, \boldsymbol{Z}_s$ (známe) „prislúchajúce“ týmto náhodným efektom sú rozmerov $n \times m_i$, $i = 1, 2, \ldots, s$. Chybový vektor $\boldsymbol{\varepsilon} \in \mathbb{R}^n$ má $\mathcal{E}(\boldsymbol{\varepsilon}) = \boldsymbol{0}$ a $\mathrm{var}(\boldsymbol{\varepsilon}) = \sigma_0^2\boldsymbol{I}$. Predpokladáme, že nie je skorelovaný s náhodným vektorom $\boldsymbol{b}$. Zrejme

$$\mathcal{E}(\boldsymbol{Y}) = \boldsymbol{X\beta}$$

a

$$\mathrm{cov}(\boldsymbol{Y}) =$$

$$= (\boldsymbol{Z}_1\vdots\ldots\vdots\boldsymbol{Z}_s\vdots\boldsymbol{I}) \begin{pmatrix} \sigma_1^2\boldsymbol{I}_{m_1,m_1} & 0 & \ldots & 0 & 0 \\ 0 & \sigma_2^2\boldsymbol{I}_{m_2,m_2} & \ldots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & \sigma_s^2\boldsymbol{I}_{m_s,m_s} & 0 \\ 0 & 0 & \ldots & 0 & \sigma_0^2\boldsymbol{I}_{n,n} \end{pmatrix} \begin{pmatrix} \boldsymbol{Z}_1' \\ \vdots \\ \boldsymbol{Z}_s' \\ \boldsymbol{I} \end{pmatrix} =$$

$$= \sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1' + \sigma_2^2\boldsymbol{Z}_2\boldsymbol{Z}_2' + \ldots + \sigma_s^2\boldsymbol{Z}_s\boldsymbol{Z}_s' + \sigma_0^2\boldsymbol{I} = \sum_{i=1}^{s} \sigma_i^2\boldsymbol{V}_i + \sigma_0^2\boldsymbol{I}.$$

V nasledujúcom budeme označovať

$$\boldsymbol{b}_0 = \boldsymbol{\varepsilon}, \quad m_0 = n, \quad \boldsymbol{Z}_0 = \boldsymbol{I}_{n,n}.$$

Potom

$$\mathrm{cov}(\boldsymbol{Y}) = \sum_{i=0}^{s} \sigma_i^2\boldsymbol{Z}_i\boldsymbol{Z}_i'.$$

Model (15) s horeuvedenými podmienkami sa nazýva *model s variančnými komponentmi* alebo *zmiešaný lineárny model*. Je špeciálnym prípadom *všeobecného lineárneho modelu s kovariančnými komponentmi* v ktorom sa predpokladá $\mathrm{cov}(\boldsymbol{Y}) = \sum_{i=1}^{s}\theta_i\boldsymbol{V}_i$, $\theta_1, \ldots, \theta_s \in \underline{\theta}$ (otvorená v $\mathbb{R}^s$), $\boldsymbol{V}_i = \boldsymbol{V}_i'$, $i = 1, 2, \ldots, s$ a matica $\sum_{i-1}^{s}\theta_i\boldsymbol{V}_i$ je pozitívne semidefinitná.

Lema 1. *V modeli (15) je $\boldsymbol{p}'\boldsymbol{\beta}$ ($\boldsymbol{p}$-daný vektor) lineárne nevychýlene odhadnuteľná funkcia (pevných) parametrov práve vtedy ak $\boldsymbol{p} \in \mu(\boldsymbol{X}') = \{\boldsymbol{X}'\boldsymbol{u}\colon \boldsymbol{u} \in \mathbb{R}^n\}$.*

Dôkaz.

$\boldsymbol{p}'\boldsymbol{\beta}$ je lineárne nevychýlene odhadnuteľná $\Longleftrightarrow$
$\exists\, \boldsymbol{l} \in \mathbb{R}^n : \mathcal{E}(\boldsymbol{l}'\boldsymbol{Y}) = \boldsymbol{p}'\boldsymbol{\beta} \ \ \forall \boldsymbol{\beta} \in \mathbb{R}^q \Longleftrightarrow$
$\exists\, \boldsymbol{l} \in \mathbb{R}^n : \boldsymbol{l}'\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{p}'\boldsymbol{\beta} \ \ \forall \boldsymbol{\beta} \in \mathbb{R}^q \Longleftrightarrow$
$\boldsymbol{l}'\boldsymbol{X} = \boldsymbol{p}' \Longleftrightarrow \boldsymbol{p} \in \mu(\boldsymbol{X}') = \{\boldsymbol{X}'\boldsymbol{u}\colon \boldsymbol{u} \in \mathbb{R}^n\}$. $\square$

Henderson v [4] navrhol v modeli (15) na základe realizácie $\boldsymbol{y}_{n,1}$ náhodného (observačného) vektora $\boldsymbol{Y}$
1. nevychýlený lineárny odhad lineárne odhadnuteľnej funkcie pevných parametrov (efektov) $\boldsymbol{p}'\boldsymbol{\beta}$,
2. lineárny prediktor vektora náhodných parametrov $(\boldsymbol{b}_1', \boldsymbol{b}_2', \dots, \boldsymbol{b}_s')'$,
3. nevychýlené kvadratické odhady variančných komponentov $\sigma_1^2, \sigma_2^2, \dots, \sigma_s^2$ a σ_0^2.

Definícia 2. *Majme maticu $\boldsymbol{A}$ rozmerov $m \times n$. Matica $\boldsymbol{A}^-$ rozmerov $n \times m$ pre ktorú platí*

$$\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{A} \tag{16}$$

je $g-inverzia$ (pseudoinverzia) matice $\boldsymbol{A}$.

Pseudoinverzia existuje ku každej matici, nie je vzťahom (16) určená jednoznačne, bližšie pozri [2], str. 67, [10] a inde.

Lema 3. $\boldsymbol{B}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{B} \Longleftrightarrow \mu(\boldsymbol{B}') \subset \mu(\boldsymbol{A}')$ *t.j.* $\exists\, \boldsymbol{D}$, *že* $\boldsymbol{B} = \boldsymbol{D}\boldsymbol{A}$.

Dôkaz. Uvedomme si najprv, že $\mu(\boldsymbol{B}') \subset \mu(\boldsymbol{A}') \Longleftrightarrow \exists\, \boldsymbol{D}'\colon \boldsymbol{B}' = \boldsymbol{A}'\boldsymbol{D}' \Longleftrightarrow \exists\, \boldsymbol{D}\colon \boldsymbol{B} = \boldsymbol{D}\boldsymbol{A}$.
Ak $\mu(\boldsymbol{B}') \subset \mu(\boldsymbol{A}')$, tak
$\boldsymbol{B}' = \boldsymbol{A}'\boldsymbol{D}' \Longrightarrow \boldsymbol{B} = \boldsymbol{D}\boldsymbol{A} \Longrightarrow \boldsymbol{B}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{D}\boldsymbol{A}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{D}\boldsymbol{A} = \boldsymbol{B}$.
Naopak, ak $\boldsymbol{B}\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{B}$, tak
$\exists\, \boldsymbol{D}(= \boldsymbol{B}\boldsymbol{A}^-)\colon \boldsymbol{D}\boldsymbol{A} = \boldsymbol{B} \Longrightarrow \mu(\boldsymbol{B}') \subset \mu(\boldsymbol{A}')$. $\square$

Poznámka 4. *Lema 3 platí aj v tvare $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{A}^-\boldsymbol{B} \Longleftrightarrow \boldsymbol{B} = \boldsymbol{A}\boldsymbol{D} \Longleftrightarrow \mu(\boldsymbol{B}) \subset \mu(\boldsymbol{A})$.*

Lema 5. $\mu(\boldsymbol{A}\ \boldsymbol{A}') = \mu(\boldsymbol{A})$.

Dôkaz. Vyplýva napr. z vety 1 v [2], str. 62.

Lema 6. *Ak* $\boldsymbol{Z} = (\boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2 \vdots \dots \vdots \boldsymbol{Z}_s)$ *a*

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix} (\boldsymbol{X} \vdots \boldsymbol{Z}) = \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{C}' & \boldsymbol{B} \end{pmatrix},$$

tak

$$\boldsymbol{G} = \begin{pmatrix} (\boldsymbol{X}'\boldsymbol{X})^- + (\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Z}\,\boldsymbol{D}^-\boldsymbol{Z}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^- & -(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Z}\,\boldsymbol{D}^- \\ -\boldsymbol{D}^-\boldsymbol{Z}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^- & \boldsymbol{D}^- \end{pmatrix},$$

je jedna g–inverzia matice $\boldsymbol{S}$, *pričom* $\boldsymbol{D}=\boldsymbol{Z}'\boldsymbol{Z}-\boldsymbol{Z}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{Z}$.

Dôkaz. Počítajme

$$\boldsymbol{SGS}=\begin{pmatrix}\boldsymbol{A} & \boldsymbol{C}\\ \boldsymbol{C}' & \boldsymbol{B}\end{pmatrix}\begin{pmatrix}\boldsymbol{A}^{-}+\boldsymbol{A}^{-}\boldsymbol{C}\boldsymbol{D}^{-}\boldsymbol{C}'\boldsymbol{A}^{-} & -\boldsymbol{A}^{-}\boldsymbol{C}\boldsymbol{D}^{-}\\ -\boldsymbol{D}^{-}\boldsymbol{C}'\boldsymbol{A}^{-} & \boldsymbol{D}^{-}\end{pmatrix}\begin{pmatrix}\boldsymbol{A} & \boldsymbol{C}\\ \boldsymbol{C}' & \boldsymbol{B}\end{pmatrix}=$$

$$=\begin{pmatrix}\mathbf{AA^{-}+AA^{-}CD^{-}C'A^{-}-CD^{-}C'A^{-}} & \vdots\, \mathbf{-AA^{-}CD^{-}+CD^{-}}\\ \mathbf{C'A^{-}+C'A^{-}CD^{-}C'A^{-}-BD^{-}C'A^{-}} & \vdots\, \mathbf{-C'A^{-}CD^{-}+BD^{-}}\end{pmatrix}\begin{pmatrix}\mathbf{A} & \mathbf{C}\\ \mathbf{C'} & \mathbf{B}\end{pmatrix}=$$

$$=\left(\begin{matrix}\mathbf{A+AA^{-}CD^{-}C'A^{-}A\ -CD^{-}C'A^{-}A\ -AA^{-}CD^{-}C'+CD^{-}C'} & \vdots\\ \mathbf{C'A^{-}A+C'A^{-}CD^{-}C'A^{-}A\ -BD^{-}C'A^{-}A\ -C'A^{-}CD^{-}C'+BD^{-}C'} & \vdots\end{matrix}\right.$$

$$\left.\begin{matrix}\vdots & \boldsymbol{AA^{-}C+AA^{-}CD^{-}C'A^{-}C-CD^{-}C'A^{-}C-AA^{-}CD^{-}B+CD^{-}B}\\ \vdots & \boldsymbol{C'A^{-}C+C'A^{-}CD^{-}C'A^{-}C-BD^{-}C'A^{-}C-C'A^{-}CD^{-}B+BD^{-}B}\end{matrix}\right).$$

Pre jednotlivé bloky matice $\boldsymbol{SGS}$ platí

$\{\boldsymbol{SGS}\}_{11}$:

$$\boldsymbol{A}+\boldsymbol{AA^{-}CD^{-}C'A^{-}A}\ -\boldsymbol{CD^{-}C'A^{-}A}\ -\boldsymbol{AA^{-}CD^{-}C'}+\boldsymbol{CD^{-}C'}=$$

$$=\boldsymbol{X'X}+\boldsymbol{X'X}(\boldsymbol{X'X})^{-}\boldsymbol{X'ZD^{-}C'A^{-}A}\ -\boldsymbol{X'ZD^{-}C'A^{-}A}-$$
$$-\boldsymbol{X'X}(\boldsymbol{X'X})^{-}\boldsymbol{X'Z}\ \boldsymbol{D^{-}C'}+\boldsymbol{X'ZD^{-}C'}=\boldsymbol{X'X}=\boldsymbol{A}$$

(lebo podľa lemy 3 je $\boldsymbol{X'X}(\boldsymbol{X'X})^{-}\boldsymbol{X'}=\boldsymbol{X'}$, čiže $\boldsymbol{AA^{-}C}=\boldsymbol{C}$).

$\{\boldsymbol{SGS}\}_{12}$:

$$\boldsymbol{AA^{-}C}+\boldsymbol{AA^{-}CD^{-}C'A^{-}C}-\boldsymbol{CD^{-}C'A^{-}C}-\boldsymbol{AA^{-}CD^{-}B}+\boldsymbol{CD^{-}B}=\boldsymbol{C}=\boldsymbol{X'Z}.$$

Podobne $\{\boldsymbol{SGS}\}_{21}=\boldsymbol{C'}=\boldsymbol{Z'X}$.

Konečne pre $\{\boldsymbol{SGS}\}_{22}$:

$$\boldsymbol{C'A^{-}C}+\boldsymbol{C'A^{-}CD^{-}C'A^{-}C}-\boldsymbol{BD^{-}C'A^{-}C}-\boldsymbol{C'A^{-}CD^{-}B}+\boldsymbol{BD^{-}B}=$$

$$=\boldsymbol{C'A^{-}C}+\{\boldsymbol{C'A^{-}C}-\boldsymbol{B}\}\{-\boldsymbol{C'A^{-}C}+\boldsymbol{B}\}^{-}\boldsymbol{C'A^{-}C}+$$
$$+\{\boldsymbol{B}-\boldsymbol{C'A^{-}C}\}\{\boldsymbol{B}-\boldsymbol{C'A^{-}C}\}^{-}\boldsymbol{B}=$$

$$=\boldsymbol{C'A^{-}C}+\{-\boldsymbol{C'A^{-}C}+\boldsymbol{B}\}\{-\boldsymbol{C'A^{-}C}+\boldsymbol{B}\}^{-}\{-\boldsymbol{C'A^{-}C}+\boldsymbol{B}\}=$$
$$=\boldsymbol{C'A^{-}C}-\boldsymbol{C'A^{-}C}+\boldsymbol{B}=\boldsymbol{Z'Z}.\quad\square$$

Poznámka 7.

(i) Pre ľubovoľnú maticu $\boldsymbol{A}$ *je* $[(\boldsymbol{A}'\boldsymbol{A})^-]'$ *tiež g-inverzia matice* $\boldsymbol{A}'\boldsymbol{A}$*, preto* $\frac{1}{2}\{(\boldsymbol{A}'\boldsymbol{A})^- + [(\boldsymbol{A}'\boldsymbol{A})^-]'\}$ *je symetrická g-inverzia matice* $\boldsymbol{A}'\boldsymbol{A}$ *(bez ohľadu na výber* $(\boldsymbol{A}'\boldsymbol{A})^-$*).*

(ii) Pre ľubovoľnú maticu $\boldsymbol{A}$ *matica* $\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^-\boldsymbol{A}'$ *nezávisí na voľbe* $(\boldsymbol{A}'\boldsymbol{A})^-$ *a* $\boldsymbol{A}(\boldsymbol{A}'\boldsymbol{A})^-\boldsymbol{A}'$ *je vždy symetrická.*

Dôkazy (i) *a* (ii) *nájdete v* [2]*, str. 69.*

(iii) Na základe (i) *a* (ii) *vždy existuje symetrická g-inverzia* $\boldsymbol{G}$ *v leme 6.*

Lema 8. *Rovnica*

$$\begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{b}}_1 \\ \vdots \\ \hat{\boldsymbol{b}}_s \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix} \boldsymbol{Y} \tag{17}$$

je vždy riešiteľná (vzhľadom na $\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{b}}_1, \dots, \hat{\boldsymbol{b}}_s$*). Jedno jej riešenie je*

$$\begin{pmatrix} \hat{\boldsymbol{\beta}} \\ \hat{\boldsymbol{b}}_1 \\ \vdots \\ \hat{\boldsymbol{b}}_s \end{pmatrix} = \boldsymbol{G} \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix} \boldsymbol{Y}, \tag{18}$$

kde $\boldsymbol{G}$ *je z lemy 6 a symetrická (existuje podľa poznámky 7). Pre toto riešenie platí:*

1. $\boldsymbol{p}'\hat{\boldsymbol{\beta}}$ *je nevychýlený odhad* $\boldsymbol{p}'\boldsymbol{\beta}$ *pre* $\boldsymbol{p} \in \mu(\boldsymbol{X}')$*.*

2. $\mathcal{E}\begin{pmatrix} \hat{\boldsymbol{b}}_1 \\ \vdots \\ \hat{\boldsymbol{b}}_s \end{pmatrix} = \boldsymbol{0}$*.*

Dôkaz. Pretože

$$\mu\left\{\begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix} (\boldsymbol{X} \vdots \boldsymbol{Z})\right\} = \mu\begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix},$$

je (17) vždy riešiteľná. Jedno riešenie je (18) (pozri napr. [2], str. 70; všeobecné riešenie pozri tamtiež).

1. Počítajme pre $\boldsymbol{p} \in \mu(\boldsymbol{X}')$ (teda $\boldsymbol{p} = \boldsymbol{X}'\boldsymbol{u}$):

$$\forall\, \boldsymbol{\beta} \in \mathbb{R}^q \quad \mathcal{E}(\boldsymbol{p}'\hat{\boldsymbol{\beta}}) = \mathcal{E}(\boldsymbol{u}'\boldsymbol{X}\{(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y} + \\ + (\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Z}\,\boldsymbol{D}^-\boldsymbol{Z}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y} - (\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Z}\,\boldsymbol{D}^-\boldsymbol{Z}'\boldsymbol{Y}\}) =$$

$$= \boldsymbol{u}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{u}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Z}\,\boldsymbol{D}^-\boldsymbol{Z}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} - \\ - \boldsymbol{u}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Z}\,\boldsymbol{D}^-\boldsymbol{Z}'\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{u}'\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{p}'\boldsymbol{\beta}$$

(využijúc lemu 5 a lemu 6).

2. Platí

$$\mathcal{E}\begin{pmatrix} \hat{\boldsymbol{b}}_1 \\ \vdots \\ \hat{\boldsymbol{b}}_s \end{pmatrix} = -\boldsymbol{D}^-\boldsymbol{Z}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{D}^-\boldsymbol{Z}'\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{0}. \qquad \square$$

Poznámka 9. *Odhad* $\boldsymbol{p}'\hat{\boldsymbol{\beta}}$ *a predikciu* $(\hat{\boldsymbol{b}}'_1, \hat{\boldsymbol{b}}'_2, \dots, \hat{\boldsymbol{b}}'_s)'$ *z lemy 8 navrhol Henderson v* [4].

Lema 10. *Ak pre náhodné vektory* $\boldsymbol{\xi}_{n,1}$ *a* $\boldsymbol{\eta}_{m,1}$ *platí* $\mathcal{E}(\boldsymbol{\xi}) = \boldsymbol{\mu}$, $\mathcal{E}(\boldsymbol{\eta}) = \boldsymbol{\nu}$ *a* $\boldsymbol{P}$ *je matica* $n \times m$, *tak*

$$\mathcal{E}(\boldsymbol{\xi}'\boldsymbol{P}\boldsymbol{\eta}) = \boldsymbol{\mu}'\boldsymbol{P}\boldsymbol{\nu} + \operatorname{tr} \boldsymbol{P}[\operatorname{cov}(\boldsymbol{\xi}, \boldsymbol{\eta})]'.$$

($\operatorname{tr} \boldsymbol{P}$ *znamená stopu matice* $\boldsymbol{P}$*).*

Dôkaz.

$$\mathcal{E}(\boldsymbol{\xi}'\boldsymbol{P}\boldsymbol{\eta}) = \mathcal{E}([\boldsymbol{\xi} - \boldsymbol{\mu} + \boldsymbol{\mu}]'\boldsymbol{P}[\boldsymbol{\eta} - \boldsymbol{\nu} + \boldsymbol{\nu}]) = \mathcal{E}([\boldsymbol{\xi} - \boldsymbol{\mu}]'\boldsymbol{P}[\boldsymbol{\eta} - \boldsymbol{\nu}]) + \boldsymbol{\mu}'\boldsymbol{P}\boldsymbol{\nu} =$$

$$= \boldsymbol{\mu}'\boldsymbol{P}\boldsymbol{\nu} + \mathcal{E}\operatorname{tr}([\boldsymbol{\xi} - \boldsymbol{\mu}]'\boldsymbol{P}[\boldsymbol{\eta} - \boldsymbol{\nu}]) = \boldsymbol{\mu}'\boldsymbol{P}\boldsymbol{\nu} + \mathcal{E}\operatorname{tr}(\boldsymbol{\eta} - \boldsymbol{\nu})(\boldsymbol{\xi} - \boldsymbol{\mu})'\boldsymbol{P} =$$

$$= \boldsymbol{\mu}'\mathbf{P}\boldsymbol{\nu} + \mathcal{E}\operatorname{tr}\mathbf{P}(\boldsymbol{\eta} - \boldsymbol{\nu})(\boldsymbol{\xi} - \boldsymbol{\mu})' = \boldsymbol{\mu}'\mathbf{P}\boldsymbol{\nu} + \operatorname{tr}\mathbf{P}\mathcal{E}(\boldsymbol{\eta} - \boldsymbol{\nu})(\boldsymbol{\xi} - \boldsymbol{\mu})' = \boldsymbol{\mu}'\mathbf{P}\boldsymbol{\nu} + \operatorname{tr}\mathbf{P}[\operatorname{cov}(\boldsymbol{\xi}, \boldsymbol{\eta})]'.$$

$\square$

Pri hľadaní odhadu variančných komponentov postupujeme zaužívanou cestou ako pri analýze rozptylu. Ukážeme si to pre prípad $s = 3$ (3 náhodné efekty plus „efekt“ chýb), teda

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}_1\boldsymbol{b}_1 + \boldsymbol{Z}_2\boldsymbol{b}_2 + \boldsymbol{Z}_3\boldsymbol{b}_3 + \boldsymbol{\varepsilon}. \tag{19}$$

Formálne predpokladáme, že aj $\boldsymbol{b}_1, \boldsymbol{b}_2$ a $\boldsymbol{b}_3$ sú pevné (nenáhodné) parametre a $\operatorname{cov}(\boldsymbol{Y}) = \sigma_0^2\boldsymbol{I}$.

Uvažujme postupnosť lineárnych modelov

$$M : \mathcal{E}(\boldsymbol{Y}) = (\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2 \vdots \boldsymbol{Z}_3)\begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix}, \quad \operatorname{cov}(\boldsymbol{Y}) = \sigma_0^2\boldsymbol{I};$$

$$M_1 : \mathcal{E}(\boldsymbol{Y}) = (\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2)\begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}, \quad \operatorname{cov}(\boldsymbol{Y}) = \sigma_0^2\boldsymbol{I};$$

$$M_2 : \mathcal{E}(\boldsymbol{Y}) = (\boldsymbol{X} \vdots \boldsymbol{Z}_1)\begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{b}_1 \end{pmatrix}, \quad \operatorname{cov}(\boldsymbol{Y}) = \sigma_0^2\boldsymbol{I};$$

$$M_3 : \mathcal{E}(\boldsymbol{Y}) = \boldsymbol{X}\boldsymbol{\beta}, \quad \operatorname{cov}(\boldsymbol{Y}) = \sigma_0^2\boldsymbol{I}.$$

Je zrejmé, že M_3 je submodelom M_2 (lebo $\boldsymbol{X} = (\boldsymbol{X} \vdots \boldsymbol{Z}_1)\begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{0} \end{pmatrix}$), podobne M_2 je submodelom M_1 (lebo $(\boldsymbol{X} \vdots \boldsymbol{Z}_1) = (\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2)\begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{0} & \boldsymbol{0} \end{pmatrix}$) a tiež M_1 je submodelom M. Predpokladajme o hodnostiach

$$n > h(\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2 \vdots \boldsymbol{Z}_3) > h(\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2) > h(\boldsymbol{X} \vdots \boldsymbol{Z}_1) > h(\boldsymbol{X}) > 0. \tag{20}$$

Pri označení z (15) položme

$$\hat{\boldsymbol{\mu}} = (\boldsymbol{X} \vdots \boldsymbol{Z}) \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} \end{pmatrix}^{-} \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix} \boldsymbol{Y},$$

$$\hat{\boldsymbol{\nu}} = (\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2) \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}_1 & \boldsymbol{X}'\boldsymbol{Z}_2 \\ \boldsymbol{Z}_1'\boldsymbol{X} & \boldsymbol{Z}_1'\boldsymbol{Z}_1 & \boldsymbol{Z}_1'\boldsymbol{Z}_2 \\ \boldsymbol{Z}_2'\boldsymbol{X} & \boldsymbol{Z}_2'\boldsymbol{Z}_1 & \boldsymbol{Z}_2'\boldsymbol{Z}_2 \end{pmatrix}^{-} \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}_1' \\ \boldsymbol{Z}_2' \end{pmatrix} \boldsymbol{Y},$$

$$\hat{\boldsymbol{\eta}} = (\boldsymbol{X} \vdots \boldsymbol{Z}_1) \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}_1 \\ \boldsymbol{Z}_1'\boldsymbol{X} & \boldsymbol{Z}_1'\boldsymbol{Z}_1 \end{pmatrix}^{-} \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}_1' \end{pmatrix} \boldsymbol{Y},$$

$$\hat{\boldsymbol{\tau}} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{Y}.$$

Podľa [2], str. 138 $\hat{\boldsymbol{\mu}}$ je BLUE (best linear unbiased estimator) (NNLO - najlepší nevychýlený lineárny odhad) $\mathcal{E}(\boldsymbol{Y})$ v modeli M, podobne $\hat{\boldsymbol{\nu}}$ je BLUE $\mathcal{E}(\boldsymbol{Y})$ v modeli M_1 atď.

S_e označme reziduálny súčet štvorcov v modeli M, t.j.

$$S_e = (\boldsymbol{Y} - \hat{\boldsymbol{\mu}})'(\boldsymbol{Y} - \hat{\boldsymbol{\mu}}).$$

Veta 11. *Veličiny* $\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\nu}}, \hat{\boldsymbol{\eta}}$ *a* $\hat{\boldsymbol{\tau}}$ *spĺňajú identitu*

$$S_e + (\hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\nu}})'(\hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\nu}}) + (\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{\eta}})'(\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{\eta}}) + (\hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\tau}})'(\hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\tau}}) + \hat{\boldsymbol{\tau}}'\hat{\boldsymbol{\tau}} = \boldsymbol{Y}'\boldsymbol{Y}. \tag{22}$$

Dôkaz. Platí

$$S_e = (\boldsymbol{Y} - \hat{\boldsymbol{\mu}})'(\boldsymbol{Y} - \hat{\boldsymbol{\mu}}) = \boldsymbol{Y}'\boldsymbol{Y} - \boldsymbol{Y}'(\boldsymbol{X} \vdots \boldsymbol{Z}) \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} \end{pmatrix}^{-} \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix} \boldsymbol{Y} = \tag{23}$$

$$= \boldsymbol{Y}'\boldsymbol{Y} - \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}}.$$

Ďalej

$$(\hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\nu}})'(\hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\nu}}) =$$

$$= \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}} - \boldsymbol{Y}'(\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2) \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}_1 & \boldsymbol{X}'\boldsymbol{Z}_2 \\ \boldsymbol{Z}_1'\boldsymbol{X} & \boldsymbol{Z}_1'\boldsymbol{Z}_1 & \boldsymbol{Z}_1'\boldsymbol{Z}_2 \\ \boldsymbol{Z}_2'\boldsymbol{X} & \boldsymbol{Z}_2'\boldsymbol{Z}_1 & \boldsymbol{Z}_2'\boldsymbol{Z}_2 \end{pmatrix}^{-} \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}_1' \\ \boldsymbol{Z}_2' \end{pmatrix} \times$$

$$\times (\boldsymbol{X} \vdots \boldsymbol{Z}) \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} \end{pmatrix}^{-} \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix} \boldsymbol{Y} -$$

$$- \boldsymbol{Y}'(\boldsymbol{X} \vdots \boldsymbol{Z}) \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} \end{pmatrix}^{-} \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix} \times$$

$$\times (\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2) \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}_1 & \boldsymbol{X}'\boldsymbol{Z}_2 \\ \boldsymbol{Z}_1'\boldsymbol{X} & \boldsymbol{Z}_1'\boldsymbol{Z}_1 & \boldsymbol{Z}_1'\boldsymbol{Z}_2 \\ \boldsymbol{Z}_2'\boldsymbol{X} & \boldsymbol{Z}_2'\boldsymbol{Z}_1 & \boldsymbol{Z}_2'\boldsymbol{Z}_2 \end{pmatrix}^{-} \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}_1' \\ \boldsymbol{Z}_2' \end{pmatrix} \boldsymbol{Y} + \hat{\boldsymbol{\nu}}'\hat{\boldsymbol{\nu}} =$$

$$= \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}} + \hat{\boldsymbol{\nu}}'\hat{\boldsymbol{\nu}} - \boldsymbol{Y}'(\boldsymbol{X}\vdots\boldsymbol{Z}_1\vdots\boldsymbol{Z}_2)\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}_1 & \boldsymbol{X}'\boldsymbol{Z}_2\\ \boldsymbol{Z}_1'\boldsymbol{X} & \boldsymbol{Z}_1'\boldsymbol{Z}_1 & \boldsymbol{Z}_1'\boldsymbol{Z}_2\\ \boldsymbol{Z}_2'\boldsymbol{X} & \boldsymbol{Z}_2'\boldsymbol{Z}_1 & \boldsymbol{Z}_2'\boldsymbol{Z}_2\end{pmatrix}^{-}\times$$

$$\times\begin{pmatrix}\boldsymbol{I}_{m_1,m_1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}_{m_1,m_4}\\ \boldsymbol{0} & \boldsymbol{I}_{m_2,m_2} & \boldsymbol{0} & \boldsymbol{0}_{m_2,m_4}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{m_3,m_3} & \boldsymbol{0}_{m_3,m_4}\end{pmatrix}\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{Z}'\end{pmatrix}\times$$

$$\times(\boldsymbol{X}\vdots\boldsymbol{Z})\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}\\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z}\end{pmatrix}^{-}\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{Z}'\end{pmatrix}\boldsymbol{Y}-$$

(24)

$$-\boldsymbol{Y}'(\boldsymbol{X}\vdots\boldsymbol{Z})\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}\\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z}\end{pmatrix}^{-}\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{Z}'\end{pmatrix}(\boldsymbol{X}\vdots\boldsymbol{Z})\begin{pmatrix}\boldsymbol{I}_{m_1,m_1} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}_{m_2,m_2} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{m_3,m_3}\\ \boldsymbol{0}_{m_4,m_1} & \boldsymbol{0}_{m_4,m_2} & \boldsymbol{0}_{m_4,m_3}\end{pmatrix}\times$$

$$\times\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}_1 & \boldsymbol{X}'\boldsymbol{Z}_2\\ \boldsymbol{Z}_1'\boldsymbol{X} & \boldsymbol{Z}_1'\boldsymbol{Z}_1 & \boldsymbol{Z}_1'\boldsymbol{Z}_2\\ \boldsymbol{Z}_2'\boldsymbol{X} & \boldsymbol{Z}_2'\boldsymbol{Z}_1 & \boldsymbol{Z}_2'\boldsymbol{Z}_2\end{pmatrix}^{-}\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{Z}_1'\\ \boldsymbol{Z}_2'\end{pmatrix}\boldsymbol{Y} = \hat{\boldsymbol{\mu}}'\hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\nu}}'\hat{\boldsymbol{\nu}}.$$

Analogicky

$$(\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{\eta}})'(\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{\eta}}) = \hat{\boldsymbol{\nu}}'\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{\eta}}'\hat{\boldsymbol{\eta}} \tag{25}$$

a

$$(\hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\tau}})'(\hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\tau}}) = \hat{\boldsymbol{\eta}}'\hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\tau}}'\hat{\boldsymbol{\tau}}. \tag{26}$$

Použitím (23)–(26) dostávame

$$S_e + (\hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\nu}})'(\hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\nu}}) + (\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{\eta}})'(\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{\eta}}) + (\hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\tau}})'(\hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\tau}}) + \hat{\boldsymbol{\tau}}'\hat{\boldsymbol{\tau}} = \boldsymbol{Y}'\boldsymbol{Y}. \qquad \square$$

Označme

$$\hat{\boldsymbol{\tau}}'\hat{\boldsymbol{\tau}} = \boldsymbol{Y}'(\boldsymbol{I} - \boldsymbol{D}_1)\boldsymbol{Y} = S_1, \quad \text{kde} \quad \boldsymbol{D}_1 = \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}', \tag{27}$$

ďalej

$$(\hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\tau}})'(\hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\tau}}) = \hat{\boldsymbol{\eta}}'\hat{\boldsymbol{\eta}} - \hat{\boldsymbol{\tau}}'\hat{\boldsymbol{\tau}} =$$

$$= \boldsymbol{Y}'(\boldsymbol{X}\vdots\boldsymbol{Z}_1)\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}_1\\ \boldsymbol{Z}_1'\boldsymbol{X} & \boldsymbol{Z}_1'\boldsymbol{Z}_1\end{pmatrix}^{-}\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{Z}_1'\end{pmatrix}\boldsymbol{Y} - \boldsymbol{Y}'(\boldsymbol{I} - \boldsymbol{D}_1)\boldsymbol{Y} =$$

$$= \boldsymbol{Y}'(\boldsymbol{X}\vdots\boldsymbol{Z}_1)\times$$

$$\times\begin{pmatrix}(\boldsymbol{X}'\boldsymbol{X})^{-} + (\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{Z}_1\boldsymbol{D}^{-}\boldsymbol{Z}_1'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-} & -(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{Z}_1\boldsymbol{D}^{-}\\ -\boldsymbol{D}^{-}\boldsymbol{Z}_1'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-} & \boldsymbol{D}^{-}\end{pmatrix}\times$$

$$\times\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{Z}_1'\end{pmatrix}\boldsymbol{Y} - \boldsymbol{Y}'(\boldsymbol{I} - \boldsymbol{D}_1)\boldsymbol{Y} =$$

$$= \boldsymbol{Y}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y} + \boldsymbol{Y}'(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}')\boldsymbol{Z}_1[\boldsymbol{Z}_1'(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}')\boldsymbol{Z}_1]^- \times$$
$$\times \boldsymbol{Z}_1'(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}')\boldsymbol{Y} - \boldsymbol{Y}'(\boldsymbol{I} - \boldsymbol{D}_1)\boldsymbol{Y} =$$

(28) $$= \boldsymbol{Y}'(\boldsymbol{D}_1 - \boldsymbol{D}_{12})\boldsymbol{Y} = S_2, \quad \text{kde} \quad \boldsymbol{D}_{12} = \boldsymbol{D}_1 - \boldsymbol{D}_1\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1)^-\boldsymbol{Z}_1'\boldsymbol{D}_1.$$

(Využili sme tvar g-inverzie matice $\boldsymbol{S}$ z lemy 6, lebo podľa poznámky 7 nezáleží na tom, ktorú g-inverziu použijeme.)

Počítajme ďalej analogicky ako v predchádzajúcom prípade

(29) $$(\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{\eta}})'(\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{\eta}}) = \hat{\boldsymbol{\nu}}'\hat{\boldsymbol{\nu}} - \hat{\boldsymbol{\eta}}'\hat{\boldsymbol{\eta}} =$$

$$= \boldsymbol{Y}'(\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2)\begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}_1 & \boldsymbol{X}'\boldsymbol{Z}_2 \\ \boldsymbol{Z}_1'\boldsymbol{X} & \boldsymbol{Z}_1'\boldsymbol{Z}_1 & \boldsymbol{Z}_1'\boldsymbol{Z}_2 \\ \boldsymbol{Z}_2'\boldsymbol{X} & \boldsymbol{Z}_2'\boldsymbol{Z}_1 & \boldsymbol{Z}_2'\boldsymbol{Z}_2 \end{pmatrix}^- \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}_1' \\ \boldsymbol{Z}_2' \end{pmatrix}\boldsymbol{Y} -$$
$$- \boldsymbol{Y}'(\boldsymbol{X} \vdots \boldsymbol{Z}_1)\begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}_1 \\ \boldsymbol{Z}_1'\boldsymbol{X} & \boldsymbol{Z}_1'\boldsymbol{Z}_1 \end{pmatrix}^- \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}_1' \end{pmatrix}\boldsymbol{Y} =$$

$$= \boldsymbol{Y}'(\boldsymbol{D}_{12} - \boldsymbol{D}_{123})\boldsymbol{Y} = S_3, \quad \text{kde} \quad \boldsymbol{D}_{123} = \boldsymbol{D}_{12} - \boldsymbol{D}_{12}\boldsymbol{Z}_2(\boldsymbol{Z}_2'\boldsymbol{D}_{12}\boldsymbol{Z}_2)^-\boldsymbol{Z}_2'\boldsymbol{D}_{12}.$$

Podobne dostaneme

(30) $$(\hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\nu}})'(\hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\nu}}) =$$

$$= \boldsymbol{Y}'(\boldsymbol{D}_{123} - \boldsymbol{D}_{1234})\boldsymbol{Y} = S_4, \quad \text{kde} \quad \boldsymbol{D}_{1234} = \boldsymbol{D}_{123} - \boldsymbol{D}_{123}\boldsymbol{Z}_3(\boldsymbol{Z}_3'\boldsymbol{D}_{123}\boldsymbol{Z}_3)^-\boldsymbol{Z}_3'\boldsymbol{D}_{123}.$$

Zo vzťahov (27)–(30) a (22) vyplýva

(31) $$S_e = \boldsymbol{Y}'\boldsymbol{Y} - S_1 - S_2 - S_3 - S_4 =$$

$$= \mathbf{Y}'\mathbf{Y} - \mathbf{Y}'(\mathbf{I} - \mathbf{D}_1)\mathbf{Y} - \mathbf{Y}'(\mathbf{D}_1 - \mathbf{D}_{12})\mathbf{Y} - \mathbf{Y}'(\mathbf{D}_{12} - \mathbf{D}_{123})\mathbf{Y} - \mathbf{Y}'(\mathbf{D}_{123} - \mathbf{D}_{1234})\mathbf{Y} =$$
$$= \boldsymbol{Y}'\boldsymbol{D}_{1234}\boldsymbol{Y}.$$

Takto sme dostali (vhodné) kvadratické formy S_e, S_1, S_2, S_3, S_4 náhodného vektora $\boldsymbol{Y}$ (toto bol „návod“ k ich dosiahnutiu).

Vráťme sa k modelu (15) pre $s = 3$. Platí

$$\boldsymbol{X}'\boldsymbol{D}_1 = \boldsymbol{X}'\boldsymbol{D}_{12} = \boldsymbol{X}'\boldsymbol{D}_{123} = \boldsymbol{X}'\boldsymbol{D}_{1234} = \mathbf{0},$$

$$\boldsymbol{Z}_1'\boldsymbol{D}_{12} = \boldsymbol{Z}_1'\boldsymbol{D}_{123} = \boldsymbol{Z}_1'\boldsymbol{D}_{1234} = \mathbf{0},$$

$$\boldsymbol{Z}_2'\boldsymbol{D}_{123} = \boldsymbol{Z}_2'\boldsymbol{D}_{1234} = \mathbf{0}$$

a

$$\boldsymbol{Z}_3'\boldsymbol{D}_{1234} = \mathbf{0}.$$

Pretože

$$\mathcal{E}(\boldsymbol{Y}) = \boldsymbol{X}\boldsymbol{\beta}$$

a

$$\operatorname{cov}(\boldsymbol{Y}) = \sigma_1^2 \boldsymbol{Z}_1 \boldsymbol{Z}_1' + \sigma_2^2 \boldsymbol{Z}_2 \boldsymbol{Z}_2' + \sigma_3^2 \boldsymbol{Z}_3 \boldsymbol{Z}_3' + \sigma_0^2 \boldsymbol{I},$$

dostávame pomocou lemy 10

$$\mathcal{E}(S_2) = \mathcal{E}(\boldsymbol{Y}'(\boldsymbol{D}_1 - \boldsymbol{D}_{12})\boldsymbol{Y}) = \tag{32}$$

$$\begin{aligned} &= \operatorname{tr}(\boldsymbol{D}_1 - \boldsymbol{D}_{12})(\sigma_1^2 \boldsymbol{Z}_1 \boldsymbol{Z}_1' + \sigma_2^2 \boldsymbol{Z}_2 \boldsymbol{Z}_2' + \sigma_3^2 \boldsymbol{Z}_3 \boldsymbol{Z}_3' + \sigma_0^2 \boldsymbol{I}) = \\ &= \sigma_1^2 \operatorname{tr} \boldsymbol{Z}_1' \boldsymbol{D}_1 \boldsymbol{Z}_1 + \sigma_2^2 \{\operatorname{tr} \boldsymbol{Z}_2' \boldsymbol{D}_1 \boldsymbol{Z}_2 - \operatorname{tr} \boldsymbol{Z}_2' \boldsymbol{D}_{12} \boldsymbol{Z}_2\} + \\ &+ \sigma_3^2 \{\operatorname{tr} \boldsymbol{Z}_3' \boldsymbol{D}_1 \boldsymbol{Z}_3 - \operatorname{tr} \boldsymbol{Z}_3' \boldsymbol{D}_{12} \boldsymbol{Z}_3\} + \sigma_0^2 \operatorname{tr}(\boldsymbol{D}_1 - \boldsymbol{D}_{12}). \end{aligned}$$

$$\mathcal{E}(S_3) = \mathcal{E}(\boldsymbol{Y}'(\boldsymbol{D}_{12} - \boldsymbol{D}_{123})\boldsymbol{Y}) = \tag{33}$$

$$= \sigma_2^2 \operatorname{tr} \boldsymbol{Z}_2' \boldsymbol{D}_{12} \boldsymbol{Z}_2 + \sigma_3^2 \{\operatorname{tr} \boldsymbol{Z}_3' \boldsymbol{D}_{12} \boldsymbol{Z}_3 - \operatorname{tr} \boldsymbol{Z}_3' \boldsymbol{D}_{123} \boldsymbol{Z}_3\} + \sigma_0^2 \operatorname{tr}(\boldsymbol{D}_{12} - \boldsymbol{D}_{123}),$$

$$\mathcal{E}(S_4) = \mathcal{E}(\boldsymbol{Y}'(\boldsymbol{D}_{123} - \boldsymbol{D}_{1234})\boldsymbol{Y}) = \tag{34}$$

$$= \sigma_3^2 \operatorname{tr} \boldsymbol{Z}_3' \boldsymbol{D}_{123} \boldsymbol{Z}_3 + \sigma_0^2 \operatorname{tr}(\boldsymbol{D}_{123} - \boldsymbol{D}_{1234}),$$

$$\mathcal{E}(S_e) = \sigma_0^2 \operatorname{tr} \boldsymbol{D}_{1234}. \tag{35}$$

Z (32) až (35) dostávame

$$\mathcal{E}\begin{pmatrix} S_2 \\ S_3 \\ S_4 \\ S_e \end{pmatrix} = \tag{36}$$

$$= \begin{pmatrix} \operatorname{tr} \boldsymbol{Z}_1' \boldsymbol{D}_1 \boldsymbol{Z}_1 & \operatorname{tr} \boldsymbol{Z}_2'(\boldsymbol{D}_1 - \boldsymbol{D}_{12}) \boldsymbol{Z}_2 & \operatorname{tr} \boldsymbol{Z}_3'(\boldsymbol{D}_1 - \boldsymbol{D}_{12}) \boldsymbol{Z}_3 & \operatorname{tr}(\boldsymbol{D}_1 - \boldsymbol{D}_{12}) \\ \boldsymbol{0} & \operatorname{tr} \boldsymbol{Z}_2' \boldsymbol{D}_{12} \boldsymbol{Z}_2 & \operatorname{tr} \boldsymbol{Z}_3'(\boldsymbol{D}_{12} - \boldsymbol{D}_{123}) \boldsymbol{Z}_3 & \operatorname{tr}(\boldsymbol{D}_{12} - \boldsymbol{D}_{123}) \\ \boldsymbol{0} & \boldsymbol{0} & \operatorname{tr} \boldsymbol{Z}_3' \boldsymbol{D}_{123} \boldsymbol{Z}_3 & \operatorname{tr}(\boldsymbol{D}_{123} - \boldsymbol{D}_{1234}) \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \operatorname{tr} \boldsymbol{D}_{1234} \end{pmatrix} \begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \sigma_3^2 \\ \sigma_0^2 \end{pmatrix} =$$

$$= \boldsymbol{E} \begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \sigma_3^2 \\ \sigma_0^2 \end{pmatrix}.$$

Teraz sa venujme riešiteľnosti systému (36). Opreme sa o známe skutočnosti:

$$\forall \boldsymbol{A}_{m,n} \quad \operatorname{tr} \boldsymbol{A}\ \boldsymbol{A}' = \sum_{i=1}^{m} \sum_{j=1}^{n} \{\boldsymbol{A}\}_{ij}^2, \tag{37}$$

$$\forall \boldsymbol{A}_{m,n} \ \ \{\mu(\boldsymbol{A})\}^{\perp} = \{\boldsymbol{A}\boldsymbol{u}\colon \boldsymbol{u} \in \mathbb{R}^n\}^{\perp} = \operatorname{Ker} \boldsymbol{A}', \tag{38}$$

kde $\perp$ je symbol pre ortogonálny doplnok a $\operatorname{Ker} \boldsymbol{A}' = \{\boldsymbol{w} \in \mathbb{R}^m\colon \boldsymbol{A}'\boldsymbol{w} = \boldsymbol{0}\}$.

Ak
$$\operatorname{tr} \boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1 = \mathbf{0} \Longrightarrow \operatorname{tr} \boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{D}_1'\boldsymbol{Z}_1 = \mathbf{0} \Longrightarrow$$
$$\Longrightarrow \boldsymbol{Z}_1'\boldsymbol{D}_1 = \mathbf{0} \Longrightarrow \boldsymbol{Z}_1'(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}') = \mathbf{0} \Longrightarrow$$
$$\Longrightarrow \operatorname{Ker} \boldsymbol{X}'(= \mu(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}')) \subset \operatorname{Ker} \boldsymbol{Z}_1' \Longrightarrow$$
$$\Longrightarrow \mu(\boldsymbol{Z}_1) \subset \mu(\boldsymbol{X}),$$
čo je spor s predpokladom (20) modelu.

Ak
$$\operatorname{tr} \boldsymbol{Z}_2'\boldsymbol{D}_{12}\boldsymbol{Z}_2 = \mathbf{0} \Longrightarrow \operatorname{tr} \boldsymbol{Z}_2'\boldsymbol{D}_{12}\boldsymbol{D}_{12}'\boldsymbol{Z}_2 = \mathbf{0} \Longrightarrow$$
$$\Longrightarrow \boldsymbol{Z}_2'\boldsymbol{D}_{12} = \mathbf{0} \Longrightarrow \boldsymbol{Z}_2'(\boldsymbol{D}_1 - \boldsymbol{D}_1\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1)^-\boldsymbol{Z}_1'\boldsymbol{D}_1) = \mathbf{0} \Longrightarrow$$
$$\Longrightarrow \boldsymbol{Z}_2'(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}' - \boldsymbol{D}_1\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1)^-\boldsymbol{Z}_1'\boldsymbol{D}_1) = \mathbf{0} \Longrightarrow$$
$$\Longrightarrow \mu(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}' - \boldsymbol{D}_1\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1)^-\boldsymbol{Z}_1'\boldsymbol{D}_1) \subset \operatorname{Ker} \boldsymbol{Z}_2' \Longrightarrow$$
$$\Longrightarrow \mu(\boldsymbol{Z}_2) \subset \operatorname{Ker}(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}' - \boldsymbol{D}_1\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1)^-\boldsymbol{Z}_1'\boldsymbol{D}_1) =$$
$$= \mu(\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}' + \boldsymbol{D}_1\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1)^-\boldsymbol{Z}_1'\boldsymbol{D}_1) =$$
$$= \mu(\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'+$$
$$+ \boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1)^-\boldsymbol{Z}_1'\boldsymbol{D}_1 - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1)^-\boldsymbol{Z}_1'\boldsymbol{D}_1) \subset \mu(\boldsymbol{X} \vdots \boldsymbol{Z}_1),$$
čo je spor s (20).

Rovnakým spôsobom ukážeme, že ak
$$\operatorname{tr} \boldsymbol{Z}_3'\boldsymbol{D}_{123}\boldsymbol{Z}_3 = \mathbf{0} \Longrightarrow \mu(\boldsymbol{Z}_3) \subset \mu(\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2 \vdots \boldsymbol{Z}_3),$$
čo je opäť v spore s (20).

Venujme sa ešte výrazu
$$\operatorname{tr} \boldsymbol{D}_{1234} = \frac{1}{\sigma_0^2}\mathcal{E}(S_e) = \frac{1}{\sigma_0^2}\mathcal{E}\left\{\boldsymbol{Y}'\left[(\boldsymbol{I} - (\boldsymbol{X} \vdots \boldsymbol{Z})\begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} \end{pmatrix}^-\begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix}\right]\boldsymbol{Y}\right\}$$
(pozri (23)).

Lema 12.
$$\mathcal{E}(S_e) = \sigma_0^2[n - h(\boldsymbol{X} \vdots \boldsymbol{Z})].$$

Dôkaz. $\boldsymbol{Z} = (\boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2 \vdots \boldsymbol{Z}_3)$. Z (23) dostávame
$$\mathcal{E}(S_e) = \mathcal{E}\left\{\boldsymbol{Y}'\left(\boldsymbol{I} - (\boldsymbol{X} \vdots \boldsymbol{Z})\begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} \end{pmatrix}^-\begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix}\right)\boldsymbol{Y}\right\} =$$
$$= \boldsymbol{\beta}'\boldsymbol{X}'\left[\boldsymbol{I} - (\boldsymbol{X} \vdots \boldsymbol{Z})\begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} \end{pmatrix}^-\begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix}\right]\boldsymbol{X}\boldsymbol{\beta}+$$

$$+\operatorname{tr}\left[\boldsymbol{I}-(\boldsymbol{X}\vdots\boldsymbol{Z})\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}\\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z}\end{pmatrix}^{-}\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{Z}'\end{pmatrix}\right]\times$$

$$\times\left[\boldsymbol{Z}\begin{pmatrix}\sigma_1^2\boldsymbol{I}_{m_1,m_1} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \sigma_2^2\boldsymbol{I}_{m_2,m_2} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \sigma_3^2\boldsymbol{I}_{m_3,m_3}\end{pmatrix}\boldsymbol{Z}'+\sigma_0^2\boldsymbol{I}\right]=$$

$$=\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta}-\boldsymbol{\beta}'(\boldsymbol{X}'\boldsymbol{X}\vdots\boldsymbol{X}'\boldsymbol{Z})\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}\\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z}\end{pmatrix}^{-}\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X}\\ \boldsymbol{Z}'\boldsymbol{X}\end{pmatrix}\boldsymbol{\beta}+$$

$$+\operatorname{tr}\boldsymbol{Z}\begin{pmatrix}\sigma_1^2\boldsymbol{I}_{m_1,m_1} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \sigma_2^2\boldsymbol{I}_{m_2,m_2} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \sigma_3^2\boldsymbol{I}_{m_3,m_3}\end{pmatrix}\boldsymbol{Z}'-$$

$$-\operatorname{tr}(\mathbf{X}\vdots\mathbf{Z})\begin{pmatrix}\mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{Z}\\ \mathbf{Z}'\mathbf{X} & \mathbf{Z}'\mathbf{Z}\end{pmatrix}^{-}\begin{pmatrix}\mathbf{X}'\mathbf{Z}\\ \mathbf{Z}'\mathbf{Z}\end{pmatrix}\begin{pmatrix}\sigma_1^2\mathbf{I}_{m_1,m_1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \sigma_2^2\mathbf{I}_{m_2,m_2} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \sigma_3^2\mathbf{I}_{m_3,m_3}\end{pmatrix}\mathbf{Z}'+$$

$$+\sigma_0^2\operatorname{tr}\boldsymbol{I}\ -\sigma_0^2\operatorname{tr}(\boldsymbol{X}\vdots\boldsymbol{Z})\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}\\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z}\end{pmatrix}^{-}\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{Z}'\end{pmatrix}=$$

$$=\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta}-\boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta}+\operatorname{tr}\boldsymbol{Z}\begin{pmatrix}\sigma_1^2\boldsymbol{I}_{m_1,m_1} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \sigma_2^2\boldsymbol{I}_{m_2,m_2} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \sigma_3^2\boldsymbol{I}_{m_3,m_3}\end{pmatrix}\boldsymbol{Z}'-$$

$$-\operatorname{tr}(\boldsymbol{Z}'\boldsymbol{X}\vdots\boldsymbol{Z}'\boldsymbol{Z})\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}\\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z}\end{pmatrix}^{-}\begin{pmatrix}\boldsymbol{X}'\boldsymbol{Z}\\ \boldsymbol{Z}'\boldsymbol{Z}\end{pmatrix}\times$$

$$\times\begin{pmatrix}\sigma_1^2\boldsymbol{I}_{m_1,m_1} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \sigma_2^2\boldsymbol{I}_{m_2,m_2} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \sigma_3^2\boldsymbol{I}_{m_3,m_3}\end{pmatrix}+$$

$$+n\sigma_0^2-\sigma_0^2\operatorname{tr}(\boldsymbol{X}\vdots\boldsymbol{Z})\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}\\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z}\end{pmatrix}^{-}\begin{pmatrix}\boldsymbol{X}'\\ \boldsymbol{Z}'\end{pmatrix}=$$

$$=\operatorname{tr}\boldsymbol{Z}'\boldsymbol{Z}\begin{pmatrix}\sigma_1^2\boldsymbol{I}_{m_1,m_1} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \sigma_2^2\boldsymbol{I}_{m_2,m_2} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \sigma_3^2\boldsymbol{I}_{m_3,m_3}\end{pmatrix}-$$

$$-\operatorname{tr}\boldsymbol{Z}'\boldsymbol{Z}\begin{pmatrix}\sigma_1^2\boldsymbol{I}_{m_1,m_1} & \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{0} & \sigma_2^2\boldsymbol{I}_{m_2,m_2} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0} & \sigma_3^2\boldsymbol{I}_{m_3,m_3}\end{pmatrix}+$$

$$+\sigma_0^2\left(n-\operatorname{tr}\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}\\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z}\end{pmatrix}\begin{pmatrix}\boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}\\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z}\end{pmatrix}^{-}\right)=\sigma_0^2\big(n-h(\boldsymbol{X}\vdots\boldsymbol{Z})\big),$$

lebo pre ľubovolnú maticu $\boldsymbol{H}$ je $\boldsymbol{H}\boldsymbol{H}^-$ (vždy) idempotentná (napr. pozri [2], str. 66) a preto $\operatorname{tr}\boldsymbol{H}\boldsymbol{H}^- = h(\boldsymbol{H}\boldsymbol{H}^-)$. Ďalej $\boldsymbol{H}\boldsymbol{H}^-\boldsymbol{H} = \boldsymbol{H}$, teda $h(\boldsymbol{H}) \geqq h(\boldsymbol{H}\boldsymbol{H}^-) \geqq h(\boldsymbol{H}\boldsymbol{H}^-\boldsymbol{H}) = h(\boldsymbol{H})$, čiže $h(\boldsymbol{H}\boldsymbol{H}^-) = h(\boldsymbol{H})$. $\square$

Ak je splnená podmienka (20), je tr $\boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1 \neq 0$, tr $\boldsymbol{Z}_2'\boldsymbol{D}_{12}\boldsymbol{Z}_2 \neq 0$, tr $\boldsymbol{Z}_3'\boldsymbol{D}_{123}\boldsymbol{Z}_3 \neq 0$ aj tr $\boldsymbol{D}_{1234} = n - h(\boldsymbol{X} \vdots \boldsymbol{Z}) \neq 0$, teda (36) je riešiteľná ($\boldsymbol{E}$ v (36) je nesingulárna). Odhad

$$\boldsymbol{E}^{-1}\begin{pmatrix} S_2 \\ S_3 \\ S_4 \\ S_e \end{pmatrix} = \begin{pmatrix} \widehat{\sigma_1^2} \\ \widehat{\sigma_2^2} \\ \widehat{\sigma_3^2} \\ \widehat{\sigma_0^2} \end{pmatrix} \tag{39}$$

je nevychýleným odhadom $(\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2)'$.

Poznámka 13. *Odhad* (39) *navrhol Henderson v* [4], *maticovo vyjadril Searle v* [12], *preformuloval Rohde* [11].

Poznámka 14.
Z predchádzajúceho je jasné, ako postupovať pri určení odhadu typu (39) *pre model* (15) *s ľubovoľným počtom s náhodných efektov.*

Skôr ako špecifikujeme odhad (39) pre konkrétne modely, uvedieme bez dôkazov ešte niekoľko základných viet ohľadne rozdelenia kvadratických foriem. Dôkazy viet nájdeme v uvedenej literatúre.

Definícia 15. *Nech $Y_1, Y_2, \dots, Y_n$ sú nezávislé $N(\mu_i, 1)$ rozdelené, $\delta = \sum_{i=1}^n \mu_i^2$.*

$$Y = Y_1^2 + Y_2^2 + \ldots + Y_n^2$$

má necentrálne $\chi^2(n, \delta)$ rozdelenie s n stupňami voľnosti a parametrom necentrality δ. Ak $\delta = 0$, Y má centrálne $\chi^2(n, 0)$ rozdelenie.

Veta 16. *Nech $\boldsymbol{Y} \sim N_p(\boldsymbol{\nu}, \boldsymbol{\Sigma})$. Náhodná premenná*

$$\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y} + 2\boldsymbol{b}'\boldsymbol{Y} + c$$

má $\chi^2(k, \delta)$ rozdelenie práve vtedy ak
(i) $\boldsymbol{\Sigma A \Sigma A \Sigma} = \boldsymbol{\Sigma A \Sigma}$, alebo ekvivalentne $(\boldsymbol{\Sigma A})^3 = (\boldsymbol{\Sigma A})^2$,
(ii) $\boldsymbol{\Sigma}(\boldsymbol{A}\,\boldsymbol{\nu} + \boldsymbol{b}) \in \mu(\boldsymbol{\Sigma A}\,\boldsymbol{\Sigma})$,
(iii) $(\boldsymbol{A\nu} + \boldsymbol{b})'\boldsymbol{\Sigma}(\boldsymbol{A\nu} + \boldsymbol{b}) = \boldsymbol{\nu}'\boldsymbol{A}\,\boldsymbol{\nu} + 2\boldsymbol{b}'\boldsymbol{\nu} + c$.
V takomto prípade

$$k = \operatorname{tr} \boldsymbol{A\Sigma},$$

$$\delta = (\boldsymbol{A\nu} + \boldsymbol{b})'\boldsymbol{\Sigma A}\,\boldsymbol{\Sigma}(\boldsymbol{A\nu} + \boldsymbol{b}).$$

Dôkaz nájdeme v [10], str. 171, [6], str. 220, [5], str. 178.

Veta 17. *Nech $\boldsymbol{Y} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$; $Q_1 = \boldsymbol{Y}'\boldsymbol{A}\,\boldsymbol{Y}$, $Q_2 = \boldsymbol{Y}'\boldsymbol{B}\boldsymbol{Y}$ dve kvadratické formy. Q_1 a Q_2 sú nezávislé práve vtedy ak*

(i) $$\boldsymbol{\Sigma A \Sigma B \Sigma} = \boldsymbol{0}, \quad \boldsymbol{\Sigma A \Sigma B \mu} = \boldsymbol{\Sigma B \Sigma A \mu} = \boldsymbol{0}, \quad \boldsymbol{\mu}'\boldsymbol{A \Sigma B \mu} = 0,$$

ak $\boldsymbol{A}$, $\boldsymbol{B}$ sú symetrické, nemusia byť p.s.d., $\boldsymbol{\Sigma}$ môže byť singulárna.

(ii) $$\boldsymbol{A \Sigma B \Sigma} = \boldsymbol{0}, \quad \boldsymbol{A \Sigma B \mu} = \boldsymbol{0},$$

ak $\boldsymbol{A}$ je p.s.d.

(iii) $$\boldsymbol{A\Sigma B} = \boldsymbol{0},$$

ak $\boldsymbol{A}$ aj $\boldsymbol{B}$ sú p.s.d.

(iv) $$\boldsymbol{A\Sigma B} = \boldsymbol{0},$$

ak $\boldsymbol{\Sigma}$ je regulárna, $\boldsymbol{A}$, $\boldsymbol{B}$ symetrické (nemusia byť p.s.d.).

Dôkaz nájdeme v [10], str. 178.

Veta 18. *Nech $\boldsymbol{Y} \sim N(\boldsymbol{\mu}, \sigma^2 \boldsymbol{H})$, $\boldsymbol{A}$, $\boldsymbol{B}$ sú symetrické. Potom*

$$\mathcal{E}(\boldsymbol{Y}'\boldsymbol{A}\ \boldsymbol{Y}\boldsymbol{Y}'\boldsymbol{B}\boldsymbol{Y}) = \sigma^4[2\,\mathrm{tr}(\boldsymbol{AHBH}) + \mathrm{tr}(\boldsymbol{AH})\,\mathrm{tr}(\boldsymbol{BH})]+ \\ + \sigma^2[\boldsymbol{\mu}'\boldsymbol{A}\ \boldsymbol{\mu}\,\mathrm{tr}(\boldsymbol{BH}) + \boldsymbol{\mu}'\boldsymbol{B}\boldsymbol{\mu}\,\mathrm{tr}(\boldsymbol{AH}) + 4\boldsymbol{\mu}'\boldsymbol{AHB}\boldsymbol{\mu} + \boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{B}\boldsymbol{\mu}],$$

$$\mathrm{cov}(\boldsymbol{a}'\boldsymbol{Y} + \boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}, \boldsymbol{b}'\boldsymbol{Y} + \boldsymbol{Y}'\boldsymbol{B}\boldsymbol{Y}) = 2\sigma^4\,\mathrm{tr}\,\boldsymbol{A}\ \boldsymbol{HBH} + \sigma^2(2\boldsymbol{A}\ \boldsymbol{\mu} + \boldsymbol{a})'\boldsymbol{H}(2\boldsymbol{B}\boldsymbol{\mu} + \boldsymbol{b}).$$

Dôkaz vykonáme pomocou [6], str. 223, [5], str. 181. Urobte ho ako cvičenie.

6 ODHADY VARIANČNÝCH KOMPONENBTOV V MODELI S JEDNÝM NÁHODNÝM EFEKTOM

Majme model

(40) $$Y_{ij} = \beta + b_i + \varepsilon_{ij}, \qquad \begin{matrix} i = 1, 2, \dots, I\,, \\ j = 1, 2, \dots, n_i\,, \end{matrix}$$

kde náhodný vektor

$$\boldsymbol{b}_1 = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_I \end{pmatrix}$$

je $N(\boldsymbol{0}_{I,1}; \sigma_1^2 \boldsymbol{I})$ rozdelený. Chybový vektor $\boldsymbol{\varepsilon} \sim N(\boldsymbol{0}, \sigma_0^2 \boldsymbol{I})$ je nezávislý od $\boldsymbol{b}$, β je číslo. Model (40) sa dá písať

$$\boldsymbol{Y} = \begin{pmatrix} Y_{11} \\ Y_{12} \\ \vdots \\ Y_{1n_1} \\ Y_{21} \\ Y_{22} \\ \vdots \\ Y_{2n_2} \\ \vdots \\ Y_{I1} \\ Y_{I2} \\ \vdots \\ Y_{In_I} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0\dots & 0 \\ 1 & 1 & 0 & 0\dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ b_1 \\ b_2 \\ \vdots \\ b_I \end{pmatrix} + \boldsymbol{\varepsilon} = (\boldsymbol{X} \vdots \boldsymbol{Z}_1) \begin{pmatrix} \beta \\ \boldsymbol{b}_1 \end{pmatrix} + \boldsymbol{\varepsilon}.$$

Zrejme $\mathcal{E}(\boldsymbol{Y}) = \boldsymbol{X}\beta = \mathbf{1}\beta$, $\mathbf{1} = (1, 1, \dots 1)'$ je $n = \sum_{i=1}^{I} n_i$-rozmerný vektor a $\mathrm{cov}(\boldsymbol{Y}) = \sigma_1^2 \boldsymbol{Z}_1 \boldsymbol{Z}_1' + \sigma_0^2 \boldsymbol{I}$. Hodnosť matice $(\boldsymbol{X} \vdots \boldsymbol{Z}_1)$ je I.

Pri odhade variančných komponentov postupujeme podľa návodu na str. 17. Máme tentoraz postupnosť submodelov

$$M: \qquad \mathcal{E}(\boldsymbol{Y}) = (\boldsymbol{X} \vdots \boldsymbol{Z}_1) \begin{pmatrix} \beta \\ \boldsymbol{b}_1 \end{pmatrix}, \quad \mathrm{cov}(\boldsymbol{Y}) = \sigma_0^2 \boldsymbol{I};$$

$$M_1: \qquad \mathcal{E}(\boldsymbol{Y}) = \boldsymbol{X}\beta, \quad \mathrm{cov}(\boldsymbol{Y}) = \sigma_0^2 \boldsymbol{I}.$$

Platí

$$\hat{\boldsymbol{\mu}} = (\boldsymbol{X} \vdots \boldsymbol{Z}_1) \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}_1 \\ \boldsymbol{Z}_1'\boldsymbol{X} & \boldsymbol{Z}_1'\boldsymbol{Z}_1 \end{pmatrix}^- \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}_1' \end{pmatrix} \boldsymbol{Y},$$

$$\hat{\boldsymbol{\nu}} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{X}'\boldsymbol{Y}.$$

Spočítajme S_2. Podľa (28)

$$S_2 = \boldsymbol{Y}'(\boldsymbol{D}_1 - \boldsymbol{D}_{12})\boldsymbol{Y},$$

kde

$$\boldsymbol{D}_1 = \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^- \boldsymbol{X}', \quad \boldsymbol{D}_{12} = \boldsymbol{D}_1 - \boldsymbol{D}_1 \boldsymbol{Z}_1 (\boldsymbol{Z}_1' \boldsymbol{D}_1 \boldsymbol{Z}_1)^- \boldsymbol{Z}_1' \boldsymbol{D}_1$$

(upozorňujeme len, že $\boldsymbol{D}_1$ a $\boldsymbol{D}_{12}$ nezávisia podľa poznámky 7 od voľby g-inverzie).

Platí

$$\boldsymbol{D}_1 = \boldsymbol{I} - \frac{1}{n} \mathbf{1}\mathbf{1}',$$

ďalej

$$\boldsymbol{Z}_1' \boldsymbol{D}_1 \boldsymbol{Z}_1 = \boldsymbol{Z}_1' \boldsymbol{Z}_1 - \frac{1}{n} \boldsymbol{Z}_1' \mathbf{1}\mathbf{1}' \boldsymbol{Z}_1 =$$

$$= \begin{pmatrix} n_1 & 0 & \dots & 0 \\ 0 & n_2 & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & n_I \end{pmatrix} - \frac{1}{n} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_I \end{pmatrix} (n_1, n_2, \dots, n_I) =$$

$$= \begin{pmatrix} n_1 - \frac{n_1^2}{n} & -\frac{n_1 n_2}{n} & \dots & -\frac{n_1 n_I}{n} \\ -\frac{n_2 n_1}{n} & n_2 - \frac{n_2^2}{n} & & -\frac{n_2 n_I}{n} \\ \vdots & & \ddots & \\ -\frac{n_I n_1}{n} & & & n_I - \frac{n_I^2}{n} \end{pmatrix}.$$

Jedna g-inverzia

$$(\boldsymbol{Z}_1' \boldsymbol{D}_1 \boldsymbol{Z}_1)^- = \begin{pmatrix} \frac{1}{n_1} & 0 & \dots & 0 \\ 0 & \frac{1}{n_2} & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \frac{1}{n_I} \end{pmatrix}$$

(dokážte ako cvičenie).

Dostávame

$$(41)\qquad \boldsymbol{D}_1-\boldsymbol{D}_{12}=(\boldsymbol{I}-\frac{1}{n}\boldsymbol{1}\boldsymbol{1}')\boldsymbol{Z}_1\begin{pmatrix}\frac{1}{n_1} & 0 & \dots & 0\\ 0 & \frac{1}{n_2} & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \frac{1}{n_I}\end{pmatrix}\boldsymbol{Z}_1'(\boldsymbol{I}-\frac{1}{n}\boldsymbol{1}\boldsymbol{1}')=$$

$$=\boldsymbol{Z}_1\begin{pmatrix}\frac{1}{n_1} & 0 & \dots & 0\\ 0 & \frac{1}{n_2} & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \frac{1}{n_I}\end{pmatrix}\boldsymbol{Z}_1'-\frac{1}{n}\boldsymbol{1}(n_1,n_2,\dots,n_I)\begin{pmatrix}\frac{1}{n_1} & 0 & \dots & 0\\ 0 & \frac{1}{n_2} & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \frac{1}{n_I}\end{pmatrix}\boldsymbol{Z}_1'-$$

$$-\frac{1}{n}\boldsymbol{Z}_1\begin{pmatrix}\frac{1}{n_1} & 0 & \dots & 0\\ 0 & \frac{1}{n_2} & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \frac{1}{n_I}\end{pmatrix}\begin{pmatrix}n_1\\ n_2\\ \vdots\\ n_I\end{pmatrix}\boldsymbol{1}'+\frac{1}{n}\boldsymbol{1}\boldsymbol{1}'=$$

$$=\boldsymbol{Z}_1\begin{pmatrix}\frac{1}{n_1} & 0 & \dots & 0\\ 0 & \frac{1}{n_2} & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \frac{1}{n_I}\end{pmatrix}\boldsymbol{Z}_1'-\frac{1}{n}\boldsymbol{1}(1,1,\dots,1)\boldsymbol{Z}_1'-\frac{1}{n}\boldsymbol{Z}_1\begin{pmatrix}1\\ 1\\ \vdots\\ 1\end{pmatrix}\boldsymbol{1}'+\frac{1}{n}\boldsymbol{1}\boldsymbol{1}'.$$

Platí

$$\operatorname{tr}(\boldsymbol{D}_1-\boldsymbol{D}_{12})=\operatorname{tr}\boldsymbol{Z}_1'\boldsymbol{Z}_1\begin{pmatrix}\frac{1}{n_1} & 0 & \dots & 0\\ 0 & \frac{1}{n_2} & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \frac{1}{n_I}\end{pmatrix}-2\frac{1}{n}\operatorname{tr}\boldsymbol{1}'\boldsymbol{Z}_1\begin{pmatrix}1\\ 1\\ \vdots\\ 1\end{pmatrix}+\frac{1}{n}\operatorname{tr}\boldsymbol{1}'\boldsymbol{1}=I-1.$$

Počítajme pomocou (41)

$$(42)\qquad S_2=\boldsymbol{Y}'(\boldsymbol{D}_1-\boldsymbol{D}_{12})\boldsymbol{Y}=\boldsymbol{Y}'\boldsymbol{Z}_1\begin{pmatrix}\frac{1}{n_1} & 0 & \dots & 0\\ 0 & \frac{1}{n_2} & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \frac{1}{n_I}\end{pmatrix}\boldsymbol{Z}_1'\boldsymbol{Y}-$$

$$-\frac{2}{n}\left(\sum_{i=1}^{I}\sum_{j=1}^{n_i}Y_{ij}\right)(1,1,\dots,1)\begin{pmatrix}\sum_{j=1}^{n_1}Y_{1j}\\ \sum_{j=1}^{n_2}Y_{2j}\\ \vdots\\ \sum_{j=1}^{n_I}Y_{Ij}\end{pmatrix}+\frac{1}{n}\left(\sum_{i=1}^{I}\sum_{j=1}^{n_i}Y_{ij}\right)^2=$$

$$=\left(\sum_{j=1}^{n_1}Y_{1j},\sum_{j=1}^{n_2}Y_{2j},\dots,\sum_{j=1}^{n_I}Y_{Ij}\right)\begin{pmatrix}\frac{1}{n_1} & 0 & \dots & 0\\ 0 & \frac{1}{n_2} & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \frac{1}{n_I}\end{pmatrix}\begin{pmatrix}\sum_{j=1}^{n_1}Y_{1j}\\ \sum_{j=1}^{n_2}Y_{2j}\\ \vdots\\ \sum_{j=1}^{n_I}Y_{Ij}\end{pmatrix}-$$

$$-\frac{1}{n}\left(\sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}\right)^2 = \sum_{i=1}^{I}\frac{1}{n_i}\left(\sum_{j=1}^{n_i} Y_{ij}\right)^2 - \frac{1}{n}\left(\sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}\right)^2 .$$

Pretože

(43) $$\hat{\boldsymbol{\nu}}'\hat{\boldsymbol{\nu}} = \boldsymbol{Y}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{Y} = \frac{1}{n}\left(\sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}\right)^2 ,$$

použitím (31) dostávame

(44) $$S_e = \sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}^2 - S_2 - \hat{\boldsymbol{\nu}}'\hat{\boldsymbol{\nu}} = \sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}^2 - \sum_{i=1}^{I}\frac{1}{n_i}\left(\sum_{j=1}^{n_i} Y_{ij}\right)^2 .$$

Pre riešenie rovníc (36) potrebujeme ešte spočítať

$$\operatorname{tr} \boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1 = \operatorname{tr} \boldsymbol{Z}_1'(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}')\boldsymbol{Z}_1 = \operatorname{tr} \boldsymbol{Z}_1'\boldsymbol{Z}_1 - \frac{1}{n}\operatorname{tr} \boldsymbol{Z}_1'\boldsymbol{X}\boldsymbol{X}'\boldsymbol{Z}_1 =$$

$$= \sum_{i=1}^{I} n_i - \frac{1}{n}\sum_{i=1}^{I} n_i^2 = n - \frac{1}{n}\sum_{i=1}^{I} n_i^2.$$

Rovnica (36) má tvar

$$\mathcal{E}\begin{pmatrix} S_2 \\ S_e \end{pmatrix} = \begin{pmatrix} n - \frac{1}{n}\sum_{i=1}^{I} n_i^2 & I-1 \\ 0 & n-I \end{pmatrix}\begin{pmatrix} \sigma_1^2 \\ \sigma_0^2 \end{pmatrix}.$$

Konečne z rovnice (39) dostávame hľadané nevychýlené odhady variančných komponentov

(45) $$\widehat{\sigma_1^2} = \frac{1}{n - \frac{1}{n}\sum_{i=1}^{I} n_i^2} S_2 - \frac{I-1}{\left(n - \frac{1}{n}\sum_{i=1}^{I} n_i^2\right)(n-I)} S_e,$$

(46) $$\widehat{\sigma_0^2} = \frac{1}{n-I} S_e,$$

kde

$$S_2 = \sum_{i=1}^{I}\frac{1}{n_i}\left(\sum_{j=1}^{n_i} Y_{ij}\right)^2 - \frac{1}{n}\left(\sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}\right)^2$$

a

$$S_e = \sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}^2 - \sum_{i=1}^{I}\frac{1}{n_i}\left(\sum_{j=1}^{n_i} Y_{ij}\right)^2 .$$

Pre vyvážený model t.j. v prípade, že $n_1 = n_2 = \ldots = n_I = t$ dostávame

$$\widehat{\sigma_1^2} = \frac{1}{t}\left\{\frac{S_2}{I-1} - \frac{S_e}{n-I}\right\} =$$

$$= \frac{It-1}{I(I-1)t^2(t-1)}\sum_{i=1}^{I}\left(\sum_{j=1}^{t} Y_{ij}\right)^2 - \frac{1}{It(t-1)}\sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}^2 -$$

$$-\frac{1}{I(I-1)t^2}\left(\sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}\right)^2, \tag{47}$$

$$\widehat{\sigma_0^2} = \frac{S_e}{n-I} =$$

$$= \frac{1}{I(t-1)}\sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}^2 - \frac{1}{It(t-1)}\sum_{i=1}^{I}\left(\sum_{j=1}^{t} Y_{ij}\right)^2. \tag{48}$$

Lema 19. *V prípade normality rozdelenia vektora* $\boldsymbol{Y}$ *má* $\frac{S_e}{\sigma_0^2}$ *vo vyváženom modeli* (40) $\chi^2(n-I,0)$ *rozdelenie.*

Dôkaz. $\boldsymbol{Y} \sim N(\boldsymbol{X}\beta; \sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1' + \sigma_0^2\boldsymbol{I})$, pričom $\mathrm{cov}(\boldsymbol{Y})$ je p.d. matica. Podľa (44) je

$$S_e = \boldsymbol{Y}'\boldsymbol{Y} - \frac{1}{t}\boldsymbol{Y}'\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Y} = \boldsymbol{Y}'(\boldsymbol{I} - \boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1')\boldsymbol{Y},$$

teda

$$\frac{S_e}{\sigma_0^2} = \boldsymbol{Y}'\left(\frac{\boldsymbol{I} - \boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'}{\sigma_0^2}\right)\boldsymbol{Y}.$$

Podľa vety 16

(i) $$\left(\frac{\boldsymbol{I} - \boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'}{\sigma_0^2}\right)(\sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1' + \sigma_0^2\boldsymbol{I})\left(\frac{\boldsymbol{I} - \boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'}{\sigma_0^2}\right) =$$

$$= \frac{1}{\sigma_0^4}\{[\sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1' - \sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1' + \sigma_0^2\boldsymbol{I} - \sigma_0^2\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'][\boldsymbol{I} - \boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1']\} =$$

$$= \frac{1}{\sigma_0^4}\{\sigma_0^2\boldsymbol{I} - \sigma_0^2\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'\} = \frac{\boldsymbol{I} - \boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'}{\sigma_0^2},$$

(ii) $$(\sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1' + \sigma_0^2\boldsymbol{I})\left(\frac{\boldsymbol{I} - \boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'}{\sigma_0^2}\right)\boldsymbol{X}\beta =$$

$$= (\boldsymbol{I} - \boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1')\boldsymbol{X}\beta \in \mu(\boldsymbol{I} - \boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'),$$

a podľa (ii)

$$\text{(iii)}\quad \beta\boldsymbol{X}'\left(\frac{\boldsymbol{I}-\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'}{\sigma_0^2}\right)(\sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1'+\sigma_0^2\boldsymbol{I})\left(\frac{\boldsymbol{I}-\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'}{\sigma_0^2}\right)\boldsymbol{X}\beta = \\ = \beta\boldsymbol{X}'\left(\frac{\boldsymbol{I}-\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'}{\sigma_0^2}\right)\boldsymbol{X}\beta.$$

Preto má $\dfrac{S_e}{\sigma_0^2}$ rozdelenie χ^2 s

$$k = \operatorname{tr}\left(\frac{\boldsymbol{I}-\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'}{\sigma_0^2}\right)(\sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1'+\sigma_0^2\boldsymbol{I}) = n-I$$

stupňami voľnosti a s parametrom necentrality

$$\delta = \beta\boldsymbol{X}'\left(\frac{\boldsymbol{I}-\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'}{\sigma_0^2}\right)(\sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1'+\sigma_0^2\boldsymbol{I})\left(\frac{\boldsymbol{I}-\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'}{\sigma_0^2}\right)\times \\ \times(\sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1'+\sigma_0^2\boldsymbol{I})\left(\frac{\boldsymbol{I}-\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-1}\boldsymbol{Z}_1'}{\sigma_0^2}\right)\boldsymbol{X}\beta = \frac{\beta^2}{\sigma_0^2}(n-n) = 0$$

(centrálne $\chi^2(n-I,0)$ rozdelenie). $\square$

Lema 20. *V prípade vyváženého modelu* (40) *(t.j.* $n_1 = n_2 = \ldots = n_I = t$*) má* $\dfrac{S_2}{t\sigma_1^2+\sigma_0^2}$ *rozdelenie* $\chi^2(I-1,0)$.

Dôkaz. Podľa (42) je

$$S_2 = \boldsymbol{Y}'\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{Z}_1)^{-}\boldsymbol{Z}_1'\boldsymbol{Y} - \frac{1}{n}\boldsymbol{Y}'\boldsymbol{X}\boldsymbol{X}'\boldsymbol{Y} = \boldsymbol{Y}'\left(\frac{1}{t}\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{n}\boldsymbol{X}\boldsymbol{X}'\right)\boldsymbol{Y}.$$

Postupujeme úplne analogicky ako v leme 19 a dokážeme (i), (ii) a (iii) podľa vety 16 pre maticu kvadratickej formy

$$\frac{\frac{1}{t}\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{n}\boldsymbol{X}\boldsymbol{X}'}{t\sigma_1^2+\sigma_0^2}.$$

Vskutku, $\dfrac{S_2}{t\sigma_1^2+\sigma_0^2}$ má $\chi^2(I-1,0)$ rozdelenie, lebo pre stupne voľnosti k platí

$$k = \operatorname{tr}\frac{\frac{1}{t}\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{n}\boldsymbol{X}\boldsymbol{X}'}{t\sigma_1^2+\sigma_0^2}(\sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1'+\sigma_0^2\boldsymbol{I}) = I-1$$

a koeficient necentrality δ je

$$\delta = \beta\boldsymbol{X}'\frac{\frac{1}{t}\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{n}\boldsymbol{X}\boldsymbol{X}'}{t\sigma_1^2+\sigma_0^2}(\sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1'+\sigma_0^2\boldsymbol{I})\frac{\frac{1}{t}\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{n}\boldsymbol{X}\boldsymbol{X}'}{t\sigma_1^2+\sigma_0^2}\times \\ \times(\sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1'+\sigma_0^2\boldsymbol{I})\frac{\frac{1}{t}\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{n}\boldsymbol{X}\boldsymbol{X}'}{t\sigma_1^2+\sigma_0^2}\boldsymbol{X}\beta = \\ = \frac{\beta^2}{t\sigma_1^2+\sigma_0^2}\boldsymbol{X}'\left(\frac{1}{t}\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{n}\boldsymbol{X}\boldsymbol{X}'\right)\boldsymbol{X} = \\ = \frac{\beta^2}{t\sigma_1^2+\sigma_0^2}\left(\frac{1}{t}\boldsymbol{1}'\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{1} - \frac{1}{n}\boldsymbol{1}'\boldsymbol{1}\boldsymbol{1}'\boldsymbol{1}\right) = 0. \qquad \square$$

Lema 21. *V prípade vyváženého modelu* (40) *a normálne rozdeleného* $\boldsymbol{Y}$ *sú*

$$\frac{S_e}{\sigma_0^2} \quad \text{a} \quad \frac{S_2}{t\sigma_1^2 + \sigma_0^2}$$

nezávislé.

Dôkaz. Pretože $\boldsymbol{Y}$ má nesingulárnu kovariančnú maticu a matice kvadratických foriem $\frac{S_e}{\sigma_0^2}$ a $\frac{S_2}{t\sigma_1^2 + \sigma_0^2}$ sú symetrické, stačí podľa vety 17 ukázať, že

$$\frac{1}{\sigma_0^2(t\sigma_1^2 + \sigma_0^2)}(\boldsymbol{I} - \frac{1}{t}\boldsymbol{Z}_1\boldsymbol{Z}_1')(\sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1' + \sigma_0^2\boldsymbol{I})(\frac{1}{t}\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{n}\boldsymbol{X}\boldsymbol{X}') = \boldsymbol{0}.$$

Toto dostaneme po krátkom výpočte. $\square$

V prípade vyváženého modelu môžeme štatistikou

$$\frac{(n-I)}{(I-1)}\frac{S_2}{S_e} \tag{49}$$

testovať hypotézu $H_0\colon \sigma_1^2 = 0$. Za platnosti H_0 má (49) $F_{I-1,n-I}$ rozdelenie (podľa lemy 19, lemy 20 a lemy 21).

Cvičenie

1. Ukážte, že vo vyváženom modeli s jedným náhodným efektom v prípade normality rozdelenia $\boldsymbol{b}$ a $\boldsymbol{\varepsilon}$ sa disperzie odhadov (47) a (48) rovnajú

$$\mathcal{D}(\widehat{\sigma_1^2}) = \frac{2(\sigma_1^2 t + \sigma_0^2)^2}{t^2(I-1)} + \frac{2\sigma_0^4}{It^2(t-1)},$$

$$\mathcal{D}(\widehat{\sigma_0^2}) = \frac{2\sigma_0^4}{I(t-1)}.$$

7 ODHADY VARIANČNÝCH KOMPONENTOV VO VYVÁŽENOM MODELI S DVOMI NÁHODNÝMI EFEKTMI A INTERAKCIOU

Vyvážený model s dvomi náhodnými efektmi a interakciou sa dá písať ako

$$Y_{ijk} = \beta + b_{1i} + b_{2j} + b_{3ij} + \varepsilon_{ijk}, \qquad \begin{aligned} i &= 1,2,\dots,I\,,\\ j &= 1,2,\dots,J\,,\\ k &= 1,2,\dots,K\,. \end{aligned} \tag{50}$$

Náhodný observačný vektor $\boldsymbol{Y} \sim N(\mathbf{1}_n\beta, \mathrm{cov}(\boldsymbol{Y}))$, $\mathbf{1} = (1, 1, \dots, 1)' \in \mathbb{R}^n$, $n = IJK$; náhodné vektory $\boldsymbol{b}_1 \sim N(\mathbf{0}_{I,1}, \sigma_1^2 \boldsymbol{I})$, $\boldsymbol{b}_2 \sim N(\mathbf{0}_{J,1}, \sigma_2^2 \boldsymbol{I})$, $\boldsymbol{b}_3 \sim N(\mathbf{0}_{IJ,1}, \sigma_3^2 \boldsymbol{I})$ aj $\boldsymbol{\varepsilon} \sim N(\mathbf{0}_{n,1}, \sigma_0^2 \boldsymbol{I})$ sú navzájom nezávislé, β je skalár (číslo, neznáma konštanta). Model (50) sa dá písať aj v tvare

(51)
$$\boldsymbol{Y}_{n,1} = (\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2 \vdots \boldsymbol{Z}_3) \begin{pmatrix} \beta \\ \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix} + \boldsymbol{\varepsilon} =$$

$$= \boldsymbol{X}\beta + \boldsymbol{Z}_1\boldsymbol{b}_1 + \boldsymbol{Z}_2\boldsymbol{b}_2 + \boldsymbol{Z}_3\boldsymbol{b}_3 + \boldsymbol{\varepsilon},$$

kde $\boldsymbol{X} = \mathbf{1}_n$, $\boldsymbol{Z}_1$ je rozmerov $n \times I$, $\boldsymbol{Z}_2$ je rozmerov $n \times J$ a $\boldsymbol{Z}_3$ je rozmerov $n \times IJ$.

$$\mathcal{E}(\boldsymbol{Y}) = \boldsymbol{X}\beta = \mathbf{1}\beta,$$

$$\mathrm{cov}(\boldsymbol{Y}) = \sigma_1^2 \boldsymbol{Z}_1\boldsymbol{Z}_1' + \sigma_2^2 \boldsymbol{Z}_2\boldsymbol{Z}_2' + \sigma_3^2 \boldsymbol{Z}_3\boldsymbol{Z}_3' + \sigma_0^2 \boldsymbol{I}.$$

Platí
$$\boldsymbol{Z}_1 = \boldsymbol{I}_{I,I} \otimes \mathbf{1}_{JK} = \boldsymbol{I}_{I,I} \otimes (\mathbf{1}_J \otimes \mathbf{1}_K)$$

($\otimes$ znamená Kroneckerov súčin matíc, t.j. ak prvky matice $\boldsymbol{A}$ sú $\{\boldsymbol{A}\}_{ij}$, prvky matice $\boldsymbol{B}$ sú $\{\boldsymbol{B}\}_{ij}$, tak prvky matice $\boldsymbol{A} \otimes \boldsymbol{B}$ sú $\{\boldsymbol{A}\}_{ij}\boldsymbol{B}$),

$$\boldsymbol{Z}_2 = \begin{pmatrix} \boldsymbol{I}_{J,J} \\ \boldsymbol{I}_{J,J} \\ \vdots \\ \boldsymbol{I}_{J,J} \end{pmatrix} \otimes \mathbf{1}_K = (\mathbf{1}_I \otimes \boldsymbol{I}_{J,J}) \otimes \mathbf{1}_K$$

a
$$\boldsymbol{Z}_3 = \boldsymbol{I}_{IJ,IJ} \otimes \mathbf{1}_K = \boldsymbol{I}_{I,I} \otimes \boldsymbol{I}_{J,J} \otimes \mathbf{1}_K.$$

Ľahko sa ukáže, že

$$\begin{aligned}
\boldsymbol{X}'\boldsymbol{X} &= IJK, \\
\boldsymbol{X}'\boldsymbol{Z}_1 &= JK(1, 1, \dots, 1)_{1,I} = JK\mathbf{1}_I', \\
\boldsymbol{X}'\boldsymbol{Z}_2 &= IK(1, 1, \dots, 1)_{1,J} = IK\mathbf{1}_J', \\
\boldsymbol{X}'\boldsymbol{Z}_3 &= K(1, 1, \dots, 1)_{1,IJ} = K\mathbf{1}_{IJ}', \\
\boldsymbol{Z}_1'\boldsymbol{Z}_1 &= JK\boldsymbol{I}_{I,I}, \\
\boldsymbol{Z}_1'\boldsymbol{Z}_2 &= K(\mathbf{1}_I \otimes \mathbf{1}_J'), \\
\boldsymbol{Z}_1'\boldsymbol{Z}_3 &= K\ (\boldsymbol{I}_{I,I} \otimes \mathbf{1}_J'), \\
\boldsymbol{Z}_2'\boldsymbol{Z}_2 &= IK\boldsymbol{I}_{J,J}, \\
\boldsymbol{Z}_2'\boldsymbol{Z}_3 &= K\ (\mathbf{1}_I' \otimes \boldsymbol{I}_{J,J}), \\
\boldsymbol{Z}_3'\boldsymbol{Z}_3 &= K\ (\boldsymbol{I}_{I,I} \otimes \boldsymbol{I}_{J,J}).
\end{aligned}$$

V ďalšom budeme potrebovať ešte nasledujúce tvrdenie:

Lema 22. *Ak m je prirodzené číslo, tak jedna g-inverzia matice*

$$\begin{pmatrix} 1-\frac{1}{m} & -\frac{1}{m} & -\frac{1}{m} & \dots & -\frac{1}{m} \\ -\frac{1}{m} & 1-\frac{1}{m} & -\frac{1}{m} & \dots & -\frac{1}{m} \\ -\frac{1}{m} & -\frac{1}{m} & 1-\frac{1}{m} & \dots & -\frac{1}{m} \\ \vdots & \vdots & & \ddots & \vdots \\ -\frac{1}{m} & -\frac{1}{m} & -\frac{1}{m} & \dots & 1-\frac{1}{m} \end{pmatrix}$$

je matica $\boldsymbol{I}_{m,m}$.

Dôkaz je jednoduchý a urobte ho ako cvičenie.
Pri odhade variančných komponentov budeme postupovať podľa návodu zo str. 17.
Z postupnosti modelov (formálne predpokladáme pevné parametre $\boldsymbol{b}_i$, $i = 1, 2, 3$)

$$\begin{aligned} M: \mathcal{E}(\boldsymbol{Y}) &= (\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2 \vdots \boldsymbol{Z}_3) \begin{pmatrix} \beta \\ \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \boldsymbol{b}_3 \end{pmatrix}, & \operatorname{cov}(\boldsymbol{Y}) &= \sigma_0^2 \boldsymbol{I}; \\ M_1: \mathcal{E}(\boldsymbol{Y}) &= (\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2) \begin{pmatrix} \beta \\ \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \end{pmatrix}, & \operatorname{cov}(\boldsymbol{Y}) &= \sigma_0^2 \boldsymbol{I}; \\ M_2: \mathcal{E}(\boldsymbol{Y}) &= (\boldsymbol{X} \vdots \boldsymbol{Z}_1) \begin{pmatrix} \beta \\ \boldsymbol{b}_1 \end{pmatrix}, & \operatorname{cov}(\boldsymbol{Y}) &= \sigma_0^2 \boldsymbol{I}; \\ M_3: \mathcal{E}(\boldsymbol{Y}) &= \boldsymbol{X}\beta, & \operatorname{cov}(\boldsymbol{Y}) &= \sigma_0^2 \boldsymbol{I}. \end{aligned}$$

dostaneme štatistiky

$$\hat{\boldsymbol{\mu}} = (\boldsymbol{X} \vdots \boldsymbol{Z}) \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z} \\ \boldsymbol{Z}'\boldsymbol{X} & \boldsymbol{Z}'\boldsymbol{Z} \end{pmatrix}^{-} \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}' \end{pmatrix} \boldsymbol{Y},$$

$$\hat{\boldsymbol{\nu}} = (\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2) \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}_1 & \boldsymbol{X}'\boldsymbol{Z}_2 \\ \boldsymbol{Z}_1'\boldsymbol{X} & \boldsymbol{Z}_1'\boldsymbol{Z}_1 & \boldsymbol{Z}_1'\boldsymbol{Z}_2 \\ \boldsymbol{Z}_2'\boldsymbol{X} & \boldsymbol{Z}_2'\boldsymbol{Z}_1 & \boldsymbol{Z}_2'\boldsymbol{Z}_2 \end{pmatrix}^{-} \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}_1' \\ \boldsymbol{Z}_2' \end{pmatrix} \boldsymbol{Y},$$

$$\hat{\boldsymbol{\eta}} = (\boldsymbol{X} \vdots \boldsymbol{Z}_1) \begin{pmatrix} \boldsymbol{X}'\boldsymbol{X} & \boldsymbol{X}'\boldsymbol{Z}_1 \\ \boldsymbol{Z}_1'\boldsymbol{X} & \boldsymbol{Z}_1'\boldsymbol{Z}_1 \end{pmatrix}^{-} \begin{pmatrix} \boldsymbol{X}' \\ \boldsymbol{Z}_1' \end{pmatrix} \boldsymbol{Y}$$

a

$$\hat{\boldsymbol{\tau}} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{Y}.$$

Platí (podľa (31))

$$S_e = \boldsymbol{Y}'\boldsymbol{Y} - S_1 - S_2 - S_3 - S_4,$$

pričom podľa (27)

$$S_1 = \boldsymbol{Y}'(\boldsymbol{I} - \boldsymbol{D}_1)\boldsymbol{Y}, \quad \text{kde} \quad \boldsymbol{D}_1 = \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}',$$

podľa (28)

$$S_2 = \boldsymbol{Y}'(\boldsymbol{D}_1 - \boldsymbol{D}_{12})\boldsymbol{Y}, \quad \text{kde} \quad \boldsymbol{D}_{12} = \boldsymbol{D}_1 - \boldsymbol{D}_1\boldsymbol{Z}_1(\boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1)^{-}\boldsymbol{Z}_1'\boldsymbol{D}_1,$$

podľa (29)

$$S_3 = \boldsymbol{Y}'(\boldsymbol{D}_{12} - \boldsymbol{D}_{123})\boldsymbol{Y}, \quad \text{kde} \quad \boldsymbol{D}_{123} = \boldsymbol{D}_{12} - \boldsymbol{D}_{12}\boldsymbol{Z}_2(\boldsymbol{Z}_2'\boldsymbol{D}_{12}\boldsymbol{Z}_2)^-\boldsymbol{Z}_2'\boldsymbol{D}_{12}$$

a podľa (30)

$$S_4 = \boldsymbol{Y}'(\boldsymbol{D}_{123} - \boldsymbol{D}_{1234})\boldsymbol{Y} = S_4, \quad \text{kde} \quad \boldsymbol{D}_{1234} = \boldsymbol{D}_{123} - \boldsymbol{D}_{123}\boldsymbol{Z}_3(\boldsymbol{Z}_3'\boldsymbol{D}_{123}\boldsymbol{Z}_3)^-\boldsymbol{Z}_3'\boldsymbol{D}_{123}.$$

Aby sme určili odhady $\widehat{\sigma_1^2}, \widehat{\sigma_2^2}, \widehat{\sigma_3^2}$ a $\widehat{\sigma_0^2}$, potrebujeme spočítať matice $\boldsymbol{D}_1, \boldsymbol{D}_{12}, \boldsymbol{D}_{123}$, $\boldsymbol{D}_{1234}$ a prvky matice $\boldsymbol{E}$ z (36) (alebo matice $\boldsymbol{E}^{-1}$ z (39)).

Lema 23. *V modeli* (50) *majú matice* $\boldsymbol{D}_1$ *a* $\boldsymbol{D}_{12}$ *(pozri* (27) *a* (28)*) tvar*

$$\boldsymbol{D}_1 = \boldsymbol{I}_{IJK,IJK} - \frac{1}{IJK}\mathbf{1}_{IJK}\mathbf{1}_{IJK}',$$

$$\boldsymbol{D}_{12} = \boldsymbol{I}_{IJK,IJK} - \frac{1}{JK}\boldsymbol{Z}_1\boldsymbol{Z}_1'.$$

Kvadratické formy S_1 *a* S_2 *sú*

$$S_1 = \boldsymbol{Y}'\left(\frac{1}{IJK}\mathbf{1}\mathbf{1}'\right)\boldsymbol{Y} = \frac{1}{IJK}\left(\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}Y_{ijk}\right)^2,$$

$$S_2 = \boldsymbol{Y}'\left(\frac{1}{JK}\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{IJK}\mathbf{1}\mathbf{1}'\right)\boldsymbol{Y} =$$

$$= \frac{1}{JK}\sum_{i=1}^{I}\left(\sum_{j=1}^{J}\sum_{k=1}^{K}Y_{ijk}\right)^2 - \frac{1}{IJK}\left(\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}Y_{ijk}\right)^2.$$

Dôkaz urobte ako cvičenie.

Lema 24. *V modeli* (50) *majú matice* $\boldsymbol{D}_{123}$ *a* $\boldsymbol{D}_{123} - \boldsymbol{D}_{1234}$ *(pozri* (29) *a* (30)*) tvar*

$$\boldsymbol{D}_{123} = \boldsymbol{I}_{IJK,IJK} - \frac{1}{JK}\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{IK}\boldsymbol{Z}_2\boldsymbol{Z}_2' + \frac{1}{IJK}\mathbf{1}\mathbf{1}'.$$

$$\boldsymbol{D}_{123} - \boldsymbol{D}_{1234} = \frac{1}{K}\boldsymbol{Z}_3\boldsymbol{Z}_3' - \frac{1}{JK}\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{IK}\boldsymbol{Z}_2\boldsymbol{Z}_2' + \frac{1}{IJK}\mathbf{1}\mathbf{1}'.$$

Dôkaz urobte ako cvičenie.

Lema 25. *V modeli* (50) *kvadratické formy* S_3 *a* S_4 *sú*

$$S_3 = \boldsymbol{Y}'\left(\frac{1}{IK}\boldsymbol{Z}_2\boldsymbol{Z}_2' - \frac{1}{IJK}\mathbf{1}\mathbf{1}'\right)\boldsymbol{Y} =$$

$$= \frac{1}{IK}\sum_{j=1}^{J}\left(\sum_{i=1}^{I}\sum_{k=1}^{K}Y_{ijk}\right)^2 - \frac{1}{IJK}\left(\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}Y_{ijk}\right)^2,$$

$$S_4 = \boldsymbol{Y}'\left(\frac{1}{K}\boldsymbol{Z}_3\boldsymbol{Z}_3' - \frac{1}{JK}\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{IK}\boldsymbol{Z}_2\boldsymbol{Z}_2' + \frac{1}{IJK}\boldsymbol{1}\boldsymbol{1}'\right)\boldsymbol{Y} =$$

$$= \frac{1}{K}\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\sum_{k=1}^{K}Y_{ijk}\right)^2 - \frac{1}{JK}\sum_{i=1}^{I}\left(\sum_{j=1}^{J}\sum_{k=1}^{K}Y_{ijk}\right)^2 -$$

$$-\frac{1}{IK}\sum_{j=1}^{J}\left(\sum_{i=1}^{I}\sum_{k=1}^{K}Y_{ijk}\right)^2 + \frac{1}{IJK}\left(\sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}Y_{ijk}\right)^2.$$

Dôkaz urobte ako cvičenie.

Lema 26. *V modeli* (50) *je kvadratická forma*

$$S_e = \boldsymbol{Y}'\left(\boldsymbol{I} - \frac{1}{K}\boldsymbol{Z}_3\boldsymbol{Z}_3'\right)\boldsymbol{Y} = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K}Y_{ijk}^2 - \frac{1}{K}\sum_{i=1}^{I}\sum_{j=1}^{J}\left(\sum_{k=1}^{K}Y_{ijk}\right)^2,$$

$\frac{S_e}{\sigma_0^2}$ *má* $\chi^2(IJ(K-1),0)$ *rozdelenie.*

Dôkaz. Využite (31) a urobte ho ako cvičenie.

Lema 27. *V modeli* (50) *platí*

$$\begin{aligned}
&\operatorname{tr}\boldsymbol{Z}_1'\boldsymbol{D}_1\boldsymbol{Z}_1 = JK(I-1),\\
&\operatorname{tr}\boldsymbol{Z}_2'(\boldsymbol{D}_1-\boldsymbol{D}_{12})\boldsymbol{Z}_2 = 0,\\
&\operatorname{tr}\boldsymbol{Z}_3'(\boldsymbol{D}_1-\boldsymbol{D}_{12})\boldsymbol{Z}_3 = K(I-1),\\
&\operatorname{tr}(\boldsymbol{D}_1-\boldsymbol{D}_{12}) = I-1,\\
&\operatorname{tr}\boldsymbol{Z}_2'\boldsymbol{D}_{12}\boldsymbol{Z}_2 = IK(J-1),\\
&\operatorname{tr}\boldsymbol{Z}_3'(\boldsymbol{D}_{12}-\boldsymbol{D}_{123})\boldsymbol{Z}_3 = K(J-1),\\
&\operatorname{tr}(\boldsymbol{D}_{12}-\boldsymbol{D}_{123}) = J-1,\\
&\operatorname{tr}\boldsymbol{Z}_3'\boldsymbol{D}_{123}\boldsymbol{Z}_3 = K(I-1)(J-1),\\
&\operatorname{tr}(\boldsymbol{D}_{123}-\boldsymbol{D}_{1234}) = (I-1)(J-1),\\
&\operatorname{tr}\boldsymbol{D}_{1234} = IJ(K-1).
\end{aligned}$$

Dôkaz. (Pri dôkaze posledného vzťahu využite (31) a lemu 12.) Urobte ho ako cvičenie.

Lema 28. *V modeli* (50) *sú kvadratické nevychýlené odhady* $\widehat{\sigma_1^2}, \widehat{\sigma_2^2}, \widehat{\sigma_3^2}$ *a* $\widehat{\sigma_0^2}$ *z rovnice* (36)

$$\begin{aligned}
\widehat{\sigma_0^2} &= \frac{S_e}{IJ(K-1)},\\
\widehat{\sigma_3^2} &= \frac{1}{K}\left(\frac{S_4}{(I-1)(J-1)} - \frac{S_e}{IJ(K-1)}\right),\\
\widehat{\sigma_2^2} &= \frac{1}{IK}\left(\frac{S_3}{J-1} - \frac{S_4}{(I-1)(J-1)}\right),\\
\widehat{\sigma_1^2} &= \frac{1}{JK}\left(\frac{S_2}{I-1} - \frac{S_4}{(I-1)(J-1)}\right).
\end{aligned}$$

Dôkaz urobte ako cvičenie.

Lema 29. *V modeli* (50) *má* $\dfrac{S_4}{\sigma_0^2 + K\sigma_3^2}$ *rozdelenie* $\chi^2((I-1)(J-1), 0)$.

Dôkaz urobte ako cvičenie.

Lema 30. *V modeli* (50) *má* $\dfrac{S_3}{\sigma_0^2 + K\sigma_3^2 + IK\sigma_2^2}$ *rozdelenie* $\chi^2(J-1, 0)$.

Dôkaz urobte ako cvičenie.

Lema 31. *V modeli* (50) *má* $\dfrac{S_2}{\sigma_0^2 + K\sigma_3^2 + JK\sigma_1^2}$ *rozdelenie* $\chi^2(I-1, 0)$.

Dôkaz urobte ako cvičenie.

Lema 32. *V modeli* (50) *sú kvadratické formy* $\dfrac{S_e}{\sigma_0^2}$ a $\dfrac{S_4}{\sigma_0^2 + K\sigma_3^2}$ *nezávislé.*

Dôkaz urobte ako cvičenie.

Lema 33. *V modeli* (50) *sú kvadratické formy* $\dfrac{S_2}{\sigma_0^2 + K\sigma_3^2 + JK\sigma_1^2}$ a $\dfrac{S_4}{\sigma_0^2 + K\sigma_3^2}$ *nezávislé.*

Dôkaz urobte ako cvičenie.

Lema 34. *V modeli* (50) *sú kvadratické formy* $\dfrac{S_3}{\sigma_0^2 + K\sigma_3^2 + IK\sigma_2^2}$ a $\dfrac{S_4}{\sigma_0^2 + K\sigma_3^2}$ *nezávislé.*

Dôkaz urobte ako cvičenie.

Testujme v modeli (50) hypotézu

$$H_0\colon \quad \sigma_3^2 = 0.$$

Za platnosti H_0 má

$$\frac{\frac{S_4}{(I-1)(J-1)}}{\frac{S_e}{IJ(K-1)}} = \frac{IJ(K-1)}{(I-1)(J-1)}\frac{S_4}{S_e} \tag{52}$$

rozdelenie $F_{(I-1)(J-1),IJ(K-1)}$ (podľa lemy 26, lemy 29 a lemy 32; $I, J, K > 1$).

Testujme v modeli (50) hypotézu

$$H_0\colon \quad \sigma_1^2 = 0.$$

Za platnosti H_0 má

$$\frac{\frac{S_2}{I-1}}{\frac{S_4}{(I-1)(J-1)}} = (J-1)\frac{S_2}{S_4} \tag{53}$$

rozdelenie $F_{I-1,(I-1)(J-1)}$ (podľa lemy 29, lemy 31 a lemy 33; $I, J > 1$).

Testujme v modeli (50) hypotézu

$$H_0\colon \quad \sigma_2^2 = 0.$$

Za platnosti H_0 má

(54) $$\frac{\frac{S_3}{J-1}}{\frac{S_4}{(I-1)(J-1)}} = (I-1)\frac{S_3}{S_4}$$

rozdelenie $F_{J-1,(I-1)(J-1)}$ (podľa lemy 29, lemy 30 a lemy 34; $I, J > 1$).

8 MINQUE ODHADY VARIANČNÝCH KOMPONENTOV

Uvažujme zmiešaný lineárny model

(55) $$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{Z}_1\boldsymbol{b}_1 + \boldsymbol{Z}_2\boldsymbol{b}_2 + \ldots + \boldsymbol{Z}_s\boldsymbol{b}_s + \boldsymbol{Z}_{s+1}\boldsymbol{b}_{s+1}.$$

(Ak $\boldsymbol{Z}_{s+1}$ označíme $\boldsymbol{Z}_0$, $\boldsymbol{b}_{s+1}$ označíme $\boldsymbol{b}_0$, σ_{s+1}^2 označíme σ_0^2, dostávame model (15).) Kovariančná matica

$$\operatorname{cov}(\boldsymbol{Y}) = \sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1' + \sigma_2^2\boldsymbol{Z}_2\boldsymbol{Z}_2' + \ldots + \sigma_s^2\boldsymbol{Z}_s\boldsymbol{Z}_s' + \sigma_{s+1}^2\boldsymbol{Z}_{s+1}\boldsymbol{Z}_{s+1}' = \sum_{i=1}^{s+1}\sigma_i^2\boldsymbol{V}_i.$$

Náhodné vektory $\boldsymbol{b}_1, \boldsymbol{b}_2, \ldots, \boldsymbol{b}_{s+1}$ sa nazývajú hypotetické (alebo tiež nepozorovateľné) premenné.

Uvažujme kvadratický odhad $\boldsymbol{Y}'\boldsymbol{AY}$ funkcie

(56) $$\boldsymbol{p}'\boldsymbol{\sigma} = (p_1, p_2, \ldots, p_{s+1})\begin{pmatrix}\sigma_1^2\\ \vdots\\ \sigma_s^2\\ \sigma_{s+1}^2\end{pmatrix},$$

kde $\boldsymbol{p}$ je daný $(s+1)$-rozmerný vektor. Požadujeme symetriu matice $\boldsymbol{A}$ kvadratickej formy $\boldsymbol{Y}'\boldsymbol{AY}$. Ďalej chceme, aby

(i) odhad $\boldsymbol{Y}'\boldsymbol{AY}$ bol invariantný, t.j.

(57) $$\forall \boldsymbol{\delta} \in \mathbb{R}^q \quad \boldsymbol{Y}'\boldsymbol{AY} = (\boldsymbol{Y} - \boldsymbol{X\delta})'\boldsymbol{A}(\boldsymbol{Y} - \boldsymbol{X\delta});$$

(ii) odhad $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ bol nevychýleným odhadom (56), t.j.

$$\forall\{\sigma_i^2 \in (0,\infty) \;\; i = 1,2,\dots,s+1\} \quad \mathcal{E}(\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}) = \sum_{i=1}^{s+1} p_i\sigma_i^2; \tag{58}$$

(iii) $\| \boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z} - \boldsymbol{\Delta} \|^2 = \mathrm{tr}(\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z} - \boldsymbol{\Delta})(\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z} - \boldsymbol{\Delta})' = \min$,
kde

$$\boldsymbol{Z} = (\boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2 \vdots \dots \vdots \boldsymbol{Z}_{s+1})$$

a

$$\boldsymbol{\Delta} = \begin{pmatrix} \frac{p_1}{m_1}\boldsymbol{I}_{m_1,m_1} & 0 & \dots & 0 \\ 0 & \frac{p_2}{m_2}\boldsymbol{I}_{m_2,m_2} & \dots & 0 \\ \vdots & & & \\ 0 & 0 & \dots & \frac{p_{s+1}}{m_{s+1}}\boldsymbol{I}_{m_{s+1},m_{s+1}}. \end{pmatrix}.$$

Odhad splňujúci (i), (ii), (iii), pričom $\boldsymbol{A} = \boldsymbol{A}'$ sa nazýva MINQUE (minimum norm quadratic unbiased estimator).

Poznámka 35. *Podľa* [2], str. 71 *platí*

$$P\{\boldsymbol{Y} \in \mu(\boldsymbol{X} \vdots \mathrm{cov}(\boldsymbol{Y}))\} = 1. \tag{59}$$

Lema 36. *Kvadratický odhad $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ ($\boldsymbol{A} = \boldsymbol{A}'$) je invariantný s pravdepodobnosťou 1 práve vtedy ak $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{0}$.*

Dôkaz. Odhad $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ je invariantný

$$\Longleftrightarrow \forall\boldsymbol{\delta} \in \mathbb{R}^q \;\; \boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y} = (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\delta})'\boldsymbol{A}(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\delta}) = \boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y} - 2\boldsymbol{\delta}'\boldsymbol{X}'\boldsymbol{A}\boldsymbol{Y} + \boldsymbol{\delta}'\boldsymbol{X}'\boldsymbol{A}\,\boldsymbol{X}\boldsymbol{\delta}$$

$$\Longleftrightarrow \;\; \boldsymbol{X}'\boldsymbol{A}\boldsymbol{Y} = \boldsymbol{0} \;\;\&\;\; \boldsymbol{X}'\boldsymbol{A}\boldsymbol{X} = \boldsymbol{0}. \tag{60}$$

Pretože podľa (59) $P\{\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{u} + \mathrm{cov}(\boldsymbol{Y})\boldsymbol{w}\} = 1$, platí (60) s pravdepodobnosťou 1 práve ak $\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X} = \boldsymbol{0}$ & $\boldsymbol{X}'\boldsymbol{A}\;\mathrm{cov}(\boldsymbol{Y}) = \boldsymbol{0}$.

Je zrejmé, že ak $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{0}$, je odhad $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ ($\boldsymbol{A} = \boldsymbol{A}'$) invariantný (s pravdepodobnosťou 1).

Naopak, ak je odhad $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ ($\boldsymbol{A} = \boldsymbol{A}'$) invariantný, teda platí (60), tak s pravdepodobnosťou 1 je

$$\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y} = \boldsymbol{Y}'[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}' + \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']'\boldsymbol{A} \\ [\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}' + \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']\boldsymbol{Y} =$$

$$= \boldsymbol{Y}'[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']'\boldsymbol{A}[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']\boldsymbol{Y}, \tag{61}$$

lebo s pravdepodobnosťou 1 je

$$\boldsymbol{Y}'[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']'\boldsymbol{A}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y} =$$

$$= \boldsymbol{Y}'[\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']'\boldsymbol{A}[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']\boldsymbol{Y} =$$
$$= \boldsymbol{Y}'[\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']'\boldsymbol{A}[\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']\boldsymbol{Y} = 0.$$

Zo vzťahu (61) vidíme, že s pravdepodobnosťou 1 sa $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ rovná

$$\boldsymbol{Y}'[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']'\boldsymbol{A}[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']\boldsymbol{Y},$$

pričom

$$[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']'\boldsymbol{A}[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']$$

je symetrická a

$$[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']'\boldsymbol{A}[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}']\boldsymbol{X} = \boldsymbol{0}. \qquad \square$$

Lema 37. *Kvadratický invariantný odhad* $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ *(*$\boldsymbol{A} = \boldsymbol{A}'$*) je nevychýleným odhadom* (56) *práve vtedy ak*

(62) $$\operatorname{tr} \boldsymbol{A}\ \boldsymbol{Z}_i\boldsymbol{Z}_i' = p_i, \qquad i = 1, 2, \ldots, s+1.$$

Dôkaz. Invariantný odhad $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ je nevychýleným odhadom (56) práve vtedy ak

$$\forall\{\sigma_i^2 \in (0,\infty) \quad i = 1, 2, \ldots, s+1\} \quad \mathcal{E}(\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}) = \boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}\boldsymbol{\beta} + \operatorname{tr} \boldsymbol{A}\ \operatorname{cov}(\boldsymbol{Y}) =$$
$$= \boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}\boldsymbol{\beta} + \sum_{i=1}^{s+1} \sigma_i^2 \operatorname{tr} \boldsymbol{A}\ \boldsymbol{Z}_i\boldsymbol{Z}_i' = \sum_{i=1}^{s+1} p_i\sigma_i^2,$$

čo je splnené práve vtedy ak

$$\operatorname{tr} \boldsymbol{A}\ \boldsymbol{Z}_i\boldsymbol{Z}_i' = p_i, \qquad i = 1, 2, \ldots, s+1. \qquad \square$$

Poznámka 38. *Keby sme mohli (izolovane) realizovať hypotetickú premennú* $\boldsymbol{b}_i$ $i \in \{1, 2, \ldots, s+1\}$, *ktorej stredná hodnota je* $\boldsymbol{0}_{m_i,1}$ *a* $\operatorname{cov}(\boldsymbol{b}_i) = \sigma_i^2\boldsymbol{I}_{m_i,m_i}$, *tak „prirodzený" odhad* σ_i^2 *by bol*

$$\widehat{\sigma_i^2} = \frac{\boldsymbol{b}_i'\boldsymbol{b}_i}{m_i}.$$

(Podľa lemy 10 je to nevychýlený odhad σ_i^2*.) „Prirodzený odhad"* (56) *je*

$$\widehat{\boldsymbol{p}'\boldsymbol{\sigma}} = (\boldsymbol{b}_1' \vdots \boldsymbol{b}_2' \vdots \ldots \vdots \boldsymbol{b}_{s+1}')\boldsymbol{\Delta} \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_{s+1} \end{pmatrix}.$$

Rozdiel medzi „prirodzeným" odhadom a invariantným nevychýleným odhadom $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ *(*$\boldsymbol{A} = \boldsymbol{A}'$*) funkcie* (56) *je*

$$\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y} - (\boldsymbol{b}_1' \vdots \boldsymbol{b}_2' \vdots \ldots \vdots \boldsymbol{b}_{s+1}')\boldsymbol{\Delta} \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_{s+1} \end{pmatrix} =$$

$$= (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})'\boldsymbol{A}\ (\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}) - (\boldsymbol{b}_1' \vdots \boldsymbol{b}_2' \vdots \dots \vdots \boldsymbol{b}_{s+1}')\boldsymbol{\Delta} \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_{s+1} \end{pmatrix} =$$

$$= (\boldsymbol{b}_1' \vdots \boldsymbol{b}_2' \vdots \dots \vdots \boldsymbol{b}_{s+1}')[\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z} - \boldsymbol{\Delta}] \begin{pmatrix} \boldsymbol{b}_1 \\ \boldsymbol{b}_2 \\ \vdots \\ \boldsymbol{b}_{s+1} \end{pmatrix}.$$

Jedna z možností minimalizovať tento rozdiel je (iii) (minimalizácia euklidovskej normy).

Veta 39. *Nech v modeli* (55) *je matica* $\widetilde{\mathbf{V}} = \boldsymbol{Z}\boldsymbol{Z}' = \boldsymbol{Z}_1\boldsymbol{Z}_1' + \boldsymbol{Z}_2\boldsymbol{Z}_2' + \dots + \boldsymbol{Z}_{s+1}\boldsymbol{Z}_{s+1}'$ *nesingulárna. Kvadratická forma* $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$, *kde*

$$(63)\quad \boldsymbol{A} = \sum_{i=1}^{s+1} \lambda_i [\widetilde{\mathbf{V}}^{-1} - \widetilde{\mathbf{V}}^{-1}\boldsymbol{X}(\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}]\boldsymbol{Z}_i\boldsymbol{Z}_i' \times \\ \times [\widetilde{\mathbf{V}}^{-1} - \widetilde{\mathbf{V}}^{-1}\boldsymbol{X}(\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}]$$

je MINQUE odhadom (56) *ak čísla* $\lambda_1, \lambda_2, \dots, \lambda_{s+1}$ *riešia systém*

$$\operatorname{tr} \boldsymbol{A}\ \boldsymbol{Z}_i\boldsymbol{Z}_i' = p_i, \quad i = 1, 2, \dots, s+1.$$

Dôkaz. $\boldsymbol{A} = \boldsymbol{A}'$. Podľa lemy 36 je $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ invariantný, lebo $\boldsymbol{A}\boldsymbol{X} = \mathbf{0}$. Podľa lemy 37 je odhad $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ nevychýlený.

Pretože

$$\begin{aligned} \| \boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z} - \boldsymbol{\Delta} \|^2 &= \operatorname{tr}(\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z} - \boldsymbol{\Delta})(\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z} - \boldsymbol{\Delta})' = \\ &= \operatorname{tr} \boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z}\boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z} - 2 \operatorname{tr} \boldsymbol{Z}'\boldsymbol{A}\boldsymbol{Z}\boldsymbol{\Delta} + \operatorname{tr} \boldsymbol{\Delta}\boldsymbol{\Delta} = \\ &= \operatorname{tr} \boldsymbol{A}\widetilde{\mathbf{V}}\boldsymbol{A}\widetilde{\mathbf{V}} - 2 \operatorname{tr} \boldsymbol{A}\boldsymbol{Z}\boldsymbol{\Delta}\boldsymbol{Z}' + \operatorname{tr} \boldsymbol{\Delta}\boldsymbol{\Delta} = \\ &= \operatorname{tr} \boldsymbol{A}\widetilde{\mathbf{V}}\boldsymbol{A}\widetilde{\mathbf{V}} - 2 \operatorname{tr} \boldsymbol{A} \sum_{i=1}^{s+1} \frac{p_i}{m_i} \boldsymbol{Z}_i\boldsymbol{Z}_i' + \operatorname{tr} \boldsymbol{\Delta}\boldsymbol{\Delta} = \\ &= \operatorname{tr} \boldsymbol{A}\widetilde{\mathbf{V}}\boldsymbol{A}\widetilde{\mathbf{V}} - 2 \sum_{i=1}^{s+1} \frac{p_i^2}{m_i} + \operatorname{tr} \boldsymbol{\Delta}\boldsymbol{\Delta} = \\ &= \operatorname{tr} \boldsymbol{A}\widetilde{\mathbf{V}}\boldsymbol{A}\widetilde{\mathbf{V}} - \operatorname{tr} \boldsymbol{\Delta}\boldsymbol{\Delta}, \end{aligned}$$

v (iii) musíme vlastne minimalizovať $\operatorname{tr} \boldsymbol{A}\widetilde{\mathbf{V}}\boldsymbol{A}\widetilde{\mathbf{V}}$, pričom $\boldsymbol{A} = \boldsymbol{A}'$, $\operatorname{tr} \boldsymbol{A}\boldsymbol{Z}_i\boldsymbol{Z}_i' = p_i$, $i = 1, 2, \dots, s+1$ a $\boldsymbol{A}\boldsymbol{X} = \mathbf{0}$.

Keby $\boldsymbol{Y}'\boldsymbol{A}_*\boldsymbol{Y}$ bol tiež MINQUE odhadom (56), tak $\boldsymbol{A}_*$ sa vždy dá napísať ako $\boldsymbol{A}_* = \boldsymbol{A} + \boldsymbol{D}$, kde $\boldsymbol{D} = \boldsymbol{D}'$, $\boldsymbol{D}\boldsymbol{X} = \mathbf{0}$ a $\operatorname{tr} \boldsymbol{D}\boldsymbol{Z}_i\boldsymbol{Z}_i' = 0$, $i = 1, 2, \dots, s+1$. Potom ale

$$\operatorname{tr} \boldsymbol{A}_*\widetilde{\mathbf{V}}\boldsymbol{A}_*\widetilde{\mathbf{V}} = \operatorname{tr}(\boldsymbol{A} + \boldsymbol{D})\widetilde{\mathbf{V}}(\boldsymbol{A} + \boldsymbol{D})\widetilde{\mathbf{V}} = \operatorname{tr} \boldsymbol{A}\widetilde{\mathbf{V}}\boldsymbol{A}\widetilde{\mathbf{V}} + \operatorname{tr} \boldsymbol{D}\widetilde{\mathbf{V}}\boldsymbol{D}\widetilde{\mathbf{V}} \geqq \operatorname{tr} \boldsymbol{A}\widetilde{\mathbf{V}}\boldsymbol{A}\widetilde{\mathbf{V}},$$

lebo

$$\operatorname{tr} \boldsymbol{A}\widetilde{\mathbf{V}}\boldsymbol{D}\widetilde{\mathbf{V}} = (\operatorname{tr} \boldsymbol{D}\widetilde{\mathbf{V}}\boldsymbol{A}\widetilde{\mathbf{V}} =)$$

$$= \operatorname{tr} \sum_{i=1}^{s+1} \lambda_i [\widetilde{\mathbf{V}}^{-1} - \widetilde{\mathbf{V}}^{-1}\boldsymbol{X}(\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}]\boldsymbol{Z}_i\boldsymbol{Z}_i' \times$$

$$\times [\widetilde{\mathbf{V}}^{-1} - \widetilde{\mathbf{V}}^{-1}\boldsymbol{X}(\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}]\widetilde{\mathbf{V}}\boldsymbol{D}\widetilde{\mathbf{V}} =$$

$$= \sum_{i=1}^{s+1} \lambda_i \operatorname{tr}[\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}]\boldsymbol{Z}_i\boldsymbol{Z}_i'[\boldsymbol{I} - \widetilde{\mathbf{V}}^{-1}\boldsymbol{X}(\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}']\boldsymbol{D} =$$

$$= \sum_{i=1}^{s+1} \lambda_i \operatorname{tr} \boldsymbol{D}\boldsymbol{Z}_i\boldsymbol{Z}_i' = 0$$

a

$$\operatorname{tr} \mathbf{D}\widetilde{\mathbf{V}}\mathbf{D}\widetilde{\mathbf{V}} = \operatorname{tr} \mathbf{DZZ'DZZ'} = \operatorname{tr} \mathbf{Z'DZZ'DZ} = \operatorname{tr}(\mathbf{Z'DZ})(\mathbf{Z'DZ})' \geqq 0. \qquad \square$$

Dôsledok 40. *Nech $(s+1)$-rozmerný vektor $\boldsymbol{g}$ je*

$$\boldsymbol{g} = \begin{pmatrix} \boldsymbol{Y}'\boldsymbol{B}\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{B}\boldsymbol{Y} \\ \boldsymbol{Y}'\boldsymbol{B}\boldsymbol{Z}_2\boldsymbol{Z}_2'\boldsymbol{B}\boldsymbol{Y} \\ \vdots \\ \boldsymbol{Y}'\boldsymbol{B}\boldsymbol{Z}_{s+1}\boldsymbol{Z}_{s+1}'\boldsymbol{B}\boldsymbol{Y} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Y}'\boldsymbol{B}\boldsymbol{V}_1\boldsymbol{B}\boldsymbol{Y} \\ \boldsymbol{Y}'\boldsymbol{B}\boldsymbol{V}_2\boldsymbol{B}\boldsymbol{Y} \\ \vdots \\ \boldsymbol{Y}'\boldsymbol{B}\boldsymbol{V}_{s+1}\boldsymbol{B}\boldsymbol{Y} \end{pmatrix}$$

a $(s+1)\times(s+1)$ matica $\boldsymbol{S}$ je

$$\boldsymbol{S} = \begin{pmatrix} \operatorname{tr} \boldsymbol{B}\boldsymbol{V}_1\boldsymbol{B}\boldsymbol{V}_1 & \operatorname{tr} \boldsymbol{B}\boldsymbol{V}_1\boldsymbol{B}\boldsymbol{V}_2 & \dots & \operatorname{tr} \boldsymbol{B}\boldsymbol{V}_1\boldsymbol{B}\boldsymbol{V}_{s+1} \\ \operatorname{tr} \boldsymbol{B}\boldsymbol{V}_2\boldsymbol{B}\boldsymbol{V}_1 & \operatorname{tr} \boldsymbol{B}\boldsymbol{V}_2\boldsymbol{B}\boldsymbol{V}_2 & \dots & \operatorname{tr} \boldsymbol{B}\boldsymbol{V}_2\boldsymbol{B}\boldsymbol{V}_{s+1} \\ \vdots & & \ddots & \\ \operatorname{tr} \boldsymbol{B}\boldsymbol{V}_{s+1}\boldsymbol{B}\boldsymbol{V}_1 & \operatorname{tr} \boldsymbol{B}\boldsymbol{V}_{s+1}\boldsymbol{B}\boldsymbol{V}_2 & \dots & \operatorname{tr} \boldsymbol{B}\boldsymbol{V}_{s+1}\boldsymbol{B}\boldsymbol{V}_{s+1} \end{pmatrix},$$

kde $\boldsymbol{B} = \widetilde{\mathbf{V}}^{-1} - \widetilde{\mathbf{V}}^{-1}\boldsymbol{X}(\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}$. MINQUE odhad funkcie $\boldsymbol{p}'\boldsymbol{\sigma} = \sum_{i=1}^{s+1} p_i\sigma_i^2$ je $\boldsymbol{p}'\boldsymbol{S}^{-}\boldsymbol{g}$.

Dôkaz. Podľa vety 39 je $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ MINQUE odhadom funkcie $\boldsymbol{p}'\boldsymbol{\sigma}$, keď

$$\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y} = \boldsymbol{Y}' \sum_{i=1}^{s+1} \lambda_i \boldsymbol{B}\boldsymbol{V}_i\boldsymbol{B}\boldsymbol{Y} = \boldsymbol{\lambda}'\boldsymbol{g}, \tag{65}$$

pričom $\boldsymbol{\lambda}' = (\lambda_1, \lambda_2, \dots, \lambda_{s+1})$ spĺňa rovnice $\operatorname{tr} \boldsymbol{A}\boldsymbol{Z}_i\boldsymbol{Z}_i' = p_i$, $i = 1, 2, \dots, s+1$ ($\boldsymbol{A} = \sum_{i=1}^{s+1} \lambda_i \boldsymbol{B}\boldsymbol{Z}_i\boldsymbol{Z}_i'\boldsymbol{B}$ je dané vzorcom (63)). Toto zapísané formou matíc znie

$$\boldsymbol{S}\boldsymbol{\lambda} = \boldsymbol{p}, \tag{66}$$

čiže $\boldsymbol{p} \in \mu(\boldsymbol{S})$. Potom ale $\boldsymbol{\lambda} = \boldsymbol{S}^{-}\boldsymbol{p}$, kde $\boldsymbol{S}^{-}$ je symetrická g-inverzia matice $\boldsymbol{S}$, je jedno riešenie (66) (pozri napr. [2], str. 70) a podľa (65) je $\boldsymbol{p}'\boldsymbol{S}^{-}\boldsymbol{g}$ MINQUE odhadom funkcie $\boldsymbol{p}'\boldsymbol{\sigma}$. $\square$

Dôsledok 41. *Nech* $\boldsymbol{\sigma}' = (\sigma_1^2, \sigma_2^2, \dots, \sigma_{s+1}^2)$. *MINQUE odhad funkcie* $\boldsymbol{p}'\boldsymbol{\sigma}$ *je* $\boldsymbol{p}'\hat{\boldsymbol{\sigma}}$, *kde* $\hat{\boldsymbol{\sigma}}$ *je riešením sústavy*

$$\boldsymbol{S}\,\hat{\boldsymbol{\sigma}} = \boldsymbol{g}. \tag{67}$$

Dôkaz. Podľa dôsledku 40 je $\boldsymbol{p}'\boldsymbol{S}^-\boldsymbol{g}$ MINQUE odhadom funkcie $\boldsymbol{p}'\boldsymbol{\sigma}$, kde $\boldsymbol{p} \in \mu(\boldsymbol{S})$. Podľa [2], str. 70 $\{\hat{\boldsymbol{\sigma}} = \boldsymbol{S}^-\boldsymbol{g} + (\boldsymbol{S}^-\boldsymbol{S} - \boldsymbol{I})\boldsymbol{z} \colon \boldsymbol{z} \in \mathbb{R}^{s+1}$, $\boldsymbol{S}^-$ je jedna (ľubovoľná) g-inverzia matice $\boldsymbol{S}\}$ sú všetky riešenia sústavy (67). Preto

$$\boldsymbol{p}'\hat{\boldsymbol{\sigma}} = \boldsymbol{p}'(\boldsymbol{S}^-\boldsymbol{g} + (\boldsymbol{S}^-\boldsymbol{S} - \boldsymbol{I})\boldsymbol{z}) = \boldsymbol{p}'\boldsymbol{S}^-\boldsymbol{g}$$

je MINQUE odhadom funkcie $\boldsymbol{p}'\boldsymbol{\sigma}$. □

Poznámka 42. *Podľa vety 18 a lemy 10 sa disperzia MINQUE odhadu* $\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}$ *v prípade* $\boldsymbol{Y} \sim N(\boldsymbol{X}\boldsymbol{\beta}; \widetilde{\mathbf{V}})$ *rovná*

$$\mathcal{D}(\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}) = \mathcal{E}(\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y} - \mathcal{E}(\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}))^2 = \mathcal{E}(\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}) - \mathcal{E}^2(\boldsymbol{Y}'\boldsymbol{A}\boldsymbol{Y}) =$$

$$= 2\operatorname{tr}\boldsymbol{A}\widetilde{\mathbf{V}}\boldsymbol{A}\widetilde{\mathbf{V}} + \operatorname{tr}^2\boldsymbol{A}\widetilde{\mathbf{V}} + 2\boldsymbol{\beta}\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}\boldsymbol{\beta}\operatorname{tr}\boldsymbol{A}\widetilde{\mathbf{V}} + 4\boldsymbol{\beta}\boldsymbol{X}'\boldsymbol{A}\widetilde{\mathbf{V}}\boldsymbol{A}\,\boldsymbol{X}\boldsymbol{\beta} +$$

$$+ (\boldsymbol{\beta}\boldsymbol{X}'\boldsymbol{A}\,\boldsymbol{X}\boldsymbol{\beta})^2 - (\boldsymbol{\beta}\boldsymbol{X}'\boldsymbol{A}\,\boldsymbol{X}\boldsymbol{\beta} + \operatorname{tr}\boldsymbol{A}\widetilde{\mathbf{V}})^2 =$$

$$= 2\operatorname{tr}\boldsymbol{A}\widetilde{\mathbf{V}}\boldsymbol{A}\widetilde{\mathbf{V}}.$$

Preto MINQUE odhad je v tomto prípade $\widetilde{\mathbf{V}}$*-LMVUIE (*$\widetilde{\mathbf{V}}$*-lokálny nevychýlený invariantný odhad s minimálnou disperziou) odhad funkcie* $\boldsymbol{p}'\boldsymbol{\sigma}$.

Cvičenie

1. Pomocou lemy 5 dokážte, že pre ľubovoľné matice $\boldsymbol{A}, \boldsymbol{B}$ vhodných rozmerov je $\mu(\boldsymbol{A}\boldsymbol{A}' + \boldsymbol{B}\boldsymbol{B}') = \mu(\boldsymbol{A} \vdots \boldsymbol{B})$.

2. Ak vo vete 39 je $\boldsymbol{X}_{s+1} = \boldsymbol{I}$, tak $\widetilde{\mathbf{V}}$ je nesingulárna. Dokážte.

9 MINQUE ODHADY VARIANČNÝCH KOMPONENTOV V MODELI S JEDNÝM NÁHODNÝM EFEKTOM

Hľadajme MINQUE odhady variančných komponentov v modeli s jedným náhodným efektom, teda v modeli (40), pričom pri označení podľa (55) sa $s = 1$, $\boldsymbol{X} = \mathbf{1}_{n,1}$

$$\boldsymbol{Z}_1 = \begin{pmatrix} \mathbf{1}_{n_1,1} \otimes \boldsymbol{e}_1' \\ \mathbf{1}_{n_2,1} \otimes \boldsymbol{e}_2' \\ \vdots \\ \mathbf{1}_{n_I,1} \otimes \boldsymbol{e}_I' \end{pmatrix}$$

(w-ta súradnica I-rozmerného vektora $\boldsymbol{e}_w$ sa rovná 1, $w \in \{1, 2, \dots, I\}$, ostatné súradnice sa rovnajú 0), $\boldsymbol{Z}_2 = \boldsymbol{I}_{n,n}$, $n = \sum_{i=1}^{I} n_i$ (σ_2^2 je v modeli (40) označené ako σ_0^2).

Lema 43. *V modeli* (40) *(pri označení matice* $\boldsymbol{B}$ *podľa dôsledku 40) platí*

$$\widetilde{\mathbf{V}}^{-1} = (\boldsymbol{Z}_1\boldsymbol{Z}_1' + \boldsymbol{I})^{-1} = \boldsymbol{I} \ - \boldsymbol{Z}_1 \begin{pmatrix} \frac{1}{n_1+1} & 0 & \dots & 0 \\ 0 & \frac{1}{n_2+1} & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \frac{1}{n_I+1} \end{pmatrix} \boldsymbol{Z}_1',$$

$$(\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}\boldsymbol{X})^{-} = \left(\sum_{i=1}^{I} \frac{n_i}{n_i+1}\right)^{-1},$$

$$\widetilde{\mathbf{V}}^{-1}\boldsymbol{X}(\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1} =$$

$$= \left(\sum_{i=1}^{I} \frac{n_i}{n_i+1}\right)^{-1} \left\{ \mathbf{1}_{n,1}\mathbf{1}_{1,n}' - \boldsymbol{Z}_1 \begin{pmatrix} \frac{n_1}{n_1+1} \\ \frac{n_2}{n_2+1} \\ \vdots \\ \frac{n_I}{n_I+1} \end{pmatrix} \mathbf{1}_{1,n}' - \right.$$

$$\left. -\mathbf{1}_{n,1} \left(\frac{n_1}{n_1+1}, \dots, \frac{n_I}{n_I+1}\right) \boldsymbol{Z}_1' + \boldsymbol{Z}_1 \begin{pmatrix} \frac{n_1}{n_1+1} \\ \frac{n_2}{n_2+1} \\ \vdots \\ \frac{n_I}{n_I+1} \end{pmatrix} \left(\frac{n_1}{n_1+1}, \dots, \frac{n_I}{n_I+1}\right) \boldsymbol{Z}_1' \right\},$$

$$\operatorname{tr} \boldsymbol{B}\boldsymbol{V}_1\boldsymbol{B}\boldsymbol{V}_1 = \operatorname{tr}(\widetilde{\mathbf{V}}^{-1} - \widetilde{\mathbf{V}}^{-1}\boldsymbol{X}(\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1})\boldsymbol{Z}_1 \times$$
$$\times \boldsymbol{Z}_1'(\widetilde{\mathbf{V}}^{-1} - \widetilde{\mathbf{V}}^{-1}\boldsymbol{X}(\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\widetilde{\mathbf{V}}^{-1})\boldsymbol{Z}_1\boldsymbol{Z}_1' =$$

$$= \sum_{i=1}^{I}\left(\frac{n_i}{n_i+1}\right)^2 - 2\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-1}\sum_{i=1}^{I}\left(\frac{n_i}{n_i+1}\right)^3 +$$
$$+\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-2}\left[\sum_{i=1}^{I}\left(\frac{n_i}{n_i+1}\right)^2\right]^2,$$

$$\operatorname{tr} \boldsymbol{B}\boldsymbol{V}_1\boldsymbol{B}\boldsymbol{V}_2 = \operatorname{tr} \boldsymbol{B}\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{B}\boldsymbol{I} = \operatorname{tr} \boldsymbol{Z}_1'\boldsymbol{B}\boldsymbol{B}\boldsymbol{Z}_1 =$$

$$= \sum_{i=1}^{I}\frac{n_i}{(n_i+1)^2} - 2\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-1}\sum_{i=1}^{I}\left(\frac{n_i}{n_i+1}\right)^2 +$$
$$+2\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-1}\sum_{i=1}^{I}\left(\frac{n_i}{n_i+1}\right)^3 + n\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-2}\sum_{i=1}^{I}\left(\frac{n_i}{n_i+1}\right)^2 -$$
$$-2\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-2}\sum_{i=1}^{I}\left(\frac{n_i}{n_i+1}\right)^2\sum_{j=1}^{I}\frac{n_j^2}{n_j+1}+$$

$$+\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-2}\sum_{i=1}^{I}\left(\frac{n_i}{n_i+1}\right)^{2}\sum_{j=1}^{I}\frac{n_j^3}{(n_j+1)^2},$$

$$\operatorname{tr}\boldsymbol{B}\boldsymbol{V}_2\boldsymbol{B}\boldsymbol{V}_2=\operatorname{tr}\boldsymbol{B}\boldsymbol{B}=$$

$$=n-2\sum_{i=1}^{I}\frac{n_i}{n_i+1}-2n\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-1}+$$

$$+6\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-1}\sum_{i=1}^{I}\frac{n_i^2}{n_i+1}-6\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-1}\sum_{i=1}^{I}\frac{n_i^3}{(n_i+1)^2}+$$

$$+\sum_{i=1}^{I}\left(\frac{n_i}{n_i+1}\right)^{2}+2\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-1}\sum_{i=1}^{I}\frac{n_i^4}{(n_i+1)^3}+$$

$$+n^2\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-2}-4\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-2}n\sum_{i=1}^{I}\frac{n_i^2}{n_i+1}+$$

$$+4\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-2}\left(\sum_{i=1}^{I}\frac{n_i^2}{n_i+1}\right)^{2}+2\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-2}n\sum_{i=1}^{I}\frac{n_i^3}{(n_i+1)^2}-$$

$$-4\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-2}\sum_{i=1}^{I}\frac{n_i^2}{n_i+1}\sum_{j=1}^{I}\frac{n_j^3}{(n_j+1)^2}+\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-2}\left(\sum_{i=1}^{I}\frac{n_i^3}{(n_i+1)^2}\right)^{2},$$

$$\boldsymbol{Y}'\boldsymbol{B}\boldsymbol{V}_1\boldsymbol{B}\boldsymbol{Y}=\boldsymbol{Y}'\boldsymbol{B}\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{B}\boldsymbol{Y}=$$

$$=\sum_{k=1}^{I}\left\{\frac{n_k}{n_k+1}\left[\sum_{j=1}^{n_k}Y_{kj}-\frac{n_k}{\sum_{i=1}^{I}\frac{n_i}{n_i+1}}\sum_{s=1}^{I}\frac{1}{n_s+1}\sum_{v=1}^{n_s}Y_{sv}\right]\right\}^{2},$$

$$\boldsymbol{Y}'\boldsymbol{B}\boldsymbol{V}_2\boldsymbol{B}\boldsymbol{Y}=\boldsymbol{Y}'\boldsymbol{B}\boldsymbol{B}\boldsymbol{Y}=$$

$$=\sum_{i=1}^{I}\sum_{j=1}^{n_i}Y_{ij}^2-\sum_{i=1}^{I}\left[\frac{1}{n_i+1}+\frac{1}{(n_i+1)^2}\right]\left(\sum_{j=1}^{n_i}Y_{ij}\right)^{2}-$$

$$-2\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-1}\left(\sum_{i=1}^{I}\frac{1}{n_i+1}\sum_{j=1}^{n_i}Y_{ij}\right)\left(\sum_{i=1}^{I}\frac{1}{(n_i+1)^2}\sum_{j=1}^{n_i}Y_{ij}\right)+$$

$$+\left(\sum_{i=1}^{I}\frac{n_i}{n_i+1}\right)^{-2}\left(\sum_{i=1}^{I}\frac{n_i}{(n_i+1)^2}\right)^{-1}\left(\sum_{i=1}^{I}\frac{1}{n_i+1}\sum_{j=1}^{n_i}Y_{ij}\right).$$

Dôkaz. Vykonáme jednoduchým ale zdĺhavým postupným výpočtom. Urobte ho ako cvičenie.

Dôsledok 44. *Vo vyváženom modeli s jedným náhodným efektom (označenie podľa* (55), $n_1 = n_2 = \ldots = n_I = t$*) platí:*

$$\operatorname{tr} \boldsymbol{B}\boldsymbol{V}_1\boldsymbol{B}\boldsymbol{V}_1 = (I-1)\left(\frac{t}{t+1}\right)^2,$$

$$\operatorname{tr} \boldsymbol{B}\boldsymbol{V}_1\boldsymbol{B}\boldsymbol{V}_2 = (I-1)\frac{t}{(t+1)^2},$$

$$\operatorname{tr} \boldsymbol{B}\boldsymbol{V}_2\boldsymbol{B}\boldsymbol{V}_2 = \frac{It^3+It^2-It-1}{(t+1)^2},$$

$$\boldsymbol{Y}'\boldsymbol{B}\boldsymbol{V}_1\boldsymbol{B}\boldsymbol{Y} = \frac{1}{(t+1)^2}\sum_{i=1}^{I}\left(\sum_{j=1}^{t} Y_{ij}\right)^2 - \frac{1}{I(t+1)^2}\left(\sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}\right)^2,$$

$$\boldsymbol{Y}'\boldsymbol{B}\boldsymbol{V}_2\boldsymbol{B}\boldsymbol{Y} = \sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}^2 - \frac{t+2}{(t+1)^2}\sum_{i=1}^{I}\left(\sum_{j=1}^{t} Y_{ij}\right)^2 - \frac{1}{It(t+1)^2}\left(\sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}\right)^2.$$

Dôkaz. Vykonáme dosadením $n_1 = n_2 = \ldots = n_I = t$ do vzorcov z lemy 43. Urobte ho ako cvičenie.

Lema 45. *Vo vyváženom modeli s jedným náhodným efektom platí*

$$\boldsymbol{S}^- = \boldsymbol{S}^{-1} = \frac{1}{It(t-1)}\begin{pmatrix} \dfrac{It^3+It^2-It-1}{(I-1)t} & -1 \\ -1 & t \end{pmatrix}$$

(označenie z dôsledku 40).

Dôkaz. Vykonáme inverziou matice $\boldsymbol{S}$, ktorej prvky sú dané v dôsledku 44. Urobte ho ako cvičenie.

Veta 46. *Vo vyváženom modeli s jedným náhodným efektom sú MINQUE odhady* $\widehat{\sigma_1^2}$ *a* $\widehat{\sigma_2^2}$ *totožné s odhadmi získanými metódou Hendersona (vzorce* (47) *a* (48)*, pričom* $\widehat{\sigma_2^2}$ *je pri našom označení totožné so* $\widehat{\sigma_0^2}$ *vo vzorci* (48)*).*

Dôkaz. Podľa dôsledku 40 sa MINQUE odhad $\widehat{\sigma_1^2}$ rovná

$$\widehat{\sigma_1^2} = (1\ ,\ 0)\boldsymbol{S}^-\boldsymbol{g},$$

pričom matica $\boldsymbol{S}^-$ je daná v leme 45 a prvky vektora $\boldsymbol{g}$ (definovaného v dôsledku 40) sú dané v dôsledku 44. Preto MINQUE odhad

$$\widehat{\sigma_1^2} = \frac{1}{It(t-1)}\left(\frac{It^3+It^2-It-1}{(I-1)t}\ ,\ -1\right)\boldsymbol{g} =$$

$$
= \frac{1}{It(t-1)} \left\{ \frac{It^3 + It^2 - It - 1}{(I-1)t} \left[\frac{1}{(t+1)^2} \sum_{i=1}^{I} \left(\sum_{j=1}^{t} Y_{ij} \right)^2 - \right. \right.
$$

$$
\left. - \frac{1}{I(t+1)^2} \left(\sum_{i=1}^{I} \sum_{j=1}^{t} Y_{ij} \right)^2 \right] -
$$

$$
\left. - \sum_{i=1}^{I} \sum_{j=1}^{t} Y_{ij}^2 + \frac{t+2}{(t+1)^2} \sum_{i=1}^{I} \left(\sum_{j=1}^{t} Y_{ij} \right)^2 + \frac{1}{It(t+1)^2} \left(\sum_{i=1}^{I} \sum_{j=1}^{t} Y_{ij} \right)^2 \right\} =
$$

$$
= \frac{It-1}{I(I-1)t^2(t-1)} \sum_{i=1}^{I} \left(\sum_{j=1}^{t} Y_{ij} \right)^2 - \frac{1}{It(t-1)} \sum_{i=1}^{I} \sum_{j=1}^{t} Y_{ij}^2 -
$$

$$
- \frac{1}{I(I-1)t^2} \left(\sum_{i=1}^{I} \sum_{j=1}^{t} Y_{ij} \right)^2 ,
$$

čo je ten istý odhad ako (47).

Podobne MINQUE odhad

$$
\widehat{\sigma_2^2} = (0\ ,\ 1) \boldsymbol{S}^{-} \boldsymbol{g} =
$$

$$
= \frac{1}{It(t-1)} (-1\ ,\ t) \times
$$

$$
\times \begin{pmatrix} \frac{1}{I(t+1)^2} \left[I \ \sum_{i=1}^{I} \left(\sum_{j=1}^{t} Y_{ij} \right)^2 - \left(\sum_{i=1}^{I} \sum_{j=1}^{t} Y_{ij} \right)^2 \right] \\ \sum_{i=1}^{I} \sum_{j=1}^{t} Y_{ij}^2 - \frac{t+2}{(t+1)^2} \sum_{i=1}^{I} \left(\sum_{j=1}^{t} Y_{ij} \right)^2 - \frac{1}{It(t+1)^2} \left(\sum_{i=1}^{I} \sum_{j=1}^{t} Y_{ij} \right)^2 \end{pmatrix} =
$$

$$
= \frac{1}{It(t-1)} \left\{ \frac{-t^2 - 2t - 1}{(t+1)^2} \sum_{i=1}^{I} \left(\sum_{j=1}^{t} Y_{ij} \right)^2 + t \sum_{i=1}^{I} \sum_{j=1}^{t} Y_{ij}^2 \right\} =
$$

$$
= \frac{1}{I(t-1)} \sum_{i=1}^{I} \sum_{j=1}^{t} Y_{ij}^2 - \frac{1}{It(t-1)} \sum_{i=1}^{I} \left(\sum_{j=1}^{t} Y_{ij} \right)^2 ,
$$

čo je ten istý odhad ako (48). $\square$

10 ODHADY VARIANČNÝCH KOMPONENTOV ZÍSKANÉ METÓDOU MAXIMÁLNEJ VIEROHODNOSTI (ML – ODHADY)

Jednou zo štandardných metód získania odhadov je metóda maximálnej vierohodnosti. Vyžaduje si predpoklad o rozdelení pravdepodobnosti pre náhodné premenné, ktorých realizáciami sú získané (namerané) údaje. Prirodzený predpoklad je normalita rozdelenia (v neposlednom rade pre matematické zvládnutie riešenia).

Majme zmiešaný lineárny model (55), teda

$$\boldsymbol{Y}_{n,1} = (\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \dots \boldsymbol{Z}_{s+1}) \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{b}_1 \\ \vdots \\ \boldsymbol{b}_{s+1} \end{pmatrix} = \boldsymbol{X}\boldsymbol{\beta} + \sum_{i=1}^{s+1} \boldsymbol{Z}_i \boldsymbol{b}_i,$$

$$\operatorname{cov}(\boldsymbol{Y}) = \sum_{i=1}^{s+1} \sigma_i^2 \boldsymbol{V}_i = \sum_{i=1}^{s+1} \sigma_i^2 \boldsymbol{Z}_i \boldsymbol{Z}_i' = \boldsymbol{V},$$

$\boldsymbol{b}_i \in \mathbb{R}^{m_i}$, $i = 1, 2, \dots, s+1$, $\mathcal{E}(\boldsymbol{b}_i) = \boldsymbol{0}$, $\operatorname{var}(\boldsymbol{b}_i) = \sigma_i^2 \boldsymbol{I}$, $i = 1, 2, \dots, s+1$, $\operatorname{cov}(\boldsymbol{b}_i, \boldsymbol{b}_j) = \boldsymbol{0}$ pre $i \neq j$ (opäť pri označení $\boldsymbol{Z}_{s+1} = \boldsymbol{Z}_0$, $\boldsymbol{b}_{s+1} = \boldsymbol{b}_0$, $\sigma_{s+1}^2 = \sigma_0^2$ dostávame model (15)). Predpokladáme $\boldsymbol{Y} \sim N_n(\boldsymbol{X}\boldsymbol{\beta},\ \boldsymbol{V})$, teda hustota rozdelenia pravdepodobnosti observačného vektora je

$$f(\boldsymbol{y}) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det \boldsymbol{V}}} e^{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})' \boldsymbol{V}^{-1} (\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})}.$$

Funkcia vierohodnosti („z jednej realizácie“ $\boldsymbol{y}$) je

$$L(\boldsymbol{\beta}, \sigma_1^2, \dots, \sigma_{s+1}^2) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{\det \boldsymbol{V}}} e^{-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})' \boldsymbol{V}^{-1} (\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})}. \tag{68}$$

Realizácie odhadov $\hat{\boldsymbol{\beta}}, \widehat{\sigma_1^2}, \dots, \widehat{\sigma_{s+1}^2}$ získané metódou maximálnej vierohodnosti *(ML–odhady)* sú také hodnoty $\hat{\boldsymbol{\beta}}_{\text{real}}, \{\widehat{\sigma_1^2}\}_{\text{real}}, \dots, \{\widehat{\sigma_{s+1}^2}\}_{\text{real}}$, pre ktoré (68) nadobúda maximum (pri realizácii $\boldsymbol{y}$).

Maximum (68) sa dosiahne aj maximalizáciou

$$l = \ln L = -\frac{1}{2} n \ln 2\pi - \frac{1}{2} \ln \det \boldsymbol{V} \ - \frac{1}{2} (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})' \boldsymbol{V}^{-1} (\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}). \tag{69}$$

Pri vypočítaní takéhoto maxima využijeme pomocné tvrdenie:

Lema 47. *Nech $\boldsymbol{a}, \boldsymbol{x}$ sú $n \times 1$ vektory,*

$$\frac{\partial \boldsymbol{a}' \boldsymbol{x}}{\partial \boldsymbol{x}} \text{ je } n \times 1 \text{ vektor so súradnicami } \left\{ \frac{\partial \boldsymbol{a}' \boldsymbol{x}}{\partial \boldsymbol{x}} \right\}_i = \frac{\partial \boldsymbol{a}' \boldsymbol{x}}{\partial x_i}, \quad i = 1, 2, \dots, n.$$

Nech $\boldsymbol{A}$ je symetrická matica $n \times n$,

$$\frac{\partial \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}}{\partial \boldsymbol{x}} \text{ je } n \times 1 \text{ vektor so súradnicami } \left\{\frac{\partial \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}}{\partial \boldsymbol{x}}\right\}_i = \frac{\partial \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}}{\partial x_i}, \quad i = 1, 2, \ldots, n.$$

Nech $\boldsymbol{B}$ je symetrická nesingulárna $n \times n$ matica, ktorej prvky b_{ij} sú funkciami premennej t, čiže $b_{ij} = b_{ij}(t)$, $i, j = 1, 2, \ldots, n$,

$$\frac{\partial \boldsymbol{B}}{\partial t} \text{ je } n \times n \text{ matica, ktorej prvky sú } \frac{\partial b_{ij}(t)}{\partial t}, \quad i, j = 1, 2, \ldots, n,$$

$$\frac{\partial \det \boldsymbol{B}}{\partial \boldsymbol{B}} \text{ je } n \times n \text{ matica, ktorej prvky sú } \frac{\partial \det \boldsymbol{B}}{\partial b_{ij}}, \quad i, j = 1, 2, \ldots, n.$$

Platí

(i)
$$\frac{\partial \boldsymbol{a}'\boldsymbol{x}}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{x}'\boldsymbol{a}}{\partial \boldsymbol{x}} = \boldsymbol{a},$$

(ii)
$$\frac{\partial \boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}}{\partial \boldsymbol{x}} = 2\boldsymbol{A}\boldsymbol{x},$$

(iii)
$$\frac{\partial \boldsymbol{B}^{-1}}{\partial t} = -\boldsymbol{B}^{-1}\frac{\partial \boldsymbol{B}}{\partial t}\boldsymbol{B}^{-1},$$

(iv)
$$\frac{\partial \ln \det \boldsymbol{B}}{\partial t} = \operatorname{tr} \boldsymbol{B}^{-1}\frac{\partial \boldsymbol{B}}{\partial t}.$$

Dôkaz. Tvrdenia (i) a (ii) dokážeme ľahko rozpísaním skalárnych súčinov (pozri tiež [9], str. 98). Pri tvrdení (iii) si pomôžeme nasledujúcim spôsobom:

Prvky matice $\boldsymbol{B}^{-1}$ označme $b_{ij}^{(-1)}$, $i, j = 1, 2, \ldots, n$. Tiež sú funkciami premennej t, čiže $b_{ij}^{(-1)} = b_{ij}^{(-1)}(t)$, $i, j = 1, 2, \ldots, n$. Pre $i, j = 1, 2, \ldots, n$ je

$$\{\boldsymbol{B}\boldsymbol{B}^{-1}\}_{i,j} = \sum_{k=1}^{n} b_{ik}(t) b_{kj}^{(-1)}(t) = \delta_{ij}$$

(δ_{ij} je tzv. Kroneckerovo delta, čiže $\delta_{ij} = 0$ pre $i \neq j$ a $\delta_{ii} = 1$).

Preto

$$0 = \frac{\partial}{\partial t}\{\boldsymbol{B}\boldsymbol{B}^{-1}\}_{ij} = \sum_{k=1}^{n} \frac{\partial}{\partial t}\left[b_{ik}(t) b_{kj}^{(-1)}(t)\right] =$$

$$= \sum_{k=1}^{n} \frac{\partial b_{ik}(t)}{\partial t} b_{kj}^{(-1)}(t) + \sum_{k=1}^{n} b_{ik}(t) \frac{\partial b_{kj}^{(-1)}(t)}{\partial t}, \quad i, j = 1, 2, \ldots, n,$$

čo v maticovom zápise je

$$\frac{\partial \boldsymbol{B}}{\partial t}\boldsymbol{B}^{-1} + \boldsymbol{B}\frac{\partial \boldsymbol{B}^{-1}}{\partial t} = \boldsymbol{0},$$

čiže

$$\frac{\partial \boldsymbol{B}^{-1}}{\partial t} = -\boldsymbol{B}^{-1}\frac{\partial \boldsymbol{B}}{\partial t}\boldsymbol{B}^{-1}.$$

Pri dôkaze (iv) vyjdeme z rovnosti

$$\frac{\partial \det \boldsymbol{B}}{\partial \boldsymbol{B}} = (\det \boldsymbol{B})(2\boldsymbol{B}^{-1} - \operatorname{diag} \boldsymbol{B}^{-1}),$$

kde

$$\operatorname{diag} \boldsymbol{B}^{-1} = \begin{pmatrix} b_{11}^{(-1)} & 0 & \dots & 0 \\ 0 & b_{22}^{(-1)} & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & b_{nn}^{(-1)} \end{pmatrix}$$

(pozri [9], str. 98). Preto

$$\frac{\partial \ln \det \boldsymbol{B}}{\partial t} = \frac{1}{\det \boldsymbol{B}}\frac{\partial \det \boldsymbol{B}}{\partial t} = \frac{1}{\det \boldsymbol{B}}\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{\partial \det \boldsymbol{B}}{\partial b_{ij}}\frac{\partial b_{ij}}{\partial t} =$$

$$= \frac{1}{\det \boldsymbol{B}}\sum_{i=1}^{n}\sum_{j=1}^{i}\det \boldsymbol{B}(2b_{ij}^{(-1)} - b_{ij}^{(-1)}\delta_{ij})\frac{\partial b_{ij}}{\partial t} = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^{(-1)}\frac{\partial b_{ij}}{\partial t} = \operatorname{tr} \boldsymbol{B}^{-1}\frac{\partial \boldsymbol{B}}{\partial t}$$

(lebo $\boldsymbol{B}$ aj $\boldsymbol{B}^{-1}$ sú symetrické matice). $\square$

Pomocou (i), (ii), (iii), a (iv) nájdeme maximum (69) riešením rovníc:

(70) $$\frac{\partial l}{\partial \boldsymbol{\beta}} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} - \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{0},$$

(71) $$\frac{\partial l}{\partial \sigma_i^2} = -\frac{1}{2}\operatorname{tr} \boldsymbol{V}^{-1}\boldsymbol{Z}_i\boldsymbol{Z}_i' + \frac{1}{2}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})'\boldsymbol{V}^{-1}\boldsymbol{Z}_i\boldsymbol{Z}_i'\boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}) = \boldsymbol{0},$$

$$i = 1, 2, \dots, s+1$$

(lebo $\dfrac{\partial \boldsymbol{V}}{\partial \sigma_i^2} = \boldsymbol{Z}_i\boldsymbol{Z}_i'$, $i = 1, 2, \dots, s+1$).

Vo všeobecnosti $\hat{\boldsymbol{\beta}}_{\text{real}}, \{\widehat{\sigma_1^2}\}_{\text{real}}, \dots \{\widehat{\sigma_{s+1}^2}\}_{\text{real}}$, ktoré maximalizujú (69) (pri realizácii $\boldsymbol{y}$) sú riešením (70) a (71). Každé riešenie (70) a (71) nemusí však byť realizáciou ML–odhadu $\hat{\boldsymbol{\beta}}, \widehat{\sigma_1^2}, \dots, \widehat{\sigma_{s+1}^2}$. Musíme sa ešte presvedčiť o matici druhých derivácií

$$
(72) \qquad \begin{pmatrix} \dfrac{\partial^2 l}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'} & \dfrac{\partial^2 l}{\partial \boldsymbol{\beta} \partial(\sigma_1^2, \ldots, \sigma_{s+1}^2)} \\ \dfrac{\partial^2 l}{\partial \begin{pmatrix} \sigma_1^2 \\ \vdots \\ \sigma_{s+1}^2 \end{pmatrix} \partial \boldsymbol{\beta}'} & \dfrac{\partial^2 l}{\partial \begin{pmatrix} \sigma_1^2 \\ \vdots \\ \sigma_{s+1}^2 \end{pmatrix} \partial(\sigma_1^2, \ldots, \sigma_{s+1}^2)} \end{pmatrix},
$$

že je pre získané riešenia (70) a (71) negatívne definitná.

$$
\frac{\partial^2 l}{\partial \boldsymbol{\beta} \partial \boldsymbol{\beta}'} \text{ je } q \times q \text{ matica, ktorej } (i,j)\text{-ty prvok je} \frac{\partial^2 l}{\partial \beta_i \partial \beta_j},
$$

$$
\frac{\partial^2 l}{\partial \boldsymbol{\beta} \partial(\sigma_1^2, \ldots, \sigma_{s+1}^2)} \text{ je } q \times (s+1) \text{ matica, ktorej } (i,j)\text{-ty prvok je} \frac{\partial^2 l}{\partial \beta_i \partial \sigma_j^2},
$$

podobne

$$
\frac{\partial^2 l}{\partial \begin{pmatrix} \sigma_1^2 \\ \vdots \\ \sigma_{s+1}^2 \end{pmatrix} \partial \boldsymbol{\beta}'} \quad \text{a} \quad \frac{\partial^2 l}{\partial \begin{pmatrix} \sigma_1^2 \\ \vdots \\ \sigma_{s+1}^2 \end{pmatrix} \partial(\sigma_1^2, \ldots, \sigma_{s+1}^2)}.
$$

Rovnako je nutné, aby

$$
(73) \qquad \{\widehat{\sigma_{s+1}^2}\}_{\text{real}} > 0, \quad \{\widehat{\sigma_i^2}\}_{\text{real}} \geqq 0 \quad i = 1, 2, \ldots, s.
$$

V takomto prípade $\hat{\boldsymbol{\beta}}_{\text{real}}, \{\widehat{\sigma_1^2}\}_{\text{real}}, \ldots \{\widehat{\sigma_{s+1}^2}\}_{\text{real}}$ sú realizáciami ML–odhadu $\hat{\boldsymbol{\beta}}$, $\widehat{\sigma_1^2} \ldots, \widehat{\sigma_{s+1}^2}$. Ak pre riešenia (70) a (71) neplatí (73), čo nastane keď niektoré $\{\widehat{\sigma_i^2}\}_{\text{real}}$, $i \in \{1, 2, \ldots, s+1\}$ sú záporné, postupujeme tak, že vynecháme v modeli tie faktory, pre ktoré $\{\widehat{\sigma_i^2}\}_{\text{real}}$ sú záporné a počítame ML–odhady v redukovanom modeli. Dá sa ukázať, že matica (72) je negatívne definitná v každom lokálnom maxime funkcie (69) pokiaľ toto maximum leží vo vnútri parametrického priestoru, t.j. ak pre riešenia (70) a (71) platí $\{\widehat{\sigma_1^2}\}_{\text{real}} > 0$, $i = 1, 2, \ldots, s+1$.

Pre riešenie $\boldsymbol{\beta}, \sigma_1^2, \ldots \sigma_{s+1}^2$ rovníc (70) a (71) platí

$$
\boldsymbol{B y} = (\boldsymbol{V}^{-1} - \boldsymbol{V}^{-1}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1})\boldsymbol{y} =
$$

$$
(74) \qquad = \boldsymbol{V}^{-1}\boldsymbol{y} - \boldsymbol{V}^{-1}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y} =
$$

$$
= \boldsymbol{V}^{-1}\boldsymbol{y} - \boldsymbol{V}^{-1}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{V}^{-1}(\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta})
$$

(využijeme lemu 3 pre maticu $\boldsymbol{V}^{-\frac{1}{2}}\boldsymbol{X}$, kde $\boldsymbol{V}^{-\frac{1}{2}} = \boldsymbol{U}\begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \frac{1}{\sqrt{\lambda_n}} \end{pmatrix}\boldsymbol{U}'$,

$\boldsymbol{V} = \boldsymbol{U}\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}\boldsymbol{U}'$, $\boldsymbol{I} = \boldsymbol{U}\boldsymbol{U}'$, $\boldsymbol{V}^{-\frac{1}{2}} = (\boldsymbol{V}^{\frac{1}{2}})^{-1}$, pozri [2], str. 64).

Pretože pre $i = 1, 2, \dots, s+1$ je

$$\text{(75)} \qquad \operatorname{tr} \boldsymbol{V}^{-1}\boldsymbol{Z}_i\boldsymbol{Z}_i' = \operatorname{tr} \boldsymbol{V}^{-1}\boldsymbol{Z}_i\boldsymbol{Z}_i'\boldsymbol{V}^{-1}\boldsymbol{V} = \operatorname{tr} \boldsymbol{V}^{-1}\boldsymbol{Z}_i\boldsymbol{Z}_i'\boldsymbol{V}^{-1}\sum_{j=1}^{s+1}\sigma_j^2\boldsymbol{Z}_j\boldsymbol{Z}_j' =$$

$$= \sum_{j=1}^{s+1}[\operatorname{tr}(\boldsymbol{V}^{-1}\boldsymbol{Z}_i\boldsymbol{Z}_i'\boldsymbol{V}^{-1}\boldsymbol{Z}_j\boldsymbol{Z}_j')]\sigma_j^2,$$

môžeme pomocou (74) a (75) písať ML–rovnice (70), (71) v ekvivalentnom tvare

$$\text{(76)} \qquad \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y},$$

(77)

$$\begin{pmatrix} \operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{V}^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1' & \operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{V}^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2' & \dots & \operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{V}^{-1}\boldsymbol{Z}_{s+1}\boldsymbol{Z}_{s+1}' \\ \operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2'\boldsymbol{V}^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1' & \operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2'\boldsymbol{V}^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2' & \dots & \operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2'\boldsymbol{V}^{-1}\boldsymbol{Z}_{s+1}\boldsymbol{Z}_{s+1}' \\ \vdots & & \ddots & \\ \operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{Z}_{s+1}\boldsymbol{Z}_{s+1}'\boldsymbol{V}^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1' & \operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{Z}_{s+1}\boldsymbol{Z}_{s+1}'\boldsymbol{V}^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2' & \dots & \operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{Z}_{s+1}\boldsymbol{Z}_{s+1}'\boldsymbol{V}^{-1}\boldsymbol{Z}_{s+1}\boldsymbol{Z}_{s+1}' \end{pmatrix} \times$$

$$\times \begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \\ \vdots \\ \sigma_{s+1}^2 \end{pmatrix} = \begin{pmatrix} \mathbf{y}'(\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1})\mathbf{Z}_1\mathbf{Z}_1'(\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1})\mathbf{y} \\ \mathbf{y}'(\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1})\mathbf{Z}_2\mathbf{Z}_2'(\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1})\mathbf{y} \\ \vdots \\ \mathbf{y}'(\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1})\mathbf{Z}_{s+1}\mathbf{Z}_{s+1}'(\mathbf{V}^{-1} - \mathbf{V}^{-1}\mathbf{X}(\mathbf{X}'\mathbf{V}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{V}^{-1})\mathbf{y} \end{pmatrix}.$$

ML–rovnice (70), (71) alebo (76), (77) sú nelineárne v parametroch $\sigma_1^2, \sigma_2^2, \dots, \sigma_{s+1}^2$ a riešia sa numericky (iteračne).

Poznámka 48. *Hartley a Rao v* [3] *zaviedli v modeli* (55) *transformáciu parametrov* $\sigma_1^2, \sigma_2^2, \dots, \sigma_{s+1}^2$ *na nové parametre*

$$\sigma_{s+1}^2, \gamma_1 = \frac{\sigma_1^2}{\sigma_{s+1}^2}, \dots, \gamma_s = \frac{\sigma_s^2}{\sigma_{s+1}^2}.$$

Preto

$$\boldsymbol{V} = \sigma_{s+1}^2\boldsymbol{H}, \quad \boldsymbol{V}^{-1} = \frac{1}{\sigma_{s+1}^2}\boldsymbol{H}^{-1}.$$

Toto vedie k preformulovaniu ML–rovníc a k ML–odhadom $\widehat{\sigma_{s+1}^2}, \hat{\gamma}_1, \dots, \hat{\gamma}_s$.
ML - rovnice pre $\boldsymbol{\beta}, \gamma_1, \dots, \gamma_s, \sigma_{s+1}^2$ *sú*

$$\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{y}$$

$$\sigma^2_{s+1} = \frac{(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})'\boldsymbol{H}^{-1}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})}{n}$$

$$\operatorname{tr}\boldsymbol{H}^{-1}\boldsymbol{Z}_i\boldsymbol{Z}_i' = \frac{(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})'\boldsymbol{H}^{-1}\boldsymbol{Z}_i\boldsymbol{Z}_i'\boldsymbol{H}^{-1}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})}{\sigma^2_{s+1}} \qquad i=1,2,\dots,s.$$

Iteračné riešenie rovníc Hartleya a Raa môže byť v mnohých prípadoch jednoduchšie ako iteračné riešenie ML–rovníc (76) a (77).

Poznámka 49. *Pre ML-odhady* $\hat{\boldsymbol{\beta}}, \widehat{\sigma^2_1}, \dots, \widehat{\sigma^2_{s+1}}$ *vieme určiť asymptotickú kovariančnú maticu (pre* $n \to \infty$*):*

$$\operatorname{cov}\left(\begin{pmatrix}\hat{\boldsymbol{\beta}}\\ \widehat{\sigma^2_1}\\ \vdots\\ \widehat{\sigma^2_{s+1}}\end{pmatrix}\right) \cong -\mathcal{E}\begin{pmatrix} \dfrac{\partial^2 l}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'} & \dfrac{\partial^2 l}{\partial\boldsymbol{\beta}\partial(\sigma^2_1,\dots,\sigma^2_{s+1})} \\ \dfrac{\partial^2 l}{\partial\begin{pmatrix}\sigma^2_1\\ \vdots\\ \sigma^2_{s+1}\end{pmatrix}\partial\boldsymbol{\beta}'} & \dfrac{\partial^2 l}{\partial\begin{pmatrix}\sigma^2_1\\ \vdots\\ \sigma^2_{s+1}\end{pmatrix}\partial(\sigma^2_1,\dots,\sigma^2_{s+1})}\end{pmatrix} =$$

$$= \mathcal{I}^{-1}\begin{pmatrix}\hat{\boldsymbol{\beta}}\\ \widehat{\sigma^2_1}\\ \vdots\\ \widehat{\sigma^2_{s+1}}\end{pmatrix}, \tag{78}$$

pričom v (78) *je* $l(\boldsymbol{\beta}, \sigma^2_1, \dots, \sigma^2_{s+1}, \boldsymbol{Y})$ *náhodnou premennou,* $\mathcal{I}\begin{pmatrix}\hat{\boldsymbol{\beta}}\\ \widehat{\sigma^2_1}\\ \vdots\\ \widehat{\sigma^2_{s+1}}\end{pmatrix}$ *je informačná matica ML–odhadov* $\hat{\boldsymbol{\beta}}, \widehat{\sigma^2_1}, \dots, \widehat{\sigma^2_{s+1}}$*. Platí*

$$\operatorname{cov}\left(\begin{pmatrix}\hat{\boldsymbol{\beta}}\\ \widehat{\sigma^2_1}\\ \vdots\\ \widehat{\sigma^2_{s+1}}\end{pmatrix}\right) \cong \begin{pmatrix}(\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1} & \boldsymbol{0}\\ \boldsymbol{0} & 2\boldsymbol{C}^{-1}\end{pmatrix},$$

kde

$$\{\boldsymbol{C}\}_{ij} = \operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{Z}_i\boldsymbol{Z}_i'\boldsymbol{V}^{-1}\boldsymbol{Z}_j\boldsymbol{Z}_j' \quad i,j=1,2,\dots,s+1.$$

Samozrejme tu vzniká otázka, čo to znamená „$n \to \infty$“ a „$\cong$“ . Hartley a Rao v [3] a Miller v [8] sa zaoberajú týmito problémami. Podrobnejšie tiež pozrite napr. v [13].

11 ML–ODHADY VO VYVÁŽENOM MODELI S JEDNÝM NÁHODNÝM EFEKTOM

Hľadajme ML–odhady $\hat{\beta}, \widehat{\sigma_1^2}, \widehat{\sigma_2^2}$ vo vyváženom modeli s jedným náhodným efektom, čiže v modeli

(79)
$$\boldsymbol{Y} = \begin{pmatrix} Y_{11} \\ Y_{12} \\ \vdots \\ Y_{1t} \\ Y_{21} \\ Y_{22} \\ \vdots \\ Y_{2t} \\ \vdots \\ Y_{I1} \\ Y_{I2} \\ \vdots \\ Y_{It} \end{pmatrix} = (\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2) \begin{pmatrix} \beta \\ \boldsymbol{b}_1 \\ \varepsilon \end{pmatrix},$$

$$\boldsymbol{X} = \boldsymbol{1}_{n,1}, \quad \boldsymbol{Z}_1 = \begin{pmatrix} \boldsymbol{1}_{t,1} \otimes \boldsymbol{e}_1' \\ \boldsymbol{1}_{t,1} \otimes \boldsymbol{e}_2' \\ \vdots \\ \boldsymbol{1}_{t,1} \otimes \boldsymbol{e}_I' \end{pmatrix}, \quad \boldsymbol{Z}_2 = \boldsymbol{I}_{n,n},$$

$$\mathrm{cov}(\boldsymbol{Y}) = \sigma_1^2 \boldsymbol{Z}_1 \boldsymbol{Z}_1' + \sigma_2^2 \boldsymbol{I} = \boldsymbol{V},$$

$\boldsymbol{Y}$ je normálne rozdelený. (Ak preznačíme $\sigma_2^2 = \sigma_0^2$, dostávame vyvážený model (40)).

Aj v tomto jednoduchom modeli sú rovnice (76) a (77) komplikované. Jednoduchšie je nájsť riešenie rovníc (70) a (71). Platí

$$[\mathrm{cov}(\boldsymbol{Y})]^{-1} = \boldsymbol{V}^{-1} = \frac{1}{\sigma_2^2}\boldsymbol{I} - \frac{\sigma_1^2}{\sigma_2^2(t\sigma_1^2 + \sigma_2^2)} \boldsymbol{Z}_1 \boldsymbol{Z}_1',$$

$$\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X} = \boldsymbol{1}' \left(\frac{1}{\sigma_2^2}\boldsymbol{I} - \frac{\sigma_1^2}{\sigma_2^2(t\sigma_1^2 + \sigma_2^2)} \boldsymbol{Z}_1 \boldsymbol{Z}_1' \right) \boldsymbol{1} = \frac{It}{(t\sigma_1^2 + \sigma_2^2)}$$

a

$$\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{Y} = \boldsymbol{1}' \left(\frac{1}{\sigma_2^2}\boldsymbol{I} - \frac{\sigma_1^2}{\sigma_2^2(t\sigma_1^2 + \sigma_2^2)} \boldsymbol{Z}_1 \boldsymbol{Z}_1' \right) \boldsymbol{Y}.$$

Pre $\hat{\beta}, \widehat{\sigma_1^2}$ a $\widehat{\sigma_2^2}$, ktoré sú riešením (70) platí

(80)
$$\hat{\beta} = (\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{V}^{-1}\boldsymbol{Y} = \frac{(t\widehat{\sigma_1^2} + \widehat{\sigma_2^2})}{It} \left(\frac{1}{\widehat{\sigma_2^2}}\boldsymbol{1}' - \frac{\widehat{\sigma_1^2}}{\widehat{\sigma_2^2}(t\widehat{\sigma_1^2} + \widehat{\sigma_2^2})} \boldsymbol{1}'\boldsymbol{Z}_1 \boldsymbol{Z}_1' \right) \boldsymbol{Y} =$$

$$= \frac{(\widehat{t\sigma_1^2+\sigma_2^2})}{It}\left(\frac{1}{\widehat{\sigma_2^2}}\mathbf{1}' - \frac{\widehat{\sigma_1^2}}{\widehat{\sigma_2^2}(\widehat{t\sigma_1^2+\sigma_2^2})}\mathbf{1}'\right)\boldsymbol{Y} = \frac{1}{It}\mathbf{1}'\boldsymbol{Y} = \frac{1}{It}\sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}.$$

Lema 50. *V modeli* (79) *platí*

$$\operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1' = \operatorname{tr}\boldsymbol{Z}_1'\boldsymbol{V}^{-1}\boldsymbol{Z}_1 = \frac{It}{(t\sigma_1^2+\sigma_2^2)},$$

$$(\boldsymbol{Y}-\boldsymbol{X}\hat{\beta})'\boldsymbol{V}^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{V}^{-1}(\boldsymbol{Y}-\boldsymbol{X}\hat{\beta}) = \boldsymbol{Y}\left[\frac{1}{(t\sigma_1^2+\sigma_2^2)^2}\left(\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{I}\mathbf{1}\mathbf{1}'\right)\right]\boldsymbol{Y},$$

kde $\hat{\beta}$ je dané vzťahom (80).

Dôkaz. Lemu dokážeme jednoduchým dosadením a úpravami, pričom využijeme rovnosti uvedené pred vzťahom (80). Dôkaz urobte ako cvičenie.

Dôsledok 51. *Nech $\hat{\beta}, \widehat{\sigma_1^2}$ a $\widehat{\sigma_2^2}$ sú riešenia ML–rovníc* (70) *a* (71) *pre $i = 1$. Platí*

(81)
$$\widehat{t\sigma_1^2} + \widehat{\sigma_2^2} = \frac{1}{It}\boldsymbol{Y}\left(\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{I}\mathbf{1}\mathbf{1}'\right)\boldsymbol{Y}.$$

Dôkaz. Vzťah (81) ľahko dostaneme ak využijeme výsledok lemy 50 a ML–rovnicu (71) pre $i = 1$. Dôkaz urobte ako cvičenie.

Lema 52. *V modeli* (79) *nech $\hat{\beta}, \widehat{\sigma_1^2}$ a $\widehat{\sigma_2^2}$ sú riešeniami rovníc* (70) *a* (71) *pre $i = 1$. Potom platí*

$$\operatorname{tr}\boldsymbol{V}^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2' = \operatorname{tr}\boldsymbol{V}^{-1} = \frac{It}{\widehat{\sigma_2^2}}\,\frac{(t-1)\widehat{\sigma_1^2}+\widehat{\sigma_2^2}}{t\widehat{\sigma_1^2}+\widehat{\sigma_2^2}},$$

$$(\boldsymbol{Y}-\boldsymbol{X}\hat{\beta})'\boldsymbol{V}^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2'\boldsymbol{V}^{-1}(\boldsymbol{Y}-\boldsymbol{X}\hat{\beta}) = \boldsymbol{Y}\left[\frac{1}{\widehat{\sigma_2^4}}\left(\boldsymbol{I} - \frac{1}{t}\boldsymbol{Z}_1\boldsymbol{Z}_1'\right)\right]\boldsymbol{Y} + \frac{I\widehat{\sigma_2^2}}{t\widehat{\sigma_1^2}+\widehat{\sigma_2^2}}.$$

Dôkaz. Lemu dokážeme podobne ako lemu 50. Užitočné je udržiavať pokope výraz $t\widehat{\sigma_1^2} + \widehat{\sigma_2^2}$. Dôkaz urobte ako cvičenie.

Dôsledok 53. *Nech $\hat{\beta}, \widehat{\sigma_1^2}$ a $\widehat{\sigma_2^2}$ sú riešeniami ML–rovníc* (70) *a* (71). *Platí*

$$\widehat{\sigma_2^2} = \boldsymbol{Y}'\left[\frac{1}{I(t-1)}\left(\boldsymbol{I} - \frac{1}{t}\boldsymbol{Z}_1\boldsymbol{Z}_1'\right)\right]\boldsymbol{Y} =$$

(82)
$$= \frac{1}{I(t-1)}\sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}^2 - \frac{1}{It(t-1)}\sum_{i=1}^{I}\left(\sum_{j=1}^{t} Y_{ij}\right)^2.$$

Dôkaz. Vzťah (82) dostaneme ak využijeme (80), (81) a lemu 52 v rovnici (71) pre $i = 2$. Dôkaz urobte ako cvičenie.

Pretože ML–odhad danej funkcie parametrov je ten istý ako táto funkcia ML – odhadov parametrov, pre ML–odhad $\widehat{\sigma_1^2}$ dostávame z (81) a (82)

$$\widehat{\sigma_1^2} = \frac{(\widehat{t\sigma_1^2 + \sigma_2^2}) - \widehat{\sigma_2^2}}{t} =$$

$$= \frac{1}{t}\left\{\frac{1}{It}\left(\boldsymbol{Y}'\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Y} - \frac{1}{I}\boldsymbol{Y}'\boldsymbol{1}\boldsymbol{1}'\boldsymbol{Y}\right)\boldsymbol{Y} - \frac{1}{I(t-1)}\boldsymbol{Y}'\boldsymbol{Y} + \frac{1}{It(t-1)}\boldsymbol{Y}'\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{Y}\right\} =$$

$$= \boldsymbol{Y}'\left[\frac{1}{It(t-1)}\left(\boldsymbol{Z}_1\boldsymbol{Z}_1' - \boldsymbol{I} - \frac{t-1}{It}\boldsymbol{1}\boldsymbol{1}'\right)\right]\boldsymbol{Y} =$$

$$= \frac{1}{It(t-1)}\sum_{i=1}^{I}\left(\sum_{j=1}^{t} Y_{ij}\right)^2 - \frac{1}{It(t-1)}\sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}^2 -$$

$$-\frac{1}{I^2t^2}\left(\sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}\right)^2. \tag{83}$$

Poznámka 54. *Samozrejme* $\{\widehat{\sigma_1^2}\}_{\text{real}}$ *a* $\{\widehat{\sigma_2^2}\}_{\text{real}}$ *sú realizácie ML–odhadov len ak* $\{\widehat{\sigma_1^2}\}_{\text{real}} \geqq 0$ a $\{\widehat{\sigma_2^2}\}_{\text{real}} > 0$.

Poznámka 55. *ML–odhad* $\widehat{\sigma_2^2}$ *je ten istý ako odhad* (48) *navrhnutý Hendersonom, či MINQUE odhad parametra* σ_2^2. *ML–odhad* $\widehat{\sigma_1^2}$ *je rôzny od Hendersonovho odhadu parametra* σ_1^2 *(Hendersonov odhad je ten istý ako MINQUE odhad).*

Samozrejme ML-odhad $\widehat{\sigma_2^2}$ je preto nevychýleným odhadom σ_2^2. Počítajme teraz strednú hodnotu ML–odhadu $\widehat{\sigma_1^2}$. Podľa lemy 10 je

$$\mathcal{E}(\widehat{\sigma_1^2}) = \frac{1}{It(t-1)}\left\{\beta^2\boldsymbol{1}'\left(\boldsymbol{Z}_1\boldsymbol{Z}_1' - \boldsymbol{I} - \frac{t-1}{It}\boldsymbol{1}\boldsymbol{1}'\right)\boldsymbol{1} + \right.$$

$$\left. + \operatorname{tr}\left(\boldsymbol{Z}_1\boldsymbol{Z}_1' - \boldsymbol{I} - \frac{t-1}{It}\boldsymbol{1}\boldsymbol{1}'\right)(\sigma_2^2\boldsymbol{I} + \sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1')\right\} =$$

$$= \frac{I-1}{I}\sigma_1^2 - \frac{\sigma_2^2}{It}. \tag{84}$$

Poznámka 56. *Z* (84) *vyplýva, že* $\widehat{\sigma_1^2}$ *nie je nevychýleným odhadom* σ_1^2.

Lema 57. *Pre kovariančnú maticu ML–odhadov* $(\widehat{\sigma_1^2}, \widehat{\sigma_2^2})'$ *platí*

$$\operatorname{cov}\left(\begin{pmatrix}\widehat{\sigma_1^2}\\ \widehat{\sigma_2^2}\end{pmatrix}\right) = \begin{pmatrix} \frac{2\sigma_2^4}{It^2(t-1)} + \frac{2(t\sigma_1^2+\sigma_2^2)^2}{It^2}\frac{I-1}{I} & -\frac{\sigma_2^4}{It(t-1)} \\ -\frac{\sigma_2^4}{It(t-1)} & \frac{2\sigma_2^4}{I(t-1)} \end{pmatrix}. \tag{85}$$

Dôkaz. Lemu dokážeme pomocou vety 18 (pozri tiež cvičenie 1 na str. 32). Opäť odporúčame udržiavať pokope výraz $t\sigma_1^2 + \sigma_2^2$. Dôkaz urobte ako cvičenie.

ML–odhady v zložitejších modeloch ako aj iné typy odhadov nájdete napr. v [13].

12 ODHADY ZÍSKANÉ METÓDOU NAJMENŠÍCH ŠTVORCOV

Majme zmiešaný lineárny model

$$\boldsymbol{Y} = \boldsymbol{X\beta} + \boldsymbol{Z}_1\boldsymbol{b}_1 + \boldsymbol{Z}_2\boldsymbol{b}_2 + \ldots + \boldsymbol{Z}_s\boldsymbol{b}_s + \boldsymbol{Z}_{s+1}\boldsymbol{b}_{s+1}$$

s rovnakým označením ako v (55) (resp. (15), pričom v (15) predpokladáme $\boldsymbol{I} = \boldsymbol{Z}_{s+1}$, $\boldsymbol{\varepsilon} = \boldsymbol{b}_{s+1}$, $\sigma_{s+1}^2 = \sigma_0^2$). Samozrejme

$$\operatorname{cov}(\boldsymbol{Y}) = \sigma_1^2\boldsymbol{Z}_1\boldsymbol{Z}_1' + \sigma_2^2\boldsymbol{Z}_2\boldsymbol{Z}_2' + \ldots + \sigma_s^2\boldsymbol{Z}_s\boldsymbol{Z}_s' + \sigma_{s+1}^2\boldsymbol{Z}_{s+1}\boldsymbol{Z}_{s+1}' = \sum_{i=1}^{s+1}\sigma_i^2\boldsymbol{V}_i.$$

Definícia 58. *Povieme, že $\hat{\boldsymbol{\beta}}$ je odhad pevných efektov získaný metódou najmenších štvorcov, ak*

$$\|\boldsymbol{X}\hat{\boldsymbol{\beta}} - \boldsymbol{Y}\|^2 = \inf_{\boldsymbol{\beta}\in\mathbb{R}^q}\|\boldsymbol{X\beta} - \boldsymbol{Y}\|^2,$$

pričom $\|\cdot\|$ je Euklidovská norma vektora, t.j. $\|\boldsymbol{z}\| = \sqrt{\boldsymbol{z}'\boldsymbol{z}}$.

Veta 59. *Ak $\hat{\boldsymbol{\beta}}$ je odhad pevných efektov získaný metódou najmenších štvorcov, tak*

$$\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y}$$

a pre lineárne nevychýlene odhadnuteľnú funkciu $\boldsymbol{p}'\boldsymbol{\beta}$ je disperzia

$$\mathcal{D}(\boldsymbol{p}'\hat{\boldsymbol{\beta}}) = \boldsymbol{p}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\sum_{i=1}^{s+1}\sigma_i^2\boldsymbol{V}_i\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{p}.$$

Dôkaz. Platí

$$\|\boldsymbol{X\beta} - \boldsymbol{Y}\|^2 = (\boldsymbol{X\beta} - \boldsymbol{Y})'(\boldsymbol{X\beta} - \boldsymbol{Y}) = \boldsymbol{\beta}'\boldsymbol{X}'\boldsymbol{X\beta} - 2\boldsymbol{Y}'\boldsymbol{X\beta} + \boldsymbol{Y}'\boldsymbol{Y}.$$

Nech

$$\inf_{\boldsymbol{\beta}\in\mathbb{R}^q}\|\boldsymbol{X\beta} - \boldsymbol{Y}\|^2 = \|\boldsymbol{X}[(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y} + \boldsymbol{\delta}] - \boldsymbol{Y}\|^2 =$$

$$= (\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y} + \boldsymbol{X\delta} - \boldsymbol{Y})'(\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y} + \boldsymbol{X\delta} - \boldsymbol{Y}) =$$

$$= \boldsymbol{Y}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y} + \boldsymbol{Y}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{X\delta} - \boldsymbol{Y}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y} +$$

$$+ \boldsymbol{\delta}'\boldsymbol{X}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y} + \boldsymbol{\delta}'\boldsymbol{X}'\boldsymbol{X\delta} - \boldsymbol{\delta}'\boldsymbol{X}'\boldsymbol{Y} - \boldsymbol{Y}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y} - \boldsymbol{Y}'\boldsymbol{X\delta} + \boldsymbol{Y}'\boldsymbol{Y} =$$

$$= \|\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y} - \boldsymbol{Y}\|^2 + \|\boldsymbol{X\delta}\|^2,$$

teda vskutku

$$\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y}.$$

Podľa lemy 1 je $\boldsymbol{p}'\boldsymbol{\beta}$ lineárne nevychýlene odhadnuteľná práve vtedy ak $\boldsymbol{p} \in \mu(\boldsymbol{X}')$, teda $\exists\boldsymbol{w} \in \mathbb{R}^n$, že $\boldsymbol{p} = \boldsymbol{X}'\boldsymbol{w}$. Preto

$$\mathcal{D}(\boldsymbol{p}'\hat{\boldsymbol{\beta}}) = \mathcal{D}(\boldsymbol{w}'\boldsymbol{X}\hat{\boldsymbol{\beta}}) = \mathcal{D}(\boldsymbol{w}'\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{Y}).$$

Dôkaz už ľahko dokončíme. $\square$

Definícia 60. *Odhad* $\hat{\boldsymbol{\sigma}} = (\widehat{\sigma_1^2}, \dots, \widehat{\sigma_{s+1}^2})'$, *pre ktorý platí*

$$\|\sum_{i=1}^{s+1} \widehat{\sigma_i^2} \boldsymbol{V}_i - \widetilde{\boldsymbol{\Sigma}}\|^2 = \min_{c_1,\dots,c_{s+1}} \|\sum_{i=1}^{s+1} c_i \boldsymbol{V}_i - \widetilde{\boldsymbol{\Sigma}}\|^2,$$

kde

$$\widetilde{\boldsymbol{\Sigma}} = (\boldsymbol{X}\hat{\boldsymbol{\beta}} - \boldsymbol{Y})(\boldsymbol{X}\hat{\boldsymbol{\beta}} - \boldsymbol{Y})',$$

($\hat{\boldsymbol{\beta}}$ *je odhad pevných efektov získaný metódou najmenších štvorcov;* $\widetilde{\boldsymbol{\Sigma}}$ *je tzv. predbežný odhad matice* $\operatorname{cov}(\boldsymbol{Y})$*;* $\|.\|$ *je Euklidovská norma matice, teda* $\|\boldsymbol{A}\| = \sqrt{\operatorname{tr} \boldsymbol{A}\boldsymbol{A}'}$*) sa nazýva odhad variančných komponentov získaný metódou najmenších štvorcov.*

Veta 61. *Označme*

$$\boldsymbol{q} = \begin{pmatrix} \operatorname{tr} \widetilde{\boldsymbol{\Sigma}}\boldsymbol{V}_1 \\ \operatorname{tr} \widetilde{\boldsymbol{\Sigma}}\boldsymbol{V}_2 \\ \vdots \\ \operatorname{tr} \widetilde{\boldsymbol{\Sigma}}\boldsymbol{V}_{s+1} \end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix} \operatorname{tr} \boldsymbol{V}_1\boldsymbol{V}_1 & \operatorname{tr} \boldsymbol{V}_1\boldsymbol{V}_2 & \dots & \operatorname{tr} \boldsymbol{V}_1\boldsymbol{V}_{s+1} \\ \operatorname{tr} \boldsymbol{V}_2\boldsymbol{V}_1 & \operatorname{tr} \boldsymbol{V}_2\boldsymbol{V}_2 & \dots & \operatorname{tr} \boldsymbol{V}_2\boldsymbol{V}_{s+1} \\ \vdots & & & \\ \operatorname{tr} \boldsymbol{V}_{s+1}\boldsymbol{V}_1 & \operatorname{tr} \boldsymbol{V}_{s+1}\boldsymbol{V}_2 & \dots & \operatorname{tr} \boldsymbol{V}_{s+1}\boldsymbol{V}_{s+1} \end{pmatrix}.$$

$\hat{\boldsymbol{\sigma}} = (\widehat{\sigma_1^2}, \dots, \widehat{\sigma_{s+1}^2})'$ *je odhad variančných komponentov získaný metódou najmenších štvorcov práve vtedy ak*

$$\boldsymbol{Q}\hat{\boldsymbol{\sigma}} = \boldsymbol{q}. \tag{86}$$

Dôkaz. Hľadajme

$$\min_{c_1,\dots,c_{s+1}} \|\sum_{i=1}^{s+1} c_i \boldsymbol{V}_i - \widetilde{\boldsymbol{\Sigma}}\|^2 =$$

$$= \min_{c_1,\dots,c_{s+1}} \operatorname{tr}(\sum_{i=1}^{s+1} c_i \boldsymbol{V}_i - \widetilde{\boldsymbol{\Sigma}})(\sum_{j=1}^{s+1} c_j \boldsymbol{V}_j - \widetilde{\boldsymbol{\Sigma}}) =$$

$$= \min_{c_1,\dots,c_{s+1}} \left\{ \sum_{i=1}^{s+1}\sum_{j=1}^{s+1} c_i c_j tr \boldsymbol{V}_i \boldsymbol{V}_j - 2\sum_{i=1}^{s+1} c_i \operatorname{tr} \widetilde{\boldsymbol{\Sigma}}\boldsymbol{V}_i + \operatorname{tr} \widetilde{\boldsymbol{\Sigma}}\widetilde{\boldsymbol{\Sigma}} \right\} =$$

$$= \min_{\boldsymbol{c}\in\mathbb{R}^{s+1}} \{\boldsymbol{c}'\boldsymbol{Q}\boldsymbol{c} - 2\boldsymbol{c}'\boldsymbol{q} + \operatorname{tr} \widetilde{\boldsymbol{\Sigma}}\widetilde{\boldsymbol{\Sigma}}\} = \Phi(\boldsymbol{c}).$$

Nutná podmienka existencie minima je

$$\frac{\partial \Phi}{\partial \boldsymbol{c}} = 2\boldsymbol{Q}\boldsymbol{c} - 2\boldsymbol{q} = \boldsymbol{0},$$

čiže ak $\hat{\boldsymbol{\sigma}}$ je odhadom variančných komponentov získaný metódou najmenších štvorcov, tak

$$\boldsymbol{Q}\hat{\boldsymbol{\sigma}} = \boldsymbol{q}.$$

Naopak, majme $\boldsymbol{u}^{(0)} \in \mathbb{R}^{s+1}$ pre ktorý platí $\boldsymbol{Q}\boldsymbol{u}^{(0)} = \boldsymbol{q}$ a $\boldsymbol{\delta}$ ľubovoľný z $\mathbb{R}^{s+1}$. Potom

$$\min_{c_1,\dots,c_{s+1}} \|\sum_{i=1}^{s+1} c_i \boldsymbol{V}_i - \widetilde{\boldsymbol{\Sigma}}\|^2 =$$

$$= \min_{\delta_1,\dots,\delta_{s+1}} \|\sum_{i=1}^{s+1} (u_i^{(0)} + \delta_i)\boldsymbol{V}_i - \widetilde{\boldsymbol{\Sigma}}\|^2 =$$

$$= \min_{\delta_1,\dots,\delta_{s+1}} \operatorname{tr}(\sum_{i=1}^{s+1} (u_i^{(0)} + \delta_i)\boldsymbol{V}_i - \widetilde{\boldsymbol{\Sigma}})(\sum_{j=1}^{s+1} (u_j^{(0)} + \delta_j)\boldsymbol{V}_j - \widetilde{\boldsymbol{\Sigma}})' =$$

$$= \min_{\boldsymbol{\delta}\in\mathbb{R}^{s+1}} \{(\boldsymbol{u}^{(0)} + \boldsymbol{\delta})'\boldsymbol{Q}(\boldsymbol{u}^{(0)} + \boldsymbol{\delta}) - 2(\boldsymbol{u}^{(0)} + \boldsymbol{\delta})'\boldsymbol{q} + \operatorname{tr} \widetilde{\boldsymbol{\Sigma}}\widetilde{\boldsymbol{\Sigma}}\} =$$

$$= \min_{\boldsymbol{\delta}\in\mathbb{R}^{s+1}} \{\boldsymbol{u}^{(0)'}\boldsymbol{Q}\boldsymbol{u}^{(0)} + 2\boldsymbol{u}^{(0)'}\boldsymbol{Q}\boldsymbol{\delta} + \boldsymbol{\delta}'\boldsymbol{Q}\boldsymbol{\delta} - 2\boldsymbol{q}'\boldsymbol{u}^{(0)} - 2\boldsymbol{q}'\boldsymbol{\delta} + \operatorname{tr} \widetilde{\boldsymbol{\Sigma}}\widetilde{\boldsymbol{\Sigma}}\} =$$

$$= \min_{\boldsymbol{\delta}\in\mathbb{R}^{s+1}} \{\boldsymbol{\delta}'\boldsymbol{Q}\boldsymbol{\delta} - \boldsymbol{u}^{(0)'}\boldsymbol{Q}\boldsymbol{u}^{(0)} + \operatorname{tr} \widetilde{\boldsymbol{\Sigma}}\widetilde{\boldsymbol{\Sigma}}\} = -\boldsymbol{u}^{(0)'}\boldsymbol{Q}\boldsymbol{u}^{(0)} + \operatorname{tr} \widetilde{\boldsymbol{\Sigma}}\widetilde{\boldsymbol{\Sigma}},$$

z čoho vyplýva, že minimum je dosiahnuté pre vektor $\hat{\boldsymbol{\sigma}} = \boldsymbol{u}^{(0)} + \boldsymbol{\delta}$, pričom $\boldsymbol{\delta}'\boldsymbol{Q}\boldsymbol{\delta} = \boldsymbol{0} \iff \boldsymbol{Q}\boldsymbol{\delta} = \boldsymbol{0}$ (lebo $\boldsymbol{Q}$ je p.s.d. matica). Teda pre odhad $\hat{\boldsymbol{\sigma}}$ variančných komponentov získaný metódou najmenších štvorcov platí $\boldsymbol{Q}\hat{\boldsymbol{\sigma}} = \boldsymbol{Q}(\boldsymbol{u}^{(0)} + \boldsymbol{\delta}) = \boldsymbol{q}$. □

Dôsledok 62. *Odhad variančných komponentov získaný metódou najmenších štvorcov existuje práve vtedy ak* $\boldsymbol{q} \in \mu(\boldsymbol{Q})$.

Dôsledok 63. *Ak* $\boldsymbol{V}_1, \dots, \boldsymbol{V}_{s+1}$ *sú lineárne nezávislé, tak odhad* $\widehat{\boldsymbol{\sigma}^2}$ *variančných komponentov získaný metódou najmenších štvorcov je*

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{Q}^{-1}\boldsymbol{q},$$

kde

$$\boldsymbol{q} = \begin{pmatrix} \boldsymbol{Y}'\boldsymbol{M_X}\boldsymbol{V}_1\boldsymbol{M_X}\boldsymbol{Y} \\ \boldsymbol{Y}'\boldsymbol{M_X}\boldsymbol{V}_2\boldsymbol{M_X}\boldsymbol{Y} \\ \vdots \\ \boldsymbol{Y}'\boldsymbol{M_X}\boldsymbol{V}_{s+1}\boldsymbol{M_X}\boldsymbol{Y} \end{pmatrix},$$

$\boldsymbol{M_X} = \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'$.

Dôkaz. Oprieme sa o fakt, že $\boldsymbol{V}_1, \dots, \boldsymbol{V}_{s+1}$ sú lineárne nezávislé práve vtedy, ak $\boldsymbol{Q}$ je regulárna matica a využijeme jednoduchú úpravu

$$q_i = \operatorname{tr} \widetilde{\boldsymbol{\Sigma}}\boldsymbol{V}_i = \operatorname{tr}(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'\boldsymbol{V}_i =$$

$$= (\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}})'\boldsymbol{V}_i(\boldsymbol{Y} - \boldsymbol{X}\hat{\boldsymbol{\beta}}) = \boldsymbol{Y}(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}')\boldsymbol{V}_i(\boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}')\boldsymbol{Y} =$$

$$= \boldsymbol{Y}\boldsymbol{M_X}\boldsymbol{V}_i\boldsymbol{M_X}\boldsymbol{Y}, \quad i = 1, 2, \dots, s+1.$$

Dôkaz urobte ako cvičenie.

Poznámka 64. *Ak* $\boldsymbol{p} = (p_1, p_2, \ldots, p_{s+1})' \in \mathbb{R}^{s+1}$, *tak odhad*

$$\widehat{\sum_{i=1}^{s+1} p_i \sigma_i^2} = \boldsymbol{p}'\hat{\boldsymbol{\sigma}},$$

kde $\boldsymbol{Q}\hat{\boldsymbol{\sigma}} = \boldsymbol{q}$.

Veta 65. *Ak* $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_{s+1}$ *sú lineárne nezávislé, tak*

$$\mathcal{E}(\widehat{\sigma_i^2}) = (\sigma_1^2, \sigma_2^2, \ldots, \sigma_{s+1}^2)\boldsymbol{R}\boldsymbol{Q}^{-1}\boldsymbol{e}_i, \quad i = 1, 2, \ldots, s+1, \tag{87}$$

kde

$$\boldsymbol{R} = \begin{pmatrix} \operatorname{tr} \boldsymbol{M_X V}_1 \boldsymbol{M_X V}_1 & \operatorname{tr} \boldsymbol{M_X V}_1 \boldsymbol{M_X V}_2 & \ldots & \operatorname{tr} \boldsymbol{M_X V}_1 \boldsymbol{M_X V}_{s+1} \\ \operatorname{tr} \boldsymbol{M_X V}_2 \boldsymbol{M_X V}_1 & \operatorname{tr} \boldsymbol{M_X V}_2 \boldsymbol{M_X V}_2 & \ldots & \operatorname{tr} \boldsymbol{M_X V}_2 \boldsymbol{M_X V}_{s+1} \\ \vdots & & \ddots & \\ \operatorname{tr} \boldsymbol{M_X V}_{s+1} \boldsymbol{M_X V}_1 & \operatorname{tr} \boldsymbol{M_X V}_{s+1} \boldsymbol{M_X V}_2 & \ldots & \operatorname{tr} \boldsymbol{M_X V}_{s+1} \boldsymbol{M_X V}_{s+1} \end{pmatrix}.$$

Dôkaz. Podľa dôsledku 63 je pre $i = 1, 2, \ldots, s+1$

$$\widehat{\sigma_i^2} = \boldsymbol{e}_i'\boldsymbol{Q}^{-1}\boldsymbol{q} = \boldsymbol{q}'\boldsymbol{Q}^{-1}\boldsymbol{e}_i,$$

teda pomocou lemy 10

$$\mathcal{E}(\widehat{\sigma_i^2}) = (\mathcal{E}(\boldsymbol{Y}'\boldsymbol{M_X V}_1 \boldsymbol{M_X Y}), \ldots, \mathcal{E}(\boldsymbol{Y}'\boldsymbol{M_X V}_{s+1}\boldsymbol{M_X Y}))\boldsymbol{Q}^{-1}\boldsymbol{e}_i =$$

$$= \left(\operatorname{tr} \boldsymbol{M_X V}_1 \boldsymbol{M_X} \sum_{i=1}^{s+1} \sigma_i^2 \boldsymbol{V}_i, \ldots, \operatorname{tr} \boldsymbol{M_X V}_{s+1} \boldsymbol{M_X} \sum_{i=1}^{s+1} \sigma_i^2 \boldsymbol{V}_i\right) \boldsymbol{Q}^{-1}\boldsymbol{e}_i =$$

$$= \left(\sum_{i=1}^{s+1} \sigma_i^2 \operatorname{tr} \boldsymbol{M_X V}_1 \boldsymbol{M_X V}_i, \ldots, \sum_{i=1}^{s+1} \sigma_i^2 \operatorname{tr} \boldsymbol{M_X V}_s \boldsymbol{M_X V}_i\right) \boldsymbol{Q}^{-1}\boldsymbol{e}_i =$$

$$= (\sigma_1^2, \sigma_2^2, \ldots, \sigma_{s+1}^2)\boldsymbol{R}\boldsymbol{Q}^{-1}\boldsymbol{e}_i. \qquad \square$$

Veta 66. *Ak* $\boldsymbol{Y} \sim N\left(\boldsymbol{X\beta}, \sum_{i=1}^{s+1} \sigma_i^2 \boldsymbol{V}_i\right)$ a $\boldsymbol{V}_1, \ldots, \boldsymbol{V}_{s+1}$ *sú lineárne nezávislé, je*

$$\mathcal{D}(\widehat{\sigma_i^2}) = 2\boldsymbol{e}_i'\boldsymbol{Q}^{-1}\boldsymbol{W}\boldsymbol{Q}^{-1}\boldsymbol{e}_i, \quad i = 1, 2, \ldots, s+1,$$

kde

$$\{\boldsymbol{W}\}_{i,j} = \sum_{k=1}^{s+1}\sum_{l=1}^{s+1} \sigma_k^2\sigma_l^2 \operatorname{tr} \boldsymbol{M_X V}_i \boldsymbol{M_X V}_k \boldsymbol{M_X V}_j \boldsymbol{M_X V}_l, \quad i, j = 1, 2, \ldots, s+1.$$

Dôkaz. Počítajme pre $i, j = 1, 2, \ldots, s+1$ použitím vety 18 a lemy 10

$$\operatorname{cov}(\boldsymbol{Y}'\boldsymbol{M_X V}_i \boldsymbol{M_X Y}, \boldsymbol{Y}'\boldsymbol{M_X V}_j \boldsymbol{M_X Y}) =$$

$$= \mathcal{E}(\boldsymbol{Y}'\boldsymbol{M_X}\boldsymbol{V}_i\boldsymbol{M_X}\boldsymbol{Y}\boldsymbol{Y}'\boldsymbol{M_X}\boldsymbol{V}_j\boldsymbol{M_X}\boldsymbol{Y}) - \mathcal{E}(\boldsymbol{Y}'\boldsymbol{M_X}\boldsymbol{V}_i\boldsymbol{M_X}\boldsymbol{Y})\mathcal{E}(\boldsymbol{Y}'\boldsymbol{M_X}\boldsymbol{V}_j\boldsymbol{M_X}\boldsymbol{Y}) =$$

$$= 2\sum_{k=1}^{s+1}\sum_{l=1}^{s+1}\sigma_k^2\sigma_l^2 \operatorname{tr} \boldsymbol{M_X}\boldsymbol{V}_i\boldsymbol{M_X}\boldsymbol{V}_k\boldsymbol{M_X}\boldsymbol{V}_j\boldsymbol{M_X}\boldsymbol{V}_l +$$

$$+ \operatorname{tr} \boldsymbol{M_X}\boldsymbol{V}_i\boldsymbol{M_X}\sum_{k=1}^{s+1}\sigma_k\boldsymbol{V}_k \operatorname{tr} \boldsymbol{M_X}\boldsymbol{V}_j\boldsymbol{M_X}\sum_{l=1}^{s+1}\sigma_l\boldsymbol{V}_l -$$

$$- \operatorname{tr} \boldsymbol{M_X}\boldsymbol{V}_i\boldsymbol{M_X}\sum_{k=1}^{s+1}\sigma_k\boldsymbol{V}_k \operatorname{tr} \boldsymbol{M_X}\boldsymbol{V}_j\boldsymbol{M_X}\sum_{l=1}^{s+1}\sigma_l\boldsymbol{V}_l =$$

$$= 2\sum_{k=1}^{s+1}\sum_{l=1}^{s+1}\sigma_k^2\sigma_l^2 \operatorname{tr} \boldsymbol{M_X}\boldsymbol{V}_i\boldsymbol{M_X}\boldsymbol{V}_k\boldsymbol{M_X}\boldsymbol{V}_j\boldsymbol{M_X}\boldsymbol{V}_l.$$

Preto

$$\operatorname{cov}(\boldsymbol{q}) = 2\boldsymbol{W}.$$

Teraz už ľahko dostávame

$$\mathcal{D}(\widehat{\sigma_i^2}) = \mathcal{D}(\boldsymbol{e}_i'\boldsymbol{Q}^{-1}\boldsymbol{q}) = \boldsymbol{e}_i'\boldsymbol{Q}^{-1}\operatorname{cov}(\boldsymbol{q})\boldsymbol{Q}^{-1}\boldsymbol{e}_i = 2\boldsymbol{e}_i'\boldsymbol{Q}^{-1}\boldsymbol{W}\boldsymbol{Q}^{-1}\boldsymbol{e}_i. \qquad \square \tag{88}$$

Iné vlastnosti odhadov získaných metódou najmenších štvorcov pozri napr. v [15].

13 ODHADY ZÍSKANÉ METÓDOU NAJMENŠÍCH ŠTVORCOV V MODELI S JEDNÝM NÁHODNÝM EFEKTOM

Majme model (40), teda

$$Y_{ij} = \beta + b_i + \boldsymbol{\varepsilon}_{ij}, \qquad \begin{aligned} i &= 1, 2, \ldots, I\,, \\ j &= 1, 2, \ldots, n_i\,, \end{aligned}$$

kde náhodný vektor

$$\boldsymbol{b}_1 = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_I \end{pmatrix} \sim N(\boldsymbol{0}_{I,1}; \sigma_1^2\boldsymbol{I})$$

a chybový vektor

$$\boldsymbol{\varepsilon} \sim N(\boldsymbol{0}, \sigma_2^2\boldsymbol{I})$$

sú nezávislé. Model zapíšeme ako

$$\boldsymbol{Y}_{n,1} = (\boldsymbol{X} \vdots \boldsymbol{Z}_1 \vdots \boldsymbol{Z}_2)\begin{pmatrix} \beta \\ \boldsymbol{b}_1 \\ \boldsymbol{\varepsilon} \end{pmatrix},$$

kde

$$\boldsymbol{X} = \mathbf{1}_{n,1}, \quad \boldsymbol{Z}_1 = \begin{pmatrix} \mathbf{1}_{n_1,1} \otimes \boldsymbol{e}_1' \\ \mathbf{1}_{n_2,1} \otimes \boldsymbol{e}_2' \\ \vdots \\ \mathbf{1}_{n_I,1} \otimes \boldsymbol{e}_I' \end{pmatrix}, \quad \boldsymbol{Z}_2 = \boldsymbol{I}_{n,n},$$

$n = \sum_{i=1}^{I} n_i$ (pozri aj kapitolu 9),

$$\mathcal{E}(\boldsymbol{Y}) = \boldsymbol{X}\beta = \mathbf{1}\beta,$$

$$\operatorname{cov}(\boldsymbol{Y}) = \sigma_1^2 \boldsymbol{Z}_1 \boldsymbol{Z}_1' + \sigma_2^2 \boldsymbol{Z}_2 \boldsymbol{Z}_2' = \sigma_1^2 \boldsymbol{V}_1 + \sigma_2^2 \boldsymbol{V}_2 = \sigma_1^2 \boldsymbol{V}_1 + \sigma_2^2 \boldsymbol{I}.$$

Pomocou vety 61, dôsledku 63, poznámky 64, vzorcov (87) a (88) spočítame odhady získané metódou najmenších štvorcov a určíme ich stredné hodnoty a disperzie. Platí

$$\operatorname{tr} \boldsymbol{V}_1 \boldsymbol{V}_1 = \operatorname{tr} \boldsymbol{Z}_1 \boldsymbol{Z}_1' \boldsymbol{Z}_1 \boldsymbol{Z}_1' = \operatorname{tr} \boldsymbol{Z}_1' \boldsymbol{Z}_1 \boldsymbol{Z}_1' \boldsymbol{Z}_1 = \sum_{i=1}^{I} n_i^2,$$

$$\operatorname{tr} \boldsymbol{V}_1 \boldsymbol{V}_2 = \operatorname{tr} \boldsymbol{V}_1 \boldsymbol{I} = \operatorname{tr} \boldsymbol{Z}_1 \boldsymbol{Z}_1' = \sum_{i=1}^{I} n_i = n,$$

$$\operatorname{tr} \boldsymbol{V}_2 \boldsymbol{V}_2 = \operatorname{tr} \boldsymbol{I}\ \boldsymbol{I} = \sum_{i=1}^{I} n_i = n.$$

Preto pre maticu $\boldsymbol{Q}$ z vety 61 a jej inverziu platí

$$\boldsymbol{Q} = \begin{pmatrix} \sum_{i=1}^{I} n_i^2 & n \\ n & n \end{pmatrix},$$

(89) $$\boldsymbol{Q}^{-1} = \frac{1}{n(\sum_{i=1}^{I} n_i^2 - n)} \begin{pmatrix} n & -n \\ -n & \sum_{i=1}^{I} n_i^2 \end{pmatrix}.$$

Ďalej budeme potrebovať vyjadriť maticu $\boldsymbol{M_X}$ (z dôsledku 63):

$$\boldsymbol{M_X} = \boldsymbol{I} - \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}' = \boldsymbol{I} - \frac{1}{n}\mathbf{1}_{n,1}\mathbf{1}_{1,n}'.$$

Spočítame aj

$$\boldsymbol{M_X}\boldsymbol{V}_1\boldsymbol{M_X} = (\boldsymbol{I} - \frac{1}{n}\mathbf{1}\mathbf{1}')\boldsymbol{Z}_1\boldsymbol{Z}_1'(\boldsymbol{I} - \frac{1}{n}\mathbf{1}\mathbf{1}') =$$

$$= \boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{n}\mathbf{1}\mathbf{1}'\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{n}\boldsymbol{Z}_1\boldsymbol{Z}_1'\mathbf{1}\mathbf{1}' + \frac{1}{n^2}\mathbf{1}\mathbf{1}'\boldsymbol{Z}_1\boldsymbol{Z}_1'\mathbf{1}\mathbf{1}'.$$

Využijúc výpočty z kapitoly 6 dostávame

(90) $$\boldsymbol{Y}'\boldsymbol{M_X}\boldsymbol{V}_1\boldsymbol{M_X}\boldsymbol{Y} =$$

$$= \boldsymbol{Y}(\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{n}\boldsymbol{1}\boldsymbol{1}'\boldsymbol{Z}_1\boldsymbol{Z}_1' - \frac{1}{n}\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{1}\boldsymbol{1}' + \frac{1}{n^2}\boldsymbol{1}\boldsymbol{1}'\boldsymbol{Z}_1\boldsymbol{Z}_1'\boldsymbol{1}\boldsymbol{1}')\boldsymbol{Y} =$$

$$= \sum_{i=1}^{I}\left(\sum_{j=1}^{n_i} Y_{ij}\right)^2 - \frac{2}{n}\sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}\sum_{k=1}^{I} n_k \sum_{l=1}^{n_k} Y_{kl} + \frac{1}{n^2}\sum_{i=1}^{I} n_i^2 \left(\sum_{k=1}^{I}\sum_{l=1}^{n_k} Y_{kl}\right)^2,$$

(91) $$\boldsymbol{Y}'\boldsymbol{M_X}\boldsymbol{V}_2\boldsymbol{M_X}\boldsymbol{Y} = \boldsymbol{Y}'\boldsymbol{M_X}\boldsymbol{Y} = \boldsymbol{Y}'(\boldsymbol{I} - \frac{1}{n}\boldsymbol{1}\boldsymbol{1}')\boldsymbol{Y} =$$

$$= \sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}^2 - \frac{1}{n}\left(\sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}\right)^2.$$

Podľa dôsledku 63, poznámky 64 a vzorcov (89), (90), (91) sa odhady jednotlivých variančných komponentov získané metódou najmenších štvorcov rovnajú

(92) $$\widehat{\sigma_1^2} = \boldsymbol{e}_1'\boldsymbol{Q}^{-1}\boldsymbol{q} = \frac{n}{n(\sum_{i=1}^{I} n_i^2 - n)}(1,\ -1)\begin{pmatrix}\boldsymbol{Y}'\boldsymbol{M_X}\boldsymbol{V}_1\boldsymbol{M_X}\boldsymbol{Y}\\ \boldsymbol{Y}'\boldsymbol{M_X}\boldsymbol{V}_2\boldsymbol{M_X}\boldsymbol{Y}\end{pmatrix} =$$

$$= \frac{1}{\sum_{i=1}^{I} n_i^2 - n}\left\{\sum_{i=1}^{I}\left(\sum_{j=1}^{n_i} Y_{ij}\right)^2 - \frac{2}{n}\sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}\sum_{k=1}^{I} n_k\sum_{l=1}^{n_k} Y_{kl} + \right.$$

$$\left. + \frac{\sum_{i=1}^{I} n_i^2 + n}{n^2}\left(\sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}\right)^2 - \sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}^2\right\},$$

(93) $$\widehat{\sigma_2^2} = -\frac{1}{\sum_{i=1}^{I} n_i^2 - n}\left\{-\sum_{i=1}^{I}\left(\sum_{j=1}^{n_i} Y_{ij}\right)^2 + \frac{2}{n}\sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}\sum_{k=1}^{I} n_k\sum_{l=1}^{n_k} Y_{kl} - \right.$$

$$\left. - \frac{2\sum_{i=1}^{I} n_i^2}{n^2}\left(\sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}\right)^2 + \frac{\sum_{s=1}^{I} n_s^2}{n}\sum_{i=1}^{I}\sum_{j=1}^{n_i} Y_{ij}^2\right\}.$$

Pre vyvážený model, teda v prípade $n_1 = n_2 = \ldots = n_I = t$ dostávame z (92)

(94) $$\widehat{\sigma_1^2} = \frac{1}{It(t-1)}\left\{\sum_{i=1}^{I}\left(\sum_{j=1}^{t} Y_{ij}\right)^2 - \frac{t-1}{It}\left(\sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}\right)^2 - \sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}^2\right\}$$

a z (93)

(95) $$\widehat{\sigma_2^2} = \frac{1}{It(t-1)}\left\{-\sum_{i=1}^{I}\left(\sum_{j=1}^{t} Y_{ij}\right)^2 + t\sum_{i=1}^{I}\sum_{j=1}^{t} Y_{ij}^2\right\} =$$

$$= \frac{1}{I(t-1)} \left\{ \sum_{i=1}^{I} \sum_{j=1}^{t} Y_{ij}^2 - \sum_{i=1}^{I} \frac{1}{t} \left(\sum_{j=1}^{t} Y_{ij} \right)^2 \right\}.$$

Z (94) je vidieť, že odhad $\widehat{\sigma_1^2}$ získaný metódou najmenších štvorcov je vo vyváženom modeli s jedným náhodným efektom ten istý ako ML–odhad a rôzny od Hendersonovho (ktorý je totožný v tomto prípade s MINQUE odhadom). Pozri tiež poznámku 55.

Z (95) vidíme, že vo vyváženom modeli s jedným náhodným efektom sú Hendersonov, MINQUE, ML aj odhad získaný metódou najmenších štvorcov parametra σ_2^2 totožné (v prípade Hendersonovho odhadu je označený ako σ_0^2). Je preto zrejmé, že odhad $\widehat{\sigma_2^2}$ získaný metódou najmenších štvorcov vo vyváženom modeli s jedným náhodným efektom, je nevychýlený.

Počítajme teraz strednú hodnotu odhadu $\widehat{\sigma_1^2}$ získaného metódou najmenších štvorcov. Podľa vety 65, vzťahu (87) je

$$\mathcal{E}(\widehat{\sigma_1^2}) = (\sigma_1^2,\ \sigma_2^2)\boldsymbol{R}\boldsymbol{Q}^{-1}\boldsymbol{e}_1',$$

kde

(96) $$\{\boldsymbol{R}\}_{11} = \operatorname{tr} \boldsymbol{M_X} \boldsymbol{V}_1 \boldsymbol{M_X} \boldsymbol{V}_1 = \operatorname{tr} \boldsymbol{M_X} \boldsymbol{Z}_1 \boldsymbol{Z}_1' \boldsymbol{M_X} \boldsymbol{Z}_1 \boldsymbol{Z}_1' =$$

$$= \operatorname{tr} \left\{ \left[\begin{pmatrix} n_1 & 0 & \dots & 0 \\ 0 & n_2 & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & n_I \end{pmatrix} - \frac{1}{n} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_I \end{pmatrix} (n_1, n_2, \dots, n_I) \right] \times \right.$$

$$\left. \times \left[\begin{pmatrix} n_1 & 0 & \dots & 0 \\ 0 & n_2 & & \\ \vdots & & \ddots & \\ 0 & 0 & \dots & n_I \end{pmatrix} - \frac{1}{n} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_I \end{pmatrix} (n_1, n_2, \dots, n_I) \right] \right\} =$$

$$= \sum_{i=1}^{I} n_i^2 - \frac{2}{n} \sum_{i=1}^{I} n_i^3 + \frac{1}{n^2} \left(\sum_{i=1}^{I} n_i^2 \right)^2 =$$

$$= \frac{1}{n} \left(\sum_{i=1}^{I} n_i \sum_{j=1}^{I} n_j^2 + \frac{1}{n} \left(\sum_{i=1}^{I} n_i^2 \right)^2 - 2 \sum_{i=1}^{I} n_i^3 \right),$$

(97) $$\{\boldsymbol{R}\}_{12} = \{\boldsymbol{R}\}_{21} = \operatorname{tr} \boldsymbol{M_X} \boldsymbol{Z}_1 \boldsymbol{Z}_1' \boldsymbol{M_X} \boldsymbol{I} = \sum_{i=1}^{I} n_i - \frac{1}{n} \sum_{i=1}^{I} n_i^2$$

a

(98) $$\{\boldsymbol{R}\}_{22} = \operatorname{tr} \boldsymbol{M_X} \boldsymbol{V}_2 \boldsymbol{M_X} \boldsymbol{V}_2 = \operatorname{tr} \boldsymbol{M_X} \boldsymbol{M_X} = n - 1.$$

Pomocou (89), (96), (97) a (98) dostávame

$$\text{(99)}\qquad \mathcal{E}(\widehat{\sigma_1^2}) = (\sigma_1^2,\ \sigma_2^2)\frac{1}{\sum_{i=1}^I n_i^2 - n}\times$$

$$\times\begin{pmatrix} \frac{1}{n}\left[n\sum_{i=1}^I n^2 + \frac{1}{n}\left(\sum_{i=1}^I n_i^2\right)^2 - 2\sum_{i=1}^I n_i^3\right] & n - \frac{1}{n}\sum_{i=1}^I n_i^2 \\ n - \frac{1}{n}\sum_{i=1}^I n_i^2 & n-1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} =$$

$$= \frac{1}{n(\sum_{i=1}^I n_i^2 - n)}(\sigma_1^2,\ \sigma_2^2)\times$$

$$\times\begin{pmatrix} (n+1)\sum_{i=1}^I n_i^2 + \frac{1}{n}\left(\sum_{i=1}^I n_i^2\right)^2 - 2\sum_{i=1}^I n_i^3 - n^2 \\ n - \sum_{i=1}^I n_i^2 \end{pmatrix} =$$

$$= \frac{\sigma_1^2}{n(\sum_{i=1}^I n_i^2 - n)}\left[(n+1)\sum_{i=1}^I n_i^2 + \frac{1}{n}\left(\sum_{i=1}^I n_i^2\right)^2 - 2\sum_{i=1}^I n_i^3 - n^2\right] - \frac{\sigma_2^2}{n}.$$

Spočítajme ešte rovnakým spôsobom

$$\text{(100)}\qquad \mathcal{E}(\widehat{\sigma_2^2}) = (\sigma_1^2,\ \sigma_2^2)\frac{1}{n(\sum_{i=1}^I n_i^2 - n)}\times$$

$$\times\begin{pmatrix} n\sum_{i=1}^I n_i^2 + \frac{1}{n}\left(\sum_{i=1}^I n_i^2\right)^2 - 2\sum_{i=1}^I n_i^3 & n^2 - \sum_{i=1}^I n_i^2 \\ n^2 - \sum_{i=1}^I n_i^2 & n(n-1) \end{pmatrix}\begin{pmatrix} -1 \\ \frac{1}{n}\sum_{i=1}^I n_i^2 \end{pmatrix} =$$

$$= \frac{1}{n(\sum_{i=1}^I n_i^2 - n)}(\sigma_1^2,\ \sigma_2^2)\begin{pmatrix} 2\sum_{i=1}^I n_i^3 + \frac{2}{n}\left(\sum_{i=1}^I n_i^2\right)^2 \\ n(\sum_{i=1}^I n_i^2 - n) \end{pmatrix} =$$

$$= \frac{2\sigma_1^2}{n(\sum_{i=1}^I n_i^2 - n)}\left[\sum_{i=1}^I n_i^3 - \frac{1}{n}\left(\sum_{i=1}^I n_i^2\right)^2\right] + \sigma_2^2.$$

Vo vyváženom modeli dostávame z (99)

$$\text{(101)}\qquad \mathcal{E}(\widehat{\sigma_1^2}) = \frac{1}{It(It^2 - Tt)}\left[(It+1)It^2 + \frac{I^2t^4}{It} - 2It^3 - I^2t^2\right] - \frac{\sigma_2^2}{It} =$$

$$= \frac{I-1}{I}\sigma_1^2 - \frac{1}{It}\sigma_2^2$$

(samozrejme vo vyváženom modeli je $\mathcal{E}(\widehat{\sigma_2^2}) = \sigma_2^2$).

Teraz spočítajme disperzie odhadov variančných komponentov získaných metódou najmenších štvorcov v modeli s jedným náhodným efektom. Potrebujeme pritom ešte nasledujúcu lemu.

Lema 67. *V modeli s jedným náhodným efektom pre maticu* $\boldsymbol{W}$ *z vety 66 platí*

$$\{\boldsymbol{W}\}_{11} = \sigma_1^4 \operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^4 + 2\sigma_1^2\sigma_2^2 \operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^3 + \sigma_2^4 \operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^2,$$

$$\{\boldsymbol{W}\}_{12} = \{\boldsymbol{W}\}_{21} = \sigma_1^4 \operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^3 + 2\sigma_1^2\sigma_2^2 \operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^2 + \sigma_2^4 \operatorname{tr} \boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1,$$

$$\{\boldsymbol{W}\}_{22} = \sigma_1^4 \operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^2 + 2\sigma_1^2\sigma_2^2 \operatorname{tr} \boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1 + \sigma_2^4 \operatorname{tr} \boldsymbol{M}_{\boldsymbol{X}},$$

$$\operatorname{tr} \boldsymbol{M}_{\boldsymbol{X}} = n - 1,$$

$$\operatorname{tr} \boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1 = n - \frac{1}{n}\sum_{i=1}^I n_i^2,$$

$$\operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^2 = \frac{1}{n}\left(n\sum_{i=1}^I n_i^2 + \frac{1}{n}\left(\sum_{i=1}^I n_i^2\right)^2 - 2\sum_{i=1}^I n_i^3\right),$$

$$\operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^3 = \sum_{i=1}^I n_i^3 - \frac{3}{n}\sum_{i=1}^I n_i^4 + \frac{3}{n^2}\sum_{i=1}^I n_i^2 \sum_{j=1}^I n_j^3 - \frac{1}{n^3}\left(\sum_{i=1}^I n_i^2\right)^3,$$

$$\operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^4 = \sum_{i=1}^I n_i^4 - \frac{4}{n}\sum_{i=1}^I n_i^5 + \frac{4}{n^2}\sum_{i=1}^I n_i^2 \sum_{j=1}^I n_j^4 + \frac{2}{n^2}\left(\sum_{i=1}^I n_i^3\right)^2 -$$
$$- \frac{4}{n^3}\left(\sum_{i=1}^I n_i^2\right)^2\left(\sum_{j=1}^I n_j^3\right) + \frac{1}{n^4}\left(\sum_{i=1}^I n_i^2\right)^4.$$

Dôkaz vykonáme jednoduchým, ale zdĺhavým dosadzovaním a spočítavaním. Urobte ho ako cvičenie.

Počítajme teraz podľa vety 66 (využijúc lemu 67):

(102)
$$\mathcal{D}(\widehat{\sigma_1^2}) = 2\boldsymbol{e}_1'\boldsymbol{Q}^{-1}\boldsymbol{W}\boldsymbol{Q}^{-1}\boldsymbol{e}_1 = \frac{2}{(\sum_{i=1}^I n_i^2 - n)^2}(\{\boldsymbol{W}\}_{11} - 2\{\boldsymbol{W}\}_{12} + \{\boldsymbol{W}\}_{22}) =$$

$$= \frac{2}{(\sum_{i=1}^I n_i^2 - n)^2}\{\sigma_1^4[\operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^4 - 2\operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^3 + \operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^2]+$$
$$+2\sigma_1^2\sigma_2^2[\operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^3 - 2\operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^2 + \operatorname{tr}\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1]+$$
$$+\sigma_2^4[\operatorname{tr}(\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1)^2 - 2\operatorname{tr}\boldsymbol{Z}_1'\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{Z}_1 + \operatorname{tr}\boldsymbol{M}_{\boldsymbol{X}}]\}.$$

Podobne

(103)
$$\mathcal{D}(\widehat{\sigma_2^2}) = 2\boldsymbol{e}_2'\boldsymbol{Q}^{-1}\boldsymbol{W}\boldsymbol{Q}^{-1}\boldsymbol{e}_2 =$$

$$= \frac{2}{(\sum_{i=1}^I n_i^2 - n)^2}\left[\{\boldsymbol{W}\}_{11} - \frac{2}{n}\sum_{i=1}^I n_i^2\{\boldsymbol{W}\}_{21} + \left(\frac{1}{n}\sum_{i=1}^I n_i^2\right)^2\{\boldsymbol{W}\}_{22}\right] =$$

$$= \frac{2}{(\sum_{i=1}^{I} n_i^2 - n)^2} \left\{ \sigma_1^4 \left[\operatorname{tr}(\boldsymbol{Z}_1' \boldsymbol{M}_{\boldsymbol{X}} \boldsymbol{Z}_1)^4 - \frac{2}{n} \sum_{i=1}^{I} n_i^2 \operatorname{tr}(\boldsymbol{Z}_1' \boldsymbol{M}_{\boldsymbol{X}} \boldsymbol{Z}_1)^3 + \right.\right.$$

$$\left. + \left(\frac{1}{n} \sum_{i=1}^{I} n_i^2 \right)^2 \operatorname{tr}(\boldsymbol{Z}_1' \boldsymbol{M}_{\boldsymbol{X}} \boldsymbol{Z}_1)^2 \right] + 2\sigma_1^2 \sigma_2^2 \left[\operatorname{tr}(\boldsymbol{Z}_1' \boldsymbol{M}_{\boldsymbol{X}} \boldsymbol{Z}_1)^3 - \right.$$

$$\left. - \left(\frac{1}{n} \sum_{i=1}^{I} n_i^2 \right) \operatorname{tr}(\boldsymbol{Z}_1' \boldsymbol{M}_{\boldsymbol{X}} \boldsymbol{Z}_1)^2 + \left(\frac{1}{n} \sum_{i=1}^{I} n_i^2 \right)^2 \operatorname{tr} \boldsymbol{Z}_1' \boldsymbol{M}_{\boldsymbol{X}} \boldsymbol{Z}_1 \right] +$$

$$\left. + \sigma_2^4 \left[\operatorname{tr}(\boldsymbol{Z}_1' \boldsymbol{M}_{\boldsymbol{X}} \boldsymbol{Z}_1)^2 - \frac{2}{n} \sum_{i=1}^{I} n_i^2 \operatorname{tr} \boldsymbol{Z}_1' \boldsymbol{M}_{\boldsymbol{X}} \boldsymbol{Z}_1 + \left(\frac{1}{n} \sum_{i=1}^{I} n_i^2 \right)^2 \operatorname{tr} \boldsymbol{M}_{\boldsymbol{X}} \right] \right\}.$$

Lema 68. *Vo vyváženom modeli s jedným náhodným efektom je*

$$\mathcal{D}(\widehat{\sigma_1^2}) = \frac{2\sigma_2^4}{It^2(t-1)} + \frac{2(t\sigma_1^2 + \sigma_2^2)^2}{It^2} \frac{I-1}{I},$$

$$\mathcal{D}(\widehat{\sigma_2^2}) = \frac{\sigma_2^4}{I(t-1)}.$$

Dôkaz. Lemu dokážeme využitím lemy 67 a dosadením $n_1 = n_2 = \ldots = n_I = t$ do (102) a (103). Dôkaz urobte ako cvičenie.

Výsledky lemy 68 porovnajte s lemou 57 a cvičením 1 na str. 32.

Pre ďalšie štúdium problematiky odhadov variančných komponentov odporúčame napr. monografiu [13].

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Obsah