Contents

CHAPTER 1

Lie groups

1. Lie groups

Definition 1.1. A *Lie group* G is a group, which is at the same time a smooth manifold in such a way that

- the multiplication $\mu : G \times G \to G$ is smooth,
- the inverse $\nu : G \to G$ is smooth.

By a *homomorphism of Lie groups* we understand a smooth group homomorphisms.

Notation. We denote by e the unit and write a^{-1} instead of $\nu(a)$. We will be using the left and right translations $\lambda_a, \rho_a : G \to G$ defined by

$$\lambda_a(b) = ab \qquad \rho_a(b) = ba$$

Theorem 1.2. *The smoothness of the inverse follows from the smoothness of the multiplication.*

Proof. The defining equation for the inverse is $\mu(a, \nu(a)) = e$. By the implicit function theorem it is enough to verify that the derivative of $\mu(a, -)$ at a^{-1} is invertible. This follows from the fact that $\mu(a, -) = \lambda_a$ has an inverse $\lambda_{a^{-1}}$. □

Remark. Every Lie group is a topological group, i.e. a group and a topological group such that the multiplication and the inverse are continuous. The fifth Hilbert problem states that every topological group G that is at the same time a (topological) manifold admits a smooth structure for which G becomes a Lie group. This was proved in 1952 (in fact the structure is even analytic). If time permits we will get to the implication $C^2 \Rightarrow C^\infty$.

Let M, N be smooth manifolds. Then the projections $p : M \times N \to M$ and $q : M \times N \to N$ provide the canonical isomorphism

$$(p_*, q_*) : T_{(x,y)}(M \times N) \xrightarrow{\cong} T_x M \times T_y N.$$

The inverse isomorphism is obtained from the inclusions

$$\begin{aligned} i_y : M &\to M \times N \\ a &\mapsto (a, y) \end{aligned} \qquad\qquad \begin{aligned} j_x : N &\to M \times N \\ b &\mapsto (x, b) \end{aligned}$$

Under the above identification the pair $(X, Y) \in T_x M \times T_y N$ corresponds to $(i_y)_* X + (j_x)_* Y \in T_{(x,y)}(M \times N)$.

Lemma 1.3. *The following formulae hold for $A, B \in T_e G$:*

$$\mu_*(A, B) = A + B, \qquad\qquad \nu_* A = -A.$$

PROOF. These are just simple calculations

$$\mu_*(A,B) = \mu((i_e)_*A + (j_e)_*B) = (\mu i_e)_*A + (\mu j_e)_*B = A + B$$

and by differentiating $e = \mu(a, \nu(a))$ in the direction $A \in T_eG$ we get

$$0 = \mu_*(A, \nu_*A) = A + \nu_*A$$

□

EXAMPLES 1.4. The classical gropups:

- The general linear group $\mathrm{GL}(n,\mathbb{R})$ - the group of invertible matrices (a_{ij}). Since $\mathrm{GL}(n,\mathbb{R}) \subseteq \mathbb{R}^{n\times n}$ can be described as $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R} - \{0\})$ it is an open subset and hence a manifold. Multiplication is clearly smooth (even algebraic).
- The general linear group $\mathrm{GL}(n,\mathbb{C})$ with coefficients in $\mathbb{C}$. We think of $\mathrm{GL}(n,\mathbb{C})$ as a subgroup of $\mathrm{GL}(2n,\mathbb{R})$ via the identification $\mathbb{C}^n = \mathbb{R}^n \oplus i\mathbb{R}^n$. The embedding becomes

  $$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

  On the other hand $\mathrm{GL}(n,\mathbb{C}) \subseteq \mathbb{C}^{n\times n}$ is again open and hence a manifold.
- The special linear groups

  $$\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det A = 1\}$$
  $$\mathrm{SL}(n,\mathbb{C}) = \{A \in \mathrm{GL}(n,\mathbb{C}) \mid \det A = 1\}$$

  are certainly closed submanifolds and also subgroups. Later we will prove

  THEOREM. *Every closed subgroup of a Lie group is a submanifold and with the submanifold smooth structure a Lie group (i.e. a Lie subgroup).*

- Let $\beta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form represented by a matrix $B = (b_{ij})$. A linear map $\alpha : \mathbb{R}^n \to \mathbb{R}^n$ is said to preserve β if

  $$\beta(\alpha x, \alpha y) = \beta(x, y) \quad \iff \quad A^T B A = B$$

  Such linear automorphisms clearly form a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$.
  - Specifically for $\beta = \langle\ ,\ \rangle$, the scalar product, we have $B = E$, the identity matrix and we obtain the orthogonal group

    $$\mathrm{O}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid A^T A = E\}$$

    and also the special orthogonal group

    $$\mathrm{SO}(n,\mathbb{R}) = \mathrm{O}(n,\mathbb{R}) \cap \mathrm{SL}(n,\mathbb{R})$$
  - Consider on $\mathbb{R}^{2n}$ the (nondegenerate antisymmetric) bilinear form

    $$\sum_{i=1}^{n} (x_i y_{n+i} - y_i x_{n+i})$$

    with its matrix $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$. The group of linear automorphisms preserving this form is called the symplectic group $\mathrm{Sp}(2n,\mathbb{R})$. Analogously we obtain $\mathrm{Sp}(2n,\mathbb{C})$.

- The unitary group $\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) \mid \bar{A}^T A = E\}$ and the special unitary group $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$. There is also a complex orthogonal group which is different from the unitary group. One of the main qualitative differences is that $\mathrm{O}(n, \mathbb{C})$ is a complex manifold and a complex Lie group (reason being that the defining equation $A^T A = E$ is holomorphic unlike that for the unitary group - it contains complex conjugation).
- The spin group $\mathrm{Spin}(n)$. We will say more about it later. It is related to $\mathrm{SO}(n, \mathbb{R})$ by a short exact sequence of groups

$$1 \to \mathbb{Z}/2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n, \mathbb{R}) \to 1.$$

- $\mathrm{Sp}(n) = \{A \in \mathrm{GL}(n, \mathbb{H}) \mid \bar{A}^T A = E\}$, the group of linear automorphisms of the quaternionic space $\mathbb{H}^n$ preserving the scalar product. Also $\mathrm{Sp}(n) = \mathrm{Sp}(2n, \mathbb{C}) \cap \mathrm{U}(2n)$.

2. Lie algebras

Definition 2.1. A vector space L over $\mathbb{R}$ is called a *Lie algebra* if there is given a bilinear map $[\ ,\ ] : L \times L \to L$ satisfying

- the antisymmetry: $[X, X] = 0$,
- the Jacobi identity: $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

From bilinearity we obtain

$$0 = [X + Y, X + Y] = [X, X] + [X, Y] + [Y, X] + [Y, Y] = [X, Y] + [Y, X]$$

implying $[Y, X] = -[X, Y]$.

Example 2.2. The vector fields on a smooth manifold M with the bracket $[X, Y]$:

$$X = \sum_i X_i \tfrac{\partial}{\partial x^i}, Y = \sum_i Y_i \tfrac{\partial}{\partial x^i} \quad \Longrightarrow \quad [X, Y] = \sum_{i,j} \left(X_j \tfrac{\partial Y_i}{\partial x^j} - Y_j \tfrac{\partial X_i}{\partial x^j} \right) \tfrac{\partial}{\partial x^i}$$

Let L be a finite dimensional Lie algebra and $e_1, \ldots, e_n$ its basis. Then $[e_i, e_j] = \sum_k c_{ij}^k e_k$. The numbers c_{ij}^k are called the structure constants of L with respect to the basis. They satisfy the following identities:

- $c_{ji}^k = -c_{ij}^k$,
- $\sum_k (c_{ij}^k c_{kl}^m + c_{jl}^k c_{ki}^m + c_{li}^k c_{kj}^m) = 0$.

Conversely, by giving the basis $e_1, \ldots, e_n$ and the structure constants c_{ij}^k satisfying the above equalities we obtain a Lie algebra L. The complete classification of Lie algebras is not yet known.

Example 2.3. Let V be a vector space and denote $L = \mathsf{hom}(V, V)$. On L we define a bracket

$$[f, g] = f \circ g - g \circ f.$$

In this way we obtatin a Lie algebra $\mathfrak{gl}(V)$.

For a Lie group G we define $\mathfrak{g} = \mathrm{Lie}(G) = T_e G$ as a vector space. Now we proceed to introduce a bracket on $\mathfrak{g}$.

DEFINITION 2.4. A vector field $X : G \to TG$ is called *left-invariant* if $(\lambda_a)_* \circ X = X \circ \lambda_a$ for any $a \in G$.

$$\begin{array}{ccc} TG & \xrightarrow{(\lambda_a)_*} & TG \\ {\scriptstyle X}\uparrow & & \uparrow{\scriptstyle X} \\ G & \xrightarrow[\lambda_a]{} & G \end{array}$$

In other words X is λ_a-related with itself which we denote by $X \sim_{\lambda_a} X$.

REMARK. The f-relatedness of vector fields X and Y has the following characterization via the flow lines, easily verified by differentiating both sides.

$$f(\mathrm{Fl}_t^X(x)) = \mathrm{Fl}_t^Y(f(x))$$

In other words f transfers the flow lines of X into the flow lines of Y. We will use this property quite often.

REMARK. Let $A \in T_eG$ be an arbitrary vector. It defines a vector field $\lambda_A : G \to TG$ by the formula $\lambda_A(a) = (\lambda_a)_* A$. This vector field is clearly left-invariant as

$$\lambda_A(ab) = (\lambda_{ab})_* A = (\lambda_a \lambda_b)_* A = (\lambda_a)_*((\lambda_b)_* A) = (\lambda_a)_*(\lambda_A(b))$$

It remains to verify its smoothness. Since $(\lambda_a)_* A = \mu_*(0_a, A)$ this is achieved by the following diagram

$$\begin{array}{ccc} TG \times TG & \xrightarrow{T\mu} & TG \\ {\scriptstyle (0,A)}\uparrow & \nearrow_{\lambda_A} & \\ G & & \end{array}$$

with $(0, A)$ being the map with components the zero section 0 and the constant map sending everything onto A.

THEOREM 2.5. *Let X, Y be left invariant vector fields. Then $X + Y$, kX and $[X, Y]$ are again left-invariant.*

PROOF. Since X and Y are λ_a related with X and Y respectively, the same is true for their sum, multiples and bracket. □

DEFINITION 2.6. The vector space $\mathfrak{g} = \mathrm{Lie}(G) = T_eG$ together with the bracket $[A, B] = [\lambda_A, \lambda_B]_e$ is called the *Lie algebra* of the Lie group G.

REMARK. For every finite dimensional Lie algebra $\mathfrak{g}$ there exists a Lie group G for which $\mathrm{Lie}(G) = \mathfrak{g}$.

We would like to explain now why this is a reasonable object of study. We have seen that the first derivative at e does not see anything from the structure of the Lie group. The second derivative does but in order to make sense of the second derivative one has to fix the coordinate charts (which we will do later and for them the second derivative will be described exactly by the Lie bracket). Without a fixed choice of the charts the second derivative only makes sense when the first derivative vanishes at that point which is not the case for the product. The way out is to "subtract from μ the sum of the two coordinates" by considering

$$[\ ,\ ] : G \times G \longrightarrow G$$
$$(a, b) \mapsto aba^{-1}b^{-1}$$

We will see shortly that the first derivative of the commutator vanishes at e and the essential part of the second derivative is exactly the Lie bracket.

NOTATION. Let X, Y be two vector fields on a manifold M. Then we denote

$$(\mathrm{Fl}_t^X)^* Y(x) = (\mathrm{Fl}_{-t}^X)_* Y(\mathrm{Fl}_t^X(x)) \in T_x M$$

the pullback of Y along the flow Fl_t^X of X. For each $x \in M$ it is defined for t small.

LEMMA 2.7. $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} (\mathrm{Fl}_t^X)^* Y(x) = (\mathrm{Fl}_{t_0}^X)^*[X,Y](x)$.

PROOF. First assume that $t_0 = 0$ and let $f : M \to \mathbb{R}$ be a smooth function. We differentiate f in the direction of the left hand side:

$$\begin{aligned}\Big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (\mathrm{Fl}_t^X)^* Y(x)\Big) f &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Big((\mathrm{Fl}_t^X)^* Y(x) f\Big)\\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Big((\mathrm{Fl}_{-t}^X)_* Y(\mathrm{Fl}_t^X(x)) f\Big)\\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Big(Y(\mathrm{Fl}_t^X(x))(f \circ \mathrm{Fl}_{-t}^X)\Big)\\ &= Y(x)(-Xf) + \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (Yf)(\mathrm{Fl}_t^X(x))\\ &= -(YXf)(x) + (XYf)(x) = \Big([X,Y](x)\Big) f\end{aligned}$$

For a general t_0 we have $(\mathrm{Fl}_t^X)^* Y(x) = (\mathrm{Fl}_{t_0}^X)^*(\mathrm{Fl}_{t-t_0}^X)^* Y(x)$. Since $(\mathrm{Fl}_{t_0}^X)^*$ is a linear map we can interchange with $\frac{\mathrm{d}}{\mathrm{d}t}$. □

COROLLARY 2.8. *The following conditions are equivalent:*

- $[X,Y] = 0$,
- $(\mathrm{Fl}_t^X)^* Y = Y$, *i.e. Y is Fl_t^X-related with itself for all t,*
- $\mathrm{Fl}_t^X \mathrm{Fl}_s^Y(x) = \mathrm{Fl}_s^Y \mathrm{Fl}_t^X(x)$, *i.e. the flow lines commute.*

In general we have $\mathrm{Fl}_{-s}^Y \mathrm{Fl}_{-t}^X \mathrm{Fl}_s^Y \mathrm{Fl}_t^X(x) = x + st[X,Y](x) + o(s,t)^2$.

PROOF. Differentiating twice we get

$$\tfrac{\partial}{\partial t}\big|_{t=0} \tfrac{\partial}{\partial s}\big|_{s=0} \mathrm{Fl}_{-s}^Y \mathrm{Fl}_{-t}^X \mathrm{Fl}_s^Y \mathrm{Fl}_t^X(x) = \tfrac{\partial}{\partial t}\big|_{t=0} \Big(-Y(x) + (\mathrm{Fl}_t^X)^* Y(x)\Big) = [X,Y](x)$$

The remaining derivatives of order at most two are clearly zero. □

EXAMPLE 2.9. Let $M = G$, a Lie group. What does $[A,B]$ for $A, B \in \mathfrak{g}$ express? Let us consider the following integral curves

- $\varphi(t)$ the flow line of λ_A with $\varphi(0) = e$,
- $\psi(t)$ the flow line of λ_B with $\psi(0) = e$.

A flow line of λ_A through a general $a \in G$ is easily $a \cdot \varphi : t \mapsto a\varphi(t)$. This follows from the λ_a-relatedness of λ_A with itself: $\frac{\mathrm{d}}{\mathrm{d}t}(a\varphi(t)) = (\lambda_a)_* \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)$. In other words

$$\mathrm{Fl}_t^{\lambda_A} = \rho_{\varphi(t)}$$

This implies that $\varphi(t_1 + t_2) = \varphi(t_1)\varphi(t_2)$ and it is a homomorphism of groups. We now compute

$$\mathrm{Fl}_{-s}^{\lambda_B} \mathrm{Fl}_{-t}^{\lambda_A} \mathrm{Fl}_s^{\lambda_B} \mathrm{Fl}_t^{\lambda_A}(x) = \varphi(t)\psi(s)\varphi(-t)\psi(-s) = \varphi(t)\psi(s)\varphi(t)^{-1}\psi(s)^{-1}.$$

In other words the group theoretic commutator $[\varphi(t), \psi(s)]$ has a Taylor polynomial

$$[\varphi(t), \psi(s)] = [A,B]st + o(s,t)^2$$

This can also be rewritten as $\mathrm{d}^2[\ ,\ ]_{(e,e)}((A,0),(0,B)) = [A,B]$. The Lie bracket thus measures the non-commutativity of the Lie group. More precisely $[A,B]=0$ if and only if all the elements $\varphi(t)$ commute with all $\psi(s)$. We will see later that the connection between commutativity of G and vanishing of the bracket works perfectly for connected Lie groups.

Definition 2.10. Let L, L' be two Lie algebras. A linear map $\varphi : L \to L'$ is called a *homomorphism of Lie algebras* if $\varphi[A,B]_L = [\varphi A, \varphi B]_{L'}$.

Theorem 2.11. *Let $f : G \to H$ be a (smooth) homomorphism of Lie groups. Then its derivative $f_* : \mathfrak{g} \to \mathfrak{h}$ at e is a homomorhpism of Lie algebras.*

Proof. Let us rewrite $f(ab) = f(a)f(b)$ using the left translations as

$$f \circ \lambda_a = \lambda_{f(a)} \circ f$$

Differentiating in the direction $A \in \mathfrak{g}$ we obtain $f_*(\lambda_a)_*A = (\lambda_{f(a)})_* f_* A$ or

$$f_*\lambda_A(a) = \lambda_{f_*A}(f(a))$$

which means that λ_A is f-related to λ_{f_*A}. Since the bracket respects relatedness, $[\lambda_A, \lambda_B]$ must be f-related to $[\lambda_{f_*A}, \lambda_{f_*B}]$. Evaluating at e yields the result. □

Definition 2.12. A smooth map $f : G \to H$ between Lie groups is a *local isomorphism* if it is both a homomorphism and a local diffeomorphism at e (i.e. the derivative $f_{*e} : \mathfrak{g} \to \mathfrak{h}$ is an isomorphism).

Two Lie groups G, H are called *locally isomorphic* if there exist neighbourhoods $U \ni e$ and $V \ni e$, in G and H respectively, together with a diffeomorphism $f : U \to V$ which satisfies:

- $f(ab) = f(a)f(b)$ whenever $a, b, ab \in U$,
- $f^{-1}(ab) = f^{-1}(a)f^{-1}(b)$ whenever $a, b, ab \in V$.

Clearly if there exists a local isomorphism $f : G \to H$ then G and H are locally isomorphic.

Theorem 2.13. *Locally isomorphic groups have isomorphic Lie algebras.* □

Example 2.14. The additive groups $\mathbb{R}$ and $\mathbb{T} = \mathrm{SU}(1)$ (the group of complex units in $\mathbb{C}$) are locally isomorphic. We think of the first as the group of translations of the line while the second is the group of rotations of the circle (or $\mathbb{C}$ for that matter). This is because there exists a local isomorphism $\mathbb{R} \to \mathbb{T}$ sending $t \mapsto e^{2\pi i t}$.

Definition 2.15. Let L, L' be Lie algebras. On their product $L \times L'$ we consider the bracket

$$[(X_1, Y_1), (X_2, Y_2)] = ([X_1, X_2]_L, [Y_1, Y_2]_{L'}).$$

We call $L \times L'$ together with this bracket the *product of Lie algebras* L and L'.

Theorem 2.16. $\mathrm{Lie}(G \times H) \cong \mathrm{Lie}(G) \times \mathrm{Lie}(H)$.

Proof. The projections $p : G \times H \to G$ and $q : G \times H \to H$ are homomorphisms and hence they induce homomorphisms of the Lie algebras in question. This means

$$p_*[(X_1, Y_1), (X_2, Y_2)] = [p_*(X_1, Y_1), p_*(X_2, Y_2)] = [X_1, X_2]$$

and similarly for q. The canonical isomorphism $(p_*, q_*) : \mathrm{Lie}(G \times H) \to \mathfrak{g} \times \mathfrak{h}$ is then an isomorphism of Lie algebras. □

REMARK. With the above Lie algebra structure $L \times L'$ forms a product in the category of Lie algebras. The previous proof is then just a demonstration of the fact that Lie is a functor and preserve products (which is obvious from the fact that this happens already at the level of tangent vector spaces at e).

What happens if we change sides? Denoting ρ_A the right-invariant vector field with value A at e the next theorem asserts that the Lie bracket defined via the right-invariant vector fields agrees with the usual one up to the minus sign.

THEOREM 2.17. *For $A, B \in \mathfrak{g}$ the following holds: $[\rho_A, \rho_B]_e = -[\lambda_A, \lambda_B]_e$.*

PROOF. Consider the opposite group G^* with multiplication $a * b = ba$. The inverse $\nu : G^* \to G$ is a group homomorphism and

$$[A, B]^* = [\lambda^*_A, \lambda^*_B]_e = [\rho_A, \rho_B]_e$$

Thus $-[\rho_A, \rho_B]_e = \nu_*[A, B]^* = [\nu_* A, \nu_* B] = [-A, -B] = [\lambda_A, \lambda_B]_e$. □

COROLLARY 2.18. *For a commutative group G the bracket on its Lie algebra is identically zero.*

3. Subgroups and subalgebras

DEFINITION 3.1. A *Lie subalgebra* $L' \subseteq L$ is a vector subspace closed under $[\ ,\ ]$.

THEOREM 3.2. *If $H \subseteq G$ is both a submanifold and a subgroup then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra.*

PROOF. In the diagram

$$\begin{array}{ccc} H \times H & \overset{\mu}{\dashrightarrow} & H \\ \big\downarrow & & \big\downarrow{\scriptstyle \iota} \\ G \times G & \underset{\mu}{\longrightarrow} & G \end{array}$$

the map μ (which exists since H is a subgroup) is smooth since H is a submanifold. Hence H is a Lie group and the inclusion $\iota : H \to G$ is a homomorphism. Thus its derivative $\iota_* : \mathfrak{h} \to \mathfrak{g}$ is a homomorphism of Lie algebras (saying that the bracket of $\mathfrak{h}$ is a restriction of the bracket on $\mathfrak{g}$) and its image is therefore a subalgebra. □

EXAMPLE 3.3. Consider $\mathbb{R}^2$. Then every line $\{(x, kx) \mid x \in \mathbb{R}\}$ (for $k \in \mathbb{R}$) is a subgroup (and a submanifold). Now consider the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Again we get subgroups for any $k \in \mathbb{R}$. For $k \in \mathbb{Q}$ this subgroup is a submanifold but not for irrational k when this subgroup is dense.

DEFINITION 3.4. A subset $A \subseteq M$ of a smooth manifold M is called an *initial submanifold* (of dimension k) if for each $x \in A$ there exists a chart

$$\varphi : U \xrightarrow{\cong} \mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$$

such that $\varphi^{-1}(\mathbb{R}^k \times \{0\})$ is exactly the path component of $U \cap A$ containing x.

Theorem 3.5. *Every initial submanifold is the image of an (essentially unique) injective immersion i satisfying the following universal property:*

$$\begin{array}{ccc} & & A \\ & \overset{g}{\nearrow} & \Big\downarrow i \\ N & \xrightarrow[f]{} & M \end{array}$$

For every smooth map $f : N \to M$ with the property $f(N) \subseteq i(A)$ the unique map $g : N \to A$ satisfying $ig = f$ is also smooth.

Proof. Let $\varphi : U \longrightarrow \mathbb{R}^m$ be a chart on N from the definition of an initial submanifold. Declare its restriction

$$C_x(U \cap A) \xrightarrow{\cong} \mathbb{R}^k \times \{0\}$$

to the path component of $U \cap A$ containing x to be a chart for A. This does endow A with a smooth structure. It differs from the subspace topology (which is inevitable) but the inclusion is clearly an injective immersion.

We verify the universal property for inclusions of initial submanifolds. Let $y \in N$ with $f(y) = x$ and V a path connected neighbourhood of y which maps into U. Since its image is also path connected it must be contained in $U \cap A$. Thus g in the chart provided by ψ is just a restriction of f and hence smooth.

Suppose now that $i' : A' \hookrightarrow M$ is another injective immersion with the same image as i. Then there exists a factorization

$$\begin{array}{ccc} A' & \xrightarrow{h} & A \\ & {\scriptstyle i'}\searrow \quad \swarrow{\scriptstyle i} & \\ & M & \end{array}$$

with h an immerison and a bijection at the same time. Since its inverse is also an immersion by the same argument h must be in fact a diffeomorphism. □

Remark. It is also true that any injective immersion i satisfying the above universal property is in fact an inclusion of an initial submanifold but we will not need this fact.

Remark. We have not proved that A has a countable basis for its topology. In fact A might well have an uncountable number of components. However each of the components of A is second countable.

Definition 3.6. A *Lie subgroup* $H \subseteq G$ is an initial submanifold which is at the same time a subgroup.

Theorem 3.7. *A Lie subgroup $H \subseteq G$ with its canonical smooth structure (and multiplication) is a Lie group. Moreover $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra.*

Proof. The whole proof is contained in the diagram

$$\begin{array}{ccc} H \times H & \overset{\mu}{\dashrightarrow} & H \\ \Big\downarrow & & \Big\downarrow \iota \\ G \times G & \xrightarrow[\mu]{} & G \end{array}$$

□

Our new definition includes the wild subgroups of the torus $\mathbb{T}^2$. In fact we are able to construct a Lie subgroup for any Lie subalgebra of $\mathfrak{g}$. To motivate our construction observe that for a Lie subgroup $H \subseteq G$ and $a \in H$ we have $T_a H = (\lambda_a)_* \mathfrak{h}$ and H is an integral submanifold of the left invariant distribution determined by $\mathfrak{h}$.

More generally for a linear subspace $P \subseteq \mathfrak{g}$ of dimension k the left translations $(\lambda_a)_* P =: \lambda_P(a) \subseteq T_a G$ form a k-dimensional distribution λ_P on G. This distribution is smooth: if $A_1, \ldots, A_k$ is a basis of P then $\lambda_{A_1}(a), \ldots, \lambda_{A_k}(a)$ is a basis of $\lambda_P(a)$.

A distribution $\mathcal{S}$ on M is called involutive if for every two vector fields $X, Y \in \mathcal{S}$ their bracket $[X, Y]$ also lies in $\mathcal{S}$.

THEOREM 3.8 (Frobenius theorem). *If $\mathcal{S}$ is involutive then for every $x \in M$ there exists a local coordinate system $y^1, \ldots, y^m$ in a neighbourhood U of x such that the vector fields $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^k}$ form a basis of the distribution $\mathcal{S}$ on U. In particular $\mathcal{S}$ is integrable.*

PROOF. Let $X_1, \ldots, X_k$ be local vector fields which, near x, span the distribution $\mathcal{S}$ and let us choose a coordinate system around x in which $X_i(x) = \frac{\partial}{\partial x^i}$. We then define a map

$$\begin{aligned} \varphi : \mathbb{R}^m \supseteq U &\longrightarrow M \\ (t^1, \ldots, t^m) &\longmapsto \mathrm{Fl}^{X_1}_{t^1} \cdots \mathrm{Fl}^{X_k}_{t^k}(0, \ldots, 0, t^{k+1}, \ldots, t^m) \end{aligned}$$

The partial derivatives at the origin clearly consist of the vectors $\frac{\partial}{\partial x^i}$ and thus φ is a local diffeomorphism.

Let us compute the partial derivative with respect to t^i for $i \leq k$ at a general point.

$$\frac{\partial \varphi}{\partial t^i} = \left(\mathrm{Fl}^{X_1}_{t^1}\right)_* \cdots \left(\mathrm{Fl}^{X_{i-1}}_{t^{i-1}}\right)_* X_i \left(\mathrm{Fl}^{X_{i+1}}_{t^{i+1}} \cdots \mathrm{Fl}^{X_m}_{t^m}(x)\right)$$

To conclude the proof it is therefore enough to show that for any Y belonging to $\mathcal{S}$ the pullbacks $(\mathrm{Fl}^Y_t)^* X_i$ also belong to $\mathcal{S}$. Denote this pullback by

$$Y_i(t) = (\mathrm{Fl}^Y_t)^* X_i(x)$$

and write $[Y, X_i] = \sum a_{ij} X_j$. By Lemma 2.7 the paths $Y_i(t)$ satisfy the following system of differential equations

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Y_i(t) = (\mathrm{Fl}^Y_t)^*[Y, X_i] = \sum a_{ij}(\mathrm{Fl}^Y_t(x)) Y_j(t)$$

We have $Y_i(0) = X_i(x) \in \mathcal{S}(x)$ and since the system is linear we must have $Y_i(t) \in \mathcal{S}(x)$ for all t. Namely applying any linear form α to this system we see that $\alpha(Y_i(t))$ satisfy the very same linear system of differential equations. Using the uniqueness and the existence of the zero solution we see that $\alpha(Y_i(0)) = 0$ for all i implies $\alpha(Y_i(t)) = 0$ for all i and t. □

By an integral submanifold we will now understand a connected initial submanifold $A \subseteq M$ for which $T_x A = \mathcal{S}_x$ for all $x \in A$. A maximal integral submanifold is one that is not contained in any bigger.

THEOREM 3.9. *If $\mathcal{S}$ is involutive then to every point $x \in M$ there exists a unique maximal integral submanifold going through that point.*

PROOF. We will obtain this initial submanifold as the set A of all points $y \in M$ which can be joined with x by a path $\gamma : I \to M$ tangent to the distribution $\mathcal{S}$, i.e. with the properties

- $\gamma(0) = x$, $\gamma(1) = y$,
- $\dot{\gamma} = \frac{\mathrm{d}}{\mathrm{d}t}\gamma \in \mathcal{S}$.

We need to verify that A is indeed an initial submanifold, maximality should be obvious. In a coordinate chart $\varphi_j : U_j \to \mathbb{R}^m$ from the Frobenius theorem $U_j \cap A$ is clearly the disjoint union

$$\bigsqcup_{(c_{k+1},\dots,c_m)\in C_j} \mathbb{R}^k \times \{(c_{k+1},\dots,c_m)\}$$

It is enough to show that each C_j is at most countable since every countable subset of $\mathbb{R}^{m-k}$ is totally disconnected (in between any two distinct x, y in a countable set $X \subseteq \mathbb{R}$ there lies some $z \notin X$). First we prove an auxiliary fact:

Let B be an integral submanifold which is second countable. Then B intersects each U_j in at most a countable number of leaves $\mathbb{R}^k \times \{(c_{k+1},\dots,c_m)\}$: if, by contradiction, the number was uncountable then choosing a point from B in each leaf we would find an uncountable discrete subset of B.

In particular every leaf of φ_j intersects at most countable number of leaves of φ_k. Now start with $A_0 = \{x\}$ and at each step "leaf complete" A_i to obtain A_{i+1}. Then $A = \bigcup A_i$ and it is second countable, hence intersects only a countable number of leaves of each φ_j. □

Let us return to a linear subspace $P \subseteq \mathfrak{g}$ and the distribution λ_P on G.

LEMMA 3.10. *λ_P is involutive if and only if P is a Lie subalgebra.*

PROOF. Since $[X, fY + gZ] = f[X,Y] + (Xf)Y + g[X,Z] + (Xg)Z$ it is enough to check the brackets of vector fields of the form λ_A with $A \in P$. But $[\lambda_A, \lambda_B] = \lambda_{[A,B]}$. □

THEOREM 3.11. *Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra. Then the maximal integral submanifold H passing through e is a Lie subgroup.*

PROOF. Let $a \in H$. Since $(\lambda_{a^{-1}})_*\lambda_{\mathfrak{h}} = \lambda_{\mathfrak{h}}$, the map $\lambda_{a^{-1}}$ preserves integral submanifolds. As $\lambda_{a^{-1}}(a) = e$ and both $a, e \in H$ we must have $\lambda_{a^{-1}}(H) = H$ and thus $a^{-1}b \in H$ for all $a, b \in H$. □

Now we tackle the uniqueness issue. First a lemma.

LEMMA 3.12. *Let $f : G \to H$ be a homomorphism of Lie groups whose derivative at identity is surjective. Then the image of f is a union of components of H.*

PROOF. The image is certainly a subgroup which is open. Since any open subgroup is necessarily also closed (its complement being a union of cosets which are open) the assertion follows. □

REMARK. Later we will use a simple variation of this lemma: Let U be a connected neighbourhood of e in a Lie group G. Then the subgroup generated by U is exactly the connected component G_e of G containing e.

THEOREM 3.13. *Let $H_1, H_2 \subseteq G$ be two connected Lie subgroups. Then $H_1 = H_2$ if and only if $\mathrm{Lie}(H_1) = \mathrm{Lie}(H_2)$.*

PROOF. Let $\mathfrak{h}$ denote the common Lie algebra of H_1 and H_2 and let H be the maximal integral submanifold of $\lambda_{\mathfrak{h}}$ passing through e. Since both H_i are also integral submanifolds they must be contained in H and the inclusions $H_i \hookrightarrow H$ are both injective and surjective by the previous lemma and thus $H = H_i$. □

4. Homomorphisms of Lie groups and algebras

LEMMA 4.1. *A group homomorphism $f : G \to H$ which is smooth near e is smooth everywhere.*

PROOF. This is a classical homogeneity argument. Denoting by U the neighbourhood of e where f is smooth pick any $a \in G$ and consider the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & H \\ {\scriptstyle\lambda_a}\downarrow & & \downarrow{\scriptstyle\lambda_{f(a)}} \\ aU & \xrightarrow[f]{} & H \end{array}$$

in which aU is a neighbourhood of a and thus f is smooth everywhere. □

The essential idea of this section is to construct homomorphisms through their graphs. Let us consider $\varphi : \mathfrak{g} \to \mathfrak{h}$, a linear map between Lie algebras. The graph of φ is the subset $\mathrm{Graph}(\varphi) = \{(A, \varphi(A)) \mid A \in \mathfrak{g}\}$.

LEMMA 4.2. $\mathrm{Graph}(\varphi)$ *is a Lie algebra if and only if ϕ is a homomorphism of Lie algebras.*

PROOF. By the definition of the bracket in the product

$$[(A, \varphi(A)), (B, \varphi(B))] = ([A, B], [\varphi(A), \varphi(B)])$$

which lies in $\mathrm{Graph}(\varphi)$ if and only if $[\varphi(A), \varphi(B)] = \varphi[A, B]$. □

Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be now a homomorphism of Lie algebras, $\mathrm{Graph}(\varphi) \subseteq \mathfrak{g} \times \mathfrak{h}$ its graph, a Lie subalgebra. There exists a unique connected Lie subgroup $F \subseteq G \times H$ with $\mathrm{Lie}(F) = \mathrm{Graph}(\varphi)$. Assuming that the composition $F \hookrightarrow G \times H \to G$ is a diffeomorphism F will be a graph of a homomorphism $f : G \to H$ with $f_* = \varphi$. In general however this projection is only a local diffeomorphism: its derivative at e is the isomorphism $\mathrm{Graph}(\varphi) \to \mathfrak{g}$ and at other points this follows from the diagram

$$\begin{array}{ccc} T_eF & \xrightarrow[\cong]{f_*} & T_eG \\ {\scriptstyle(\lambda_a)_*}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle(\lambda_{f(a)})_*} \\ T_aF & \xrightarrow[f_*]{} & T_{f(a)}G \end{array}$$

DEFINITION 4.3. A continuous map $f : X \to Y$ is a *covering* if for each $y \in Y$ there exists its neighbourhood U such that

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\cong} & \coprod_{c \in C} U \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle\sum_{c\in C}\mathrm{id}} & \\ & U & \end{array}$$

LEMMA 4.4. *Every local isomorphism of Lie groups is a covering.*

PROOF. Let $f : G \to H$ be the local isomorphism, $U \ni a$, $V \ni b$ open neighbourhoods for which $f|_U : U \xrightarrow{\cong} V$ with inverse g. Then we will show that

$$f^{-1}(V) = \bigsqcup_{k \in \ker f} k \cdot U$$

Therefore let $x \in f^{-1}(V)$. Then $x = (x \cdot g(f(x))^{-1}) \cdot g(f(x))$ is the decomposition. Also $kx = k'x'$ implies that $x(x')^{-1} = k^{-1}k' \in \ker f$ and thus $f(x) = f(x')$. Since f in injective on U, $x = x'$ and necessarily $k = k'$.

The proof is finished by recalling that the image of f is a union of components (so that for any b the a above exists). □

THEOREM 4.5. *Let X be a path connected and locally simply connected topological space. Then X is simply connected if and only if every connected covering of X is a global homeomorphism.*

Before going into the proof we draw a corollary:

THEOREM 4.6. *Let G be a simply connected Lie group, H any Lie group. Then for every homomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$ of Lie algebras there exists a unique homomorphism of Lie groups $f : G \to H$ with the property $f_* = \varphi$. For connected G the uniqueness part is still valid.*

PROOF. The above constructed homomorphism $F \to G$ is a covering and according to the previous theorem a diffeomorphism. Thus F is the graph of f. □

COROLLARY 4.7. *Two simply connected Lie groups G and H are isomorphic if and only if their Lie algebras are isomorphic.* □

The assumption of simple connectivity is essential: the canonical projection map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = \mathbb{T}$ is a homomorphism but there is no non-trivial homomorphism in the opposite direction despite the fact $\operatorname{Lie}\mathbb{R} = \operatorname{Lie}\mathbb{T}$.

PROOF OF THEOREM 4.5. Let us construct the universal covering of X. Set

$$\tilde{X} = \{[\gamma] \mid \gamma : (I, 0) \to (X, x)\}$$

where $[\gamma]$ denotes the class with respect to homotopies preserving *both* endpoints. The projection $p : \tilde{X} \to X$ sends $[\gamma] \mapsto \gamma(1)$. Then clearly

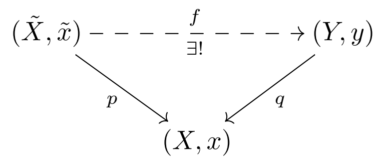
- $p^{-1}(x) \cong \pi_1(X, x)$.
- p is a covering: Let U be a simply connected neighbourhood of x'. Then

$$p^{-1}(U) \cong \bigsqcup_{\substack{[\gamma] \\ \gamma(0)=x \\ \gamma(1)=x'}} [\gamma] * \underbrace{\{[\delta] \mid \delta : (I, 0) \to (U, y)\}}_{\substack{\text{in bijection with } U \text{ by} \\ \text{simple connectivity}}}$$

  This bijection defines a topology on $\tilde{X}$ for which p is a covering. Therefore $\tilde{X}$ is a smooth manifold if X was to start with (again we leave out the proof that $\tilde{X}$ is second countable).

  REMARK. We have shown that $\pi_1(X, x)$ is at most countable since $p^{-1}(x)$ is discrete and X second countable.

- p is universal: let $q : Y \to X$ be a covering with connected Y and let $y \in q^{-1}(x)$. Then there exists a unique $f : \tilde{X} \to Y$ satisfying $qf = p$ and $f(\tilde{x}) = y$ where $\tilde{x} = [x] \in \tilde{X}$ is the class of the constant path



  This is about the path lifting property: the path $\gamma : (I, 0) \to (X, x)$ has a unique continuous lift to $(\tilde{X}, \tilde{x})$, namely $t \mapsto [\gamma|_{[0,t]}]$. Denote the unique lift to (Y, y) by $\tilde{\gamma}$. Since the lifts must be preserved f must send $[\gamma] \mapsto \tilde{\gamma}(1)$.
- If $\pi_1(X, x) = \{e\}$ then $\tilde{X} \to X$ is a homeomorphism: it is a local homeomorphism from the definition of a covering and surjective from the path connectedness of X. We will prove injectivity. Let $p[\gamma] = p[\delta]$, i.e.

$$\gamma, \delta : (I, 0, 1) \to (X, x, x')$$

  The concatenation $\gamma * \delta^{-1}$ is a loop in X, hence contractible to a point which gives $[\gamma] = [\delta]$.

□

5. The exponential map

Definition 5.1. By a *one-parameter subgroup* in G we understand a homomorphism $\gamma : \mathbb{R} \to G$.

Theorem 5.2. *For every $A \in \mathfrak{g}$ there exists a unique one-parameter subgroup $\gamma_A : \mathbb{R} \to G$ such that $\dot{\gamma}_A(0) = A$.*

Proof. $\mathbb{R}$ is simply connected and $\operatorname{Lie} \mathbb{R} = \mathbb{R}$ with the trivial bracket and thus a homomorphism $\mathbb{R} \to \mathfrak{g}$ of Lie algebras is the same thing as a linear map. □

The one-parameter subgroup γ_A is an integral curve of λ_A and more generally for every $a \in G$ the curve $t \mapsto a \cdot \gamma_A(t)$ is:

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} a\gamma_A(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} a\gamma_A(t_0)\gamma_A(t - t_0) = (\lambda_{a\gamma_A(t_0)})_* A = \lambda_A(a \cdot \gamma_A(t_0))$$

Theorem 5.3. *The flow of the left-invariant vector field λ_A is*

$$\mathrm{Fl}_t^{\lambda_A}(a) = a\gamma_A(t) = \rho_{\gamma_A(t)}(a)$$

Moreover λ_A is complete (the integral curves are defined for all $t \in \mathbb{R}$).

Definition 5.4. The map $\exp : \mathfrak{g} \to G$ sending $A \mapsto \gamma_A(1)$ is called the *exponential map* of the Lie group G.

Example 5.5. For $G = (\mathbb{R}^+, \cdot)$ the associated Lie algebra is $\operatorname{Lie} G = \mathbb{R}$, the left-invariant vector field $\lambda_A(a) = (\lambda_a)_* A = aA$. The equation for the flow is

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\gamma_A = \gamma_A A$$

and its solution is clearly $\gamma_A(t) = e^{tA}$. Hence $\exp(A) = e^A$.

EXAMPLE 5.6. More generally for $G = \mathrm{GL}(n, \mathbb{R})$ the exponential map is

$$\begin{aligned} \exp : \mathfrak{gl}(n, \mathbb{R}) &\longrightarrow \mathrm{GL}(n, \mathbb{R}) \\ A &\longmapsto e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k \end{aligned}$$

THEOREM 5.7. *It holds* $\exp(tA) = \gamma_A(t)$.

PROOF. $\gamma_A(t) = \mathrm{Fl}^{\lambda_A}_{t\cdot 1}(e) = \mathrm{Fl}^{t\cdot\lambda_A}_1(e) = \mathrm{Fl}^{\lambda_{tA}}_1(e) = \exp(tA)$. □

THEOREM 5.8. *The map* $\exp : \mathfrak{g} \to G$ *is smooth and a diffeomorphism on a neighbourhood of* 0.

PROOF. The vector field λ_A depends smoothly on A and thus also exp. We compute the derivative of exp by considering a curve $t \mapsto tA$ in $\mathfrak{g}$. Its image under exp is $t \mapsto \exp(tA) = \gamma_A(t)$ whose derivative at 0 is $\dot{\gamma}_A(0) = A$. We conclude that $\exp_* = \mathrm{id} : \mathfrak{g} \to \mathfrak{g}$. □

THEOREM 5.9. *For every homomorphism of Lie groups the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\ f\ } & H \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathfrak{g} & \xrightarrow[f_*]{} & \mathfrak{h} \end{array}$$

PROOF. $f(\gamma_A(t))$ is a one-parameter subgroup with initial speed f_*A and thus equal to $\gamma_{f_*A}(t)$. Evaluating at $t = 1$ yields the result. □

LEMMA 5.10. *Let* $f : G \to H$ *be a homomorphism of Lie groups with* G *connected and let* $K \subseteq H$ *be a Lie subgroup. Then* $f(G) \subseteq K$ *if and only if* $f_*(\mathfrak{g}) \subseteq \mathfrak{k}$.

PROOF. Suppose that $f_*(\mathfrak{g}) \subseteq \mathfrak{k}$. Then $f(\exp(\mathfrak{g})) = \exp(f_*(\mathfrak{g})) \subseteq \exp(\mathfrak{k}) \subseteq K$. Since $\exp(\mathfrak{g})$ is a neighbourhood of e in G, $f^{-1}(K)$ is an open subgroup of G. As G is connected $f^{-1}(K)$ must equal G. □

THEOREM 5.11. *Let* $\varphi : \mathbb{R} \to G$ *be a continuous group homomorphism. Then* φ *is smooth.*

PROOF. In a neighbourhood of $0 \in \mathbb{R}$ we can write uniquely $\varphi(t) = \exp(A(t))$ with $X(t)$ a continuous path in $\mathfrak{g}$ starting at 0. We would like to show that $X(t)$ is linear. Let $\varphi[-t_0, t_0] \subseteq \exp U$ where U is a ball centered at 0 and such that exp is a diffeomorphism on $2U$. Let $n \in \mathbb{N}$. We will show that $kX\left(\frac{t_0}{n}\right) = X\left(k\frac{t_0}{n}\right)$ for $0 \le k \le n$ by induction on k. For $k = 0$ or $k = 1$ this is clear. Assuming the statement true for k write

$$(k+1)X\left(\tfrac{t_0}{n}\right) = kX\left(\tfrac{t_0}{n}\right) + X\left(\tfrac{t_0}{n}\right) \in 2U$$

Since

$$\begin{aligned} \exp\left((k+1)X\left(\tfrac{t_0}{n}\right)\right) &= \left(\exp X\left(\tfrac{t_0}{n}\right)\right)^{k+1} = \varphi\left(\tfrac{t_0}{n}\right)^{k+1} \\ &= \varphi\left((k+1)\tfrac{t_0}{n}\right) = \exp\left(X\left((k+1)\tfrac{t_0}{n}\right)\right) \end{aligned}$$

and exp is injective on $2U$ this finishes the induction step. As a particular case $nX\left(\frac{t_0}{n}\right) = X(t_0)$ and thus $X\left(\frac{k}{n}t_0\right) = \frac{k}{n}X(t_0)$ which easily holds also for all integers k with $|k| \le n$. From continuity $X(rt_0) = rX(t_0)$ for all $r \in [-1, 1]$. Since $\varphi|_{[-t_0,t_0]}$

is now linear and hence smooth, it is smooth everywhere by the usual argument (homogeneity). □

THEOREM 5.12. *Let G, H be Lie groups and $f : G \to H$ a continuous group homomorphism between them. Then f is smooth.*

PROOF. Pick a basis $A_1, \dots, A_m$ in $\mathfrak{g}$ and define a map $\varphi : \mathbb{R}^m \to G$ by

$$(t_1, \dots, t_m) \mapsto \exp(t_1 A_1) \cdots \exp(t_m A_m)$$

Clearly φ is a diffeomorphism near 0. It is called a coordinate chart of a second kind (the first kind is exp itself). The composition $f\varphi$ is the map

$$(t_1, \dots, t_m) \mapsto f(\exp(t_1 A_1)) \cdots f(\exp(t_m A_m))$$

which is smooth: each continuous one-parameter subgroup $f(\exp(t_i A_i))$ is smooth by the previous theorem and so is their product. Again we can globalize by homogeneity. □

THEOREM 5.13 (The closed subgroup theorem). *Let $H \subseteq G$ be a subgroup (in the algebraic sense) which is also closed as a subspace of a Lie group G. Then H is a submanifold and thus a Lie subgroup.*

PROOF. We divide the proof into a few steps:

- Define
$$\mathfrak{h} = \{\dot\gamma(0) \mid \gamma : (\mathbb{R}, 0) \to (G, e) \text{ a smooth curve}\}$$
Then $\mathfrak{h}$ is a linear subspace since $\dot\gamma_1(0) + \dot\gamma_2(0) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} (\gamma_1(t) \cdot \gamma_2(t))$ and $k\dot\gamma(0) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \gamma(kt)$.
- Let $A_n \in \mathfrak{g}$ be a sequence converging to A and let $t_n > 0$ converge to $0 \in \mathbb{R}$. We claim that if $\exp(t_n An) \in H$ then $\exp(tA) \in H$ for all $t \in \mathbb{R}$. We may suppose that $t > 0$. Choose $m_n \in \mathbb{N}$ in such a way that $|t - m_n t_n|$ is minimal. Then $|t - m_n t_n| \to 0$ and consequently $m_n t_n A_n \to tA$. But $\exp(m_n t_n A_n) = \exp(t_n A_n)^{m_n} \in H$ and since H is closed it follows that $\exp(tA) \in H$ too.
- We show that $\mathfrak{h} = \{A \in \mathfrak{g} \mid \exp(tA) \in H \; \forall t \in \mathbb{R}\}$. The inclusion $\supseteq$ follows from the definition of $\mathfrak{h}$. For the reverse inclusion let $A \in \mathfrak{g}$ be $\dot\gamma(0)$ for some curve $\gamma : \mathbb{R} \to H$. For t small we write $\gamma(r) = \exp(A(t))$. Then
$$A = \dot\gamma(0) = \exp_*(\dot A(0)) = \dot A(0) = \lim_{n\to\infty} \frac{A(\frac{1}{n})}{\frac{1}{n}}$$
Setting $A_n = nA\left(\frac{1}{n}\right) \to A$ and $t_n = \frac{1}{n}$ we have
$$\exp(t_n A_n) = \exp\left(A\left(\tfrac{1}{n}\right)\right) = \gamma\left(\tfrac{1}{n}\right) \in H.$$
and by the previous point $\exp(tA) \in H$ for all $t \in \mathbb{R}$.
- Let $\mathfrak{k} \subseteq \mathfrak{g}$ be a linear subspace complementary to $\mathfrak{h}$. We claim that there exists a neighbourhood $0 \in W \subseteq \mathfrak{k}$ such that $\exp(W) \cap H = \{e\}$. By contradiction let $B_n \to 0$ be a sequence in $\mathfrak{k}$ such that $\exp(B_n) \in H$. With respect to some norm on $\mathfrak{k}$ consider $A_n = \frac{B_n}{|B_n|}$. By passing to a subsequence we may assume that A_n converges to some $A \in \mathfrak{k}$. Putting $t_n = |B_n|$ we have $\exp(t_n A_n) = \exp(B_n) \in H$ and thus $\exp(tA) \in H$ for all $t \in \mathbb{R}$. By the previous point $A \in \mathfrak{h}$, a contradiction to $A \in \mathfrak{k}$.

- Define $\varphi : \mathfrak{h} \times \mathfrak{k} \to G$ by $(A, B) \mapsto \exp A \cdot \exp B$. We will show that there exists a neighbourhood $0 \in V \subseteq \mathfrak{h}$ for which the restriction
$$\varphi : V \times W \xrightarrow{\cong} U \subseteq G$$
is a diffeomorphism onto its image U (which is easy) and such that
$$U \cap H = \varphi(V \times \{0\}).$$
Therefore let $x \in U \cap H$ be in the image, $x = \exp A \cdot \exp B$. As both $x, \exp A \in H$, also $\exp B \in H$. By the previous point $B = 0$.

Thus we found a chart at e which flattens out H. Charts at other points are obtained by translation. □

6. Homogeneous spaces

Definition 6.1. By a *left action* of a Lie group G on a smooth manifold M we understand a smooth map $\ell : G \times M \to M$ satisfying $\ell_e = \mathrm{id}$ and $\ell_a \circ \ell_b = \ell_{ab}$ where we write $\ell_a = \ell(a, -)$. The algebraic content is a homomorphism $G \to \mathrm{Diff}(M)$.

The *right action* $r : M \times G \to M$ has to satisfy $r_e = \mathrm{id}$ and $r_a \circ r_b = r_{ba}$.

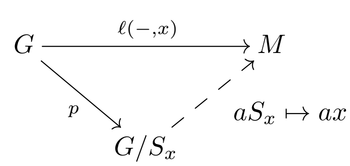
We will write $\ell_a(x) = a \cdot x$ and $r_a(x) = xa$.

Remark. A right action of G is the same as a left action of the opposite group G^*.

Definition 6.2. The orbit of a point $x \in M$ is the subset $Gx = \{ax \mid a \in G\}$. We call the action *transitive* if there is only one orbit in M or equivalently if $Gx = M$ for every $x \in M$.

The *stabilizer subgroup* of a point $x \in M$ is the (closed) subgroup
$$S_x = \{a \in G \mid ax = x\}.$$
The action is called *free* if the stabilizer subgroup of each point is trivial, $S_x = \{e\}$ for every $x \in M$. The action is called *effective* if $\ell_a = \ell_b$ implies $a = b$, i.e. if the homomorphism $G \to \mathrm{Diff}(M)$ is injective.

Set theoretically the action yields a diagram



an injective map

and if the action is transitive then $G/S_x \to M$ is even a bijection. Naturally G/S_x is a topological space, a quotient of G:
$$U \subseteq G/S_x \text{ is open} \iff p^{-1}(U) \subseteq G \text{ is open.}$$

Theorem 6.3. *Let $H \subseteq G$ be a closed subgroup of a Lie group G. Then there exists a unique smooth structure on G/H for which $p : G \to G/H$ is a submersion.*

PROOF. First we will demonstrate uniqueness in a more general context. The idea here is that surjective submersions are quotient objects:

$$\begin{array}{ccc} M & \xrightarrow{g} & P \\ {\scriptstyle f}\downarrow & \nearrow_{h} & \\ N & & \end{array}$$

If f is a surjective submersion and g any smooth map which factors through f set-theoretically, i.e. such that $\ker f \subseteq \ker g$ (or more concretely $f(x) = f(x')$ implies $g(x) = g(x')$), then the unique map h satisfying $g = hf$ is smooth. This follows easily from the fact that f admits smooth local sections (and h is thus a composition of g with such a section).

The uniqueness now follows formally since in the diagram

$$\begin{array}{ccc} & G & \\ {\scriptstyle p}\swarrow & & \searrow{\scriptstyle p} \\ G/H & \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} & G/H \end{array} \qquad \longleftarrow \text{possibly different smooth structures}$$

the unique factorization maps are the identity maps and the fact that they are both smooth means precisely that the two smooth structures coincide.

It remains to prove the existence. Let $\mathfrak{k} \subseteq \mathfrak{g}$ be a linear subspace complementary to $\mathfrak{h}$. There are neighbourhoods $0 \in V \subseteq \mathfrak{k}$, $0 \in W \subseteq \mathfrak{h}$ and $e \in U \subseteq G$ such that

$$\begin{aligned} \varphi : V \times W &\longrightarrow U \\ (A, B) &\longmapsto \exp A \cdot \exp B \end{aligned}$$

is a diffeomorphism and $U \cap H = \varphi(\{0\} \times W)$. Let $0 \in V' \subseteq V$ be such that

$$(\exp V')^{-1} \cdot (\exp V) \subseteq U$$

which is possible by continuity of the operations. Suppose now that $A_1, A_2 \in V'$ are such that $(\exp A_1) \cdot H = (\exp A_2) \cdot H$. Then $(\exp A_1)^{-1} \cdot \exp A_2 \in U \cap H$ and is equal to $\exp B$ for a unique $B \in W$. Multiplying back

$$\varphi(A_2, 0) = \varphi(A_1, B)$$

which implies $A_1 = A_2$ and $B = 0$. This says that the map

$$\begin{aligned} f : V' \times H &\longrightarrow G \\ (A, b) &\longmapsto (\exp A) \cdot b \end{aligned}$$

is injective. Since it is also a local diffeomorphism on $V' \times (\exp W)$ by translation it is so everywhere and f is in fact a diffeomorphism onto its image.

We have now identified a neighbourhood of $H \subseteq G$ with a product $V' \times H$ and in such a way that the cosets $a \cdot H$ lying in this "chart" are of the form $\{A\} \times H$. Thus the map

$$\psi : V' \cong V' \times \{e\} \hookrightarrow V' \times H \hookrightarrow G \xrightarrow{p} G/H$$

embeds V' as a neighbourhood of the coset $eH \in G/H$. We therefore declare it a chart on G/H. In this way the map p becomes the projection $V' \times H \to V'$ and thus a submersion. To get a chart near arbitrary aH redefine f as

$$\begin{aligned} f_a : V' \times H &\longrightarrow G \\ (A, b) &\longmapsto (\exp A) \cdot a \cdot b \end{aligned}$$

The transition map between the resulting charts $\psi_{a'}$ and ψ_a is the composition

$$V' \xrightarrow{\exp} U \xrightarrow{\rho_{a'a^{-1}}} U \xrightarrow{f^{-1}} V' \times H \longrightarrow V'$$

with all arrows smooth and $\rho_{a'a^{-1}}$ only locally defined. □

Definition 6.4. The manifold G/H is called a *homogeneous space*.

Theorem 6.5. *The orbit of each point is an immersed submanifold (i.e. image of an injective immersion).*

Proof. Consider the diagram

$$\begin{array}{ccc} G & \xrightarrow{\ell(-,x)} & M \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle f} & \\ & G/S_x & \end{array}$$

with the map f smooth by the previous theorem. We need to show that it is an immersion (on the other hand it is injective almost by the definition of S_x). Suppose first that for $A \in \mathfrak{g}$ its image p_*A is sent to $0 \in T_xM$ by f_*. Then $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \exp(tA)x = 0$. On the other hand

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} \exp(tA)x &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} \exp(t_0A)\exp((t-t_0)A)x \\ &= (\ell_{\exp(t_0A)})_* \underbrace{\tfrac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} \exp((t-t_0)A)x}_{0} = 0 \end{aligned}$$

Thus $\exp(tA)x = x$ for all $t \in \mathbb{R}$ and $\exp(tA) \in S_x$ implying that $A \in \ker p_*$ and $p_*A = 0$. This finishes the proof that f is an immersion at eS_x. At other points this is guaranteed by the homogeneity:

$$\begin{array}{ccccc} eS_x & \qquad & G/S_x & \xrightarrow{f} & M \\ \big\downarrow & & {\scriptstyle \ell_a}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle \ell_a} \\ aS_x & & G/S_x & \xrightarrow[f]{} & M \end{array}$$

□

Example 6.6. Fix $v \in \mathbb{R}^2$ and consider the following action of $\mathbb{R}$ on $\mathbb{R}^2$

$$\begin{aligned} \mathbb{R} \times \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ (t, u) &\longmapsto u + tv \end{aligned}$$

Clearly the orbit of u is the line $u + \mathbb{R}v$. Passing to the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ we see that orbits need not be embedded submanifolds.

Remark. In general every orbit is an initial submanifold.

Corollary 6.7. *For a transitive action the map $f : G/S_x \to M$ is a diffeomorphism.*

PROOF. From Sard's theorem it easily follows that smooth bijections exist only between manifolds of the same dimension. Hence the immersion f is in fact a local diffeomorphism. Being also bijective it is a diffeomorphism by the inverse function theorem. □

EXAMPLES 6.8. Examples of homogeneous spaces:

- Let V be a vector space. Then $\mathrm{GL}(V)$ acts transitively on $V - \{0\}$ and thus $V - \{0\} \cong \mathrm{GL}(V)/S_v$ where $v \in V - \{0\}$.
- The sphere S^{n-1} with the action of $O(n)$ is a homogeneous space, $S^{n-1} \cong O(n)/O(n-1)$ where $O(n-1)$ is thought of as a subgroup of $O(n)$ consisting of block matrices
$$O(n-1) \cong \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in O(n) \;\middle|\; A \in O(n-1) \right\}$$
- The n-dimensional affine space is acted upon by the group
$$GA(n) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(n+1) \;\middle|\; A \in \mathrm{GL}(n), v \in \mathbb{R} \right\}$$
of affine transformations, namely we identify a point $x \in \mathbb{R}^n$ with a vector $\left(\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right)$ in $\mathbb{R}^{n+1}$ and then
$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + v \\ 1 \end{pmatrix}$$
The origin is preserved exactly by the subgroup
$$\mathrm{GL}(n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in GA(n) \;\middle|\; A \in \mathrm{GL}(n) \right\}$$
describing $\mathbb{R}^n$ as $GA(n)/\mathrm{GL}(n)$. Similarly with $\mathrm{GL}(n)$ replaced by $O(n)$ we arrive at $\mathbb{R}^n \cong Euc(n)/O(n)$ with $Euc(n)$ denoting the group of (not necessarily origin preserving) isometries of $\mathbb{R}^n$.
- The Stiefel manifold (of orthonormal k-frames in V)
$$S_k(V) = \{(v_1, \dots, v_k) \mid \langle v_i, v_j \rangle = \delta_{ij}\}$$
has as examples $S_1(V)$, the unit sphere in V, $S_n(\mathbb{R}^n) = O(n)$. For general $S_k(\mathbb{R}^n)$ there is a natural action of $O(n)$ componentwise:
$$A(v_1, \dots, v_k) = (Av_1, \dots, Av_k)$$
The stabilizer of the k-tuple $(e_1, \dots, e_k)$ of the first k vectors of the standard basis is clearly
$$O(n-k) \cong \left\{ \begin{pmatrix} E & 0 \\ 0 & C \end{pmatrix} \in O(n) \;\middle|\; C \in O(n-k) \right\}$$
Thus $S_k(\mathbb{R}^n) \cong O(n)/O(n-k)$.
- The Grassmann manifold $G_k(V)$ of all k-dimensional subspaces of V is naturally a quotient of $S_k(V)$, namely by the means of the map
$$\begin{aligned} S_k(V) &\longrightarrow G_k(V) \\ (v_1, \dots, v_k) &\longmapsto \mathrm{span}(v_1, \dots, v_k) \end{aligned}$$

The $O(n)$-action on $S_k(\mathbb{R}^n)$ passes to $G_k(\mathbb{R}^n)$ with the stabilizer of $\mathbb{R}^k$ being

$$O(k) \times O(n-k) \cong \left\{ \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \in O(n) \;\middle|\; B \in O(k), C \in O(n-k) \right\}$$

and thus providing $G_k(\mathbb{R}^n) \cong O(n)/O(k) \times O(n-k)$.

Theorem 6.9. *Let $N \subseteq G$ be a closed normal subgroup. Then G/N with its canonical smooth structure is a Lie group.*

Proof. The left vertical arrow in

$$\begin{array}{ccc} G \times G & \xrightarrow{\ \mu\ } & G \\ {\scriptstyle p\times p}\downarrow & & \downarrow{\scriptstyle p} \\ G/N \times G/N & \dashrightarrow & G/N \end{array}$$

is a surjective submersion therefore the dotted arrow (the multiplication in G/N) is smooth. □

7. The adjoint representation

Definition 7.1. By a *representation* of G we understand a left action of G on a vector space V by linear maps (automorphisms), i.e. for which each $\ell_a : V \to V$ is linear. Equivalently $\rho : G \to \mathrm{GL}(V)$ is a (smooth) homomorphism of Lie groups.

Definition 7.2. A representation of a Lie algebra L on a vector space V is a homomorphism $\pi : L \to \mathfrak{gl}(V)$ of Lie algebras. More concretely π is a linear map for which $\pi[X,Y](v) = \pi X \circ \pi Y(v) - \pi Y \circ \pi X(v)$.

Definition 7.3. A linear subspace $W \subseteq V$ is called *invariant* with respect to a representation ρ if $\rho(a)(W) \subseteq W$ for all $a \in G$. Analogously it is called *invariant* with respect to a representation π if $\pi(X)(W) \subseteq W$ for all $X \in L$.

Theorem 7.4. *Let G be a connected Lie group and ρ its representation on V, $\rho_* : \mathfrak{g} \to V$ the induced representation of $\mathfrak{g}$. Then $W \subseteq V$ is invariant with respect to ρ if and only if it is invariant with respect to ρ_*.*

Proof. Consider the following subgroup of $\mathrm{GL}(V)$

$$\mathrm{GL}(V, W) = \{\varphi \in \mathrm{GL}(V) \mid \varphi(W) \subseteq W\}.$$

It is easy to show that

$$\mathfrak{gl}(V, W) = \mathrm{Lie}(\mathrm{GL}(V, W)) = \{\varphi \in \mathfrak{gl}(V) \mid \varphi(W) \subseteq W\}.$$

The statement then becomes a special case of Lemma 5.10. □

Let $\ell : G \times M \to M$ be a left action and $x \in M$ its fixed point (i.e. $S_x = G$). Then $\rho : G \to \mathrm{GL}(T_x M)$ given by $a \mapsto (\ell_a)_{*x}$ is smooth by

$$\begin{array}{ccc} TG \times TM & \xrightarrow{\ \ell_*\ } & TM \\ {\scriptstyle 0\times \mathrm{id}}\uparrow & \nearrow{\scriptstyle \rho^\sharp} & \\ G \times T_x M & & \end{array}$$

and consequently a representation of G on T_xM. We apply these general considerations to the action of G on itself via conjugation (inner automorphisms):

$$(a,b) \longmapsto \mathrm{int}_a\, b = aba^{-1}$$

Now $e \in G$ is a fixed point and we define $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ as above

$$\mathrm{Ad}(a)B = (\mathrm{int}_a)_* B$$

Moreover $\mathrm{Ad}(a) \in \mathsf{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ since int_a is a homomorphism of Lie groups. We denote the induced representation by $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ (in fact $\mathsf{Der}(\mathfrak{g})$).

THEOREM 7.5. *For each $A, B \in \mathfrak{g}$ it holds* $\mathrm{ad}(A)(B) = [A,B]$.

PROOF. We compute

$$\begin{aligned} \mathrm{ad}(A)(B) &= \tfrac{\partial}{\partial s}\big|_{s=0} \mathrm{Ad}(\exp(sA))(B) \\ &= \tfrac{\partial}{\partial s}\big|_{s=0} \tfrac{\partial}{\partial t}\big|_{t=0} \mathrm{int}_{\exp(sA)} \exp(tB) \\ &= \tfrac{\partial}{\partial s}\big|_{s=0} \tfrac{\partial}{\partial t}\big|_{t=0} \exp(sA)\exp(tB)\exp(-sA) \\ &= \tfrac{\partial}{\partial s}\big|_{s=0} \tfrac{\partial}{\partial t}\big|_{t=0} \mathrm{Fl}^{\lambda_A}_{-s}\, \mathrm{Fl}^{\lambda_B}_{t}\, \mathrm{Fl}^{\lambda_A}_{s}(e) \\ &= \tfrac{\partial}{\partial s}\big|_{s=0} (\mathrm{Fl}^{\lambda_A}_{-s})_* \lambda_B(\mathrm{Fl}^{\lambda_A}_{s}(e)) = [\lambda_A, \lambda_B]_e = [A,B] \end{aligned}$$

□

THEOREM 7.6. *If $H \subseteq G$ is a normal subgroup then $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal, i.e. a linear subspace such that $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$ (meaning $[A,B] \in \mathfrak{h}$ for all $A \in \mathfrak{g}$ and $B \in \mathfrak{h}$).*

PROOF. Since $aHa^{-1} \subseteq H$ or $\mathrm{int}_a H \subseteq H$ we differentiate to get $\mathrm{Ad}(a)(\mathfrak{h}) \subseteq \mathfrak{h}$ and finally $\mathrm{ad}(\mathfrak{g})\mathfrak{h} \subseteq \mathfrak{h}$. □

THEOREM 7.7. *Let H be a connected Lie subgroup of a connected Lie group G such that $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal. Then H is a normal subgroup.*

PROOF. We have $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}, \mathfrak{h})$. Since G is connected $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}, \mathfrak{h})$. It is enough to show that $\mathrm{int}_a(\exp tB) \in H$ for all $B \in \mathfrak{h}$ since the subgroup generated by such elements is the whole group H. But $\exp tB$ is a one-parameter subgroup and int_a a homomorphism, thus $\mathrm{int}_a \exp tB$ is also a one-parameter subgroup in G with initial speed $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \mathrm{int}_a \exp tB = \mathrm{Ad}(a)B \in \mathfrak{h}$. □

THEOREM 7.8. *Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then its kernel is a closed normal subgroup $K \subseteq G$ and its Lie algebra $\mathfrak{k}$ is the kernel of $\varphi_* : \mathfrak{g} \to \mathfrak{h}$.*

PROOF. $A \in \mathfrak{k}$ iff $\exp tA \in K$ iff $\exp(t \cdot \varphi_* A) = \varphi(\exp tA) = e$ iff $\varphi_* A = 0$. □

DEFINITION 7.9. The *centre* C of a Lie group G is the set

$$C = \{a \in G \mid ab = ba\ \forall b \in G\}$$

In other words, C is the kernel of $\mathrm{int} : G \to \mathsf{Aut}(G)$.

THEOREM 7.10. *The centre of a connected Lie group G is the kernel of the adjoint representation* Ad.

PROOF. $a \in C$ iff $\mathrm{int}_a(G) = e$ iff $\mathrm{Ad}(a)\mathfrak{g} = 0$ iff $\mathrm{Ad}(a) = 0$. □

DEFINITION 7.11. The *centre* of a Lie group L is the ideal

$$Z = \{X \in L \mid [X,Y] = 0\ \forall Y \in L\}$$

In other words, Z is the kernel of $\mathrm{ad} : L \to \mathfrak{gl}(L)$.

THEOREM 7.12. *For a connected Lie group G, the centre Z of $\mathfrak{g}$ the Lie algebra of the centre C of G.*

PROOF. Since $C = \ker(\mathrm{Ad})$, its Lie algebra $\mathrm{Lie}(C) = \ker(\mathrm{ad})$. □

REMARK. If the centre of L is zero then L can be embedded into $\mathfrak{gl}(L)$ via the representation ad.

THEOREM 7.13 (Ado). *Every Lie algebra can be embedded into $\mathfrak{gl}(V)$ for some finite-dimensional vector space V.*

COROLLARY 7.14. *Every Lie algebra is induced from some Lie group.*

PROOF. By Ado's theorem $L \subseteq \mathfrak{gl}(n)$. Since $\mathfrak{gl}(n) = \mathrm{Lie}(\mathrm{GL}(n))$ one can find a Lie subgroup of $\mathrm{GL}(n)$ corresponding to L. □

8. Fundamental vector fields

Consider a left action $\ell : G \times M \to M$. To every vector $A \in \mathfrak{g}$ we associate a vector field ℓ_A on M by $\ell_A(x) = (\ell(-,x))_* A$. As usual ℓ_A is smooth and is called the *fundamental vector field* on M corresponding to $A \in \mathfrak{g}$. Analogously we define fundamental vector fields for right actions.

THEOREM 8.1. *In the case of a left action of G on M it holds $[\ell_A, \ell_B] = \ell_{-[A,B]}$. For the right action we obtain $[r_A, r_B] = r_{[A,B]}$.*

PROOF. On $M \times G$ consider the vector field $(0, \lambda_A)(x,a) = (0_x, \lambda_A(a))$.

$$r_{*(x,a)}(0, \lambda_A) = (r(x,-))_{*a}\lambda_A = (r(xa,-))_{*e}A = r_A(xa)$$

says that $(0, \lambda_A)$ is r-related to r_A. As the same is true for B we obtain for the brackets that $[(0,\lambda_A),(0,\lambda_B)]$ is r-related to $[r_A, r_B]$. But

$$[(0,\lambda_A),(0,\lambda_B)] = ([0,0],[\lambda_A,\lambda_B]) = (0, \lambda_{[A,B]})$$

which is r-related to $r_{[A,B]}$. Thus $[r_A, r_B] = r_{[A,B]}$. □

The last theorem can be expressed by saying that $r : \mathfrak{g} \to \mathfrak{X}M$, $A \mapsto r_A$ is a homomorphism of Lie algebras. The left action gives an antihomomorphism.

DEFINITION 8.2. By a *right infinitesimal action* of a Lie group G on a manifold M we understand a homomorphism $R : \mathfrak{g} \to \mathfrak{X}M$ of Lie groups. A right infinitesimal action is called *complete* if R_A is a complete vector field for each $A \in \mathfrak{g}$. Analogously a *left infinitesimal action* is an antihomomorphism.

EXAMPLE 8.3. The fundamental vector fields are complete: $r(x, \exp tA) = x \exp tA$ is an integral curve through x defined for all $t \in \mathbb{R}$.

REMARK. A left action is a homomorphism of Lie groups $G \to \mathrm{Diff}(M)$ (with infinite dimensional target). The induced Lie algebra homomorphism is $\mathfrak{g} \to \mathrm{Lie}(\mathrm{Diff}(M))$, the latter being $\mathfrak{X}M$ but with the opposite bracket. As for finite dimensional Lie groups we can "integrate" a homomorphism of Lie groups but here under additional requirements - the completeness.

THEOREM 8.4. *For a complete right infinitesimal action $R : \mathfrak{g} \to \mathfrak{X}M$ of a simply connected Lie group G on M there exists a unique right action $r : M \times G \to M$ of G on M such that R_A is its fundamental vector field r_A.*

REMARKS.

- The simple connectivity is necessary: for the action of $G = \mathbb{R}$ on itself by translations the infinitesimal action $r_t = t$ passes to an infinitesimal action of $\mathbb{R}$ on the quotient $\mathbb{R}/\mathbb{Z}$ for which no action exists.
- The theorem holds locally without the assumptions of completeness and simple connectivity.
- The usual translation between left and right yields an analogous statement for left actions.

PROOF. Let first r be an action of G on M. Let S_x denote the following submanifold

$$S_x = \{(xa, a) \mid a \in G\} \subseteq M \times G$$

The tangent space of S_x is

$$TS_x = \{(r_A(xa), \lambda_A(a)) \mid a \in G, A \in \mathfrak{g}\}$$

Thus S_x is an integral submanifold of the distribution $\langle (r_A, \lambda_A) \mid A \in \mathfrak{g}\rangle$.

Let us now start the actual proof of the theorem by considering the distribution $D = \langle (R_A, \lambda_A) \mid A \in \mathfrak{g}\rangle$. Then D is involutive since

$$[(R_A, \lambda_A), (R_B, \lambda_B)] = ([R_A, R_B], [\lambda_A, \lambda_B]) = (R_{[A,B]}, \lambda_{[A,B]}).$$

Let S_x be the maximal integral submanifold of D through $(x, e) \in M \times G$. We claim now that $p_x : S_x \hookrightarrow M \times G \to G$ is a diffeomorphism.

First we show that it is a covering. Fix $a \in G$ and consider an arbitrary $(y, a) \in M \times G$. The computation

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_0} \underbrace{(\mathrm{Fl}^{R_A}_t(y), a \exp tA)}_{\gamma(t)} = (R_A(\mathrm{Fl}^{R_A}_{t_0}(y)), \lambda_A(a \exp t_0 A)) \in D$$

shows that $\gamma(t) \in S_A$ since it is tangent to the distribution D. Let $U \subseteq \mathfrak{g}$ be an open ball centered at 0 on which exp is a diffeomorphism. If $(y, a) \in S_x$ then also $(\mathrm{Fl}^{R_A}_1(y), a \exp A) \in S_x$ for all $A \in U$ and such points form an open neighbourhood on which p_x is a diffeomorphism onto $a \exp U$. If $(z, b) \in S_x$ is arbitrary with $b \in a \exp U$ then $b = a \exp A$ and thus the above subset considered for $(\mathrm{Fl}^{R_{-A}}_1(z), b \exp(-A))$ contains (z, b). This finishes the proof that p_x is a covering and in fact a diffeomorphism as G is simply connected.

We define for $x \in M$ and $a \in G$ the action by the requirement

$$(xa, a) \in S_x$$

By the previous part there is a unique choice for xa. We need to show that r is smooth but first let us prove the axioms of an action. Clearly $xe = x$ as S_x is an integral manifold through (x, e). Consider now a left action of G on $M \times G$ by $a(y, b) = (y, ab)$. The distribution D is invariant under this action (as $(\mathrm{id}, \lambda_a)_*(R_A, \lambda_A) = (R_A, \lambda_A)$) and thus also its maximal integral submanifolds. The requirement for our action r can be then rewritten as

$$S_x = aS_{xa} = a(bS_{(xa)b})) = (ab)S_{(xa)b}$$

As also $S_x = (ab)S_{x(ab)}$ the maximal integral submanifolds $S_{(xa)b}$ and $S_{x(ab)}$ must also be equal proving $(xa)b = x(ab)$.

A word about smoothness... □

DEFINITION 8.5. Consider two actions r and r' of a Lie group G on manifolds M and M'. A map $f : M \to M'$ is called *equivariant* if $f(xa) = f(x)a$.

THEOREM 8.6. *If $f : M \to M'$ is equivariant then r_A is f-related to r'_A.*

PROOF. The requirement from the definition is $f \circ r(x, -) = r'(x, -) \circ f$. Applying the derivatives of both sides to A we get $f_* r_A = r'_A f$. □

THEOREM 8.7. *Let $f : M \to M'$ be a smooth map such that r_A is f-related to r'_A. If G is connected then f is equivariant.*

PROOF. Consider the set $H \subseteq G$ of all $a \in G$ for which $f(xa) = f(x)a$ for all $x \in M$. Then H is clearly a subgroup and thus we only need that it contains a neighbourhood of e. But $f(x \exp tA) = f(\mathrm{Fl}_t^{r_A}(x)) = \mathrm{Fl}_t^{r'_A}(f(x)) = f(x) \exp tA$, hence $\exp \mathfrak{g} \subseteq H$ and H is open and therefore equal to G. □

9. Locally isomorphic Lie groups

Let G be a connected Lie group. Recall that the universal covering of G is

$$\begin{array}{ccc} \tilde{G} & = & \{[\gamma] \mid \gamma : (I, 0) \to (G, e)\} \\ {\scriptstyle p}\downarrow & & \downarrow \\ G & & \gamma(1) \end{array}$$

with $[\gamma]$ the homotopy class of γ relative to the boundary. $\tilde{G}$ is simply connected: firstly $\pi_1 \tilde{G} \to \pi_1 G$ (this works for any covering) since we can lift homotopies and constant paths lift to constant paths. The image consists exactly of the classes of loops that lift to loops. For $\tilde{G}$ if $\gamma : I \to G$ lifts to a loop its endpoints must be equal $\tilde{e} = [\gamma]$ and the image is therefore trivial.

We give $\tilde{G}$ a structure of a Lie group: let $\gamma, \delta : (I, 0) \to (G, e)$ be two paths. Define their product to be the path

$$(\gamma \cdot \delta)(t) = \gamma(t)\delta(t)$$

which easily passes to homotopy classes rel ∂I.

THEOREM 9.1. *The above multiplication on $\tilde{G}$ describes a structure of a Lie group for which the projection $p : \tilde{G} \to G$ is a local isomorphism (i.e. a homomorphism and a local diffeomorphism).*

PROOF. The unit and inverses are also pointwise. The diagram

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & \dashrightarrow & \tilde{G} \\ \downarrow & & \downarrow {\scriptstyle \text{local diffeomorphism}} \\ G \times G & \xrightarrow[\text{smooth}]{} & G \end{array}$$

shows that the dotted arrow (the multiplication in $\tilde{G}$) is smooth. □

There is an action of $\pi_1 G$ on $\tilde{G}$, $\pi_1 G \times \tilde{G} \to \tilde{G}$ given by

$$([\alpha], [\gamma]) \longmapsto [\alpha] \cdot [\gamma] = [\alpha * \gamma]$$

which respects the projection $p : \tilde{G} \to G$. Let $\Gamma \subseteq \pi_1 G$ be a subgroup and consider

$$p_\Gamma : \tilde{G}/\Gamma \to G$$

where $\tilde{G}/\Gamma$ is the space of orbits of the restriction of the action to Γ. Locally

$$\begin{array}{ccc} U \times \pi_1 G & \hookrightarrow & \tilde{G} \\ \downarrow & & \downarrow p \\ U & \hookrightarrow & G \end{array}$$

and the action of Γ is by left multiplication in $\pi_1 G$. Thus the projection p_Γ from $\tilde{G}/\Gamma$ to G is locally of the form

$$U \times (\pi_1 G/\Gamma) \to U$$

and in particular is a covering of G.

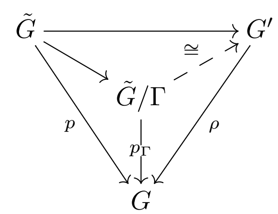
THEOREM 9.2. *Let G be a connected Lie group. Then the mapping*

$$\{\textit{subgroups } \Gamma \subseteq \pi_1 G\} \longrightarrow \left\{ \begin{array}{l} \textit{local isomorphisms } \rho : G' \to G \\ \textit{with } G' \textit{ any connected Lie group} \end{array} \right\} /\textit{iso}$$

$$\Gamma \longmapsto (p_\Gamma : \tilde{G}/\Gamma \to G)$$

is a bijection with inverse $\rho \longmapsto \operatorname{im}(\pi_1 \rho : \pi_1 G' \to \pi_1 G)$.

PROOF. The image of $\pi_1 p_\Gamma$ consists of those loops that lift to loops in $\tilde{G}/\Gamma$. These are precisely those in Γ. In the opposite direction any ρ fits into the diagram



with $\Gamma = \operatorname{im}(\pi_1 \rho)$. The top arrow exists by universality of $\tilde{G}$. The dotted arrow exists since loops in Γ lift to loops in G'. It is an isomorphism of Lie groups. □

REMARK. We will show in the tutorial that $\pi_1 G \hookrightarrow \tilde{G}$ is a homomorphism and the action of $\pi_1 G$ on $\tilde{G}$ is by left translations, i.e. $\tilde{G}/\Gamma$ is a quotient of $\tilde{G}$ by (a central subgroup) Γ.

EXAMPLE 9.3 (The universal covering of a commutative connected Lie group G). Since $\operatorname{Lie} G = \mathbb{R}^n$ with zero bracket it is also the Lie algebra of the simply connected Lie group $\mathbb{R}^n$ (with vector addition) and thus $\tilde{G} = \mathbb{R}^n$. Therefore $G \cong \mathbb{R}^n/\Gamma$ where Γ is some discrete subgroup of $\mathbb{R}^n$. We will show now that $\Gamma = \mathbb{Z}^k \subseteq \mathbb{R}^n$ in some coordinates on $\mathbb{R}^n$.

First reduction is to the case $n = k$, namely we have $\operatorname{span} \Gamma = \mathbb{R}^k \subseteq \mathbb{R}^n$ and Γ is still discrete in $\mathbb{R}^k$. We must show that $\Gamma = \mathbb{Z}^k$ in some coordinates on $\mathbb{R}^k$.

We start an induction by $k = 1$ which we proved in the tutorial. For the induction step we may assume that $\Gamma \subseteq \mathbb{R} \times \mathbb{R}^k = \mathbb{R}^{k+1}$ is such that the intersection $\Gamma \cap \mathbb{R} \neq 0$ with the first coordinate axis is nonzero. Since it is also discrete it is generated by some a_0. In $\mathbb{R}^k = \mathbb{R}^{k+1}/\mathbb{R}$ consider its subgroup $\Gamma/\langle a_0\rangle$. We show by contradiction that it is discrete. Namely assume the existence of a sequence $\alpha_n = (\beta_n, \gamma_n) \in \Gamma$ with $\gamma_n \to 0$ in $\mathbb{R}^k$. By adding a suitable multiple of a_0 to each α_n we may assume that $\beta_n \in [-a_0/2, a_0/2]$ and by extracting a subsequence we may further assume that α_n converges. But then $\alpha_{n+1} - \alpha_n \in \Gamma$ converges to 0, a contradiction with Γ being discrete. By the induction hypothesis $\Gamma/\langle a_0\rangle = \langle \tilde{a}_1, \dots, \tilde{a}_k\rangle$. We choose for each $\tilde{a}_i$ an element $a_i \in \Gamma$ representing it. Then the suitable basis in which $\Gamma = \mathbb{Z}^{k+1}$ is formed by $(a_0, a_1, \dots, a_k)$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}a_0 & \longrightarrow & \mathbb{Z}\{a_0, a_1, \dots, a_k\} & \longrightarrow & \mathbb{Z}\{\tilde{a}_1, \dots, \tilde{a}_k\} & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \\ 0 & \longrightarrow & \Gamma \cap \mathbb{R} & \longrightarrow & \Gamma & \longrightarrow & \Gamma/(\Gamma \cap \mathbb{R}) & \longrightarrow & 0 \end{array}$$

COROLLARY 9.4. *The only compact connected commutative Lie group of dimension k is the torus $\mathbb{T}^k = (S^1)^k$.*

EXAMPLE 9.5. For $n \geq 3$ we have $\pi_1 \operatorname{SO}(n) \cong \mathbb{Z}/2$. Therefore $\operatorname{SO}(n)$ possesses a two-sheeted universal covering which is denoted by $\operatorname{Spin}(n) = \widetilde{\operatorname{SO}}(n)$. We will show geometrically that $\pi_1 \operatorname{SO}(n) = \mathbb{Z}/2$ in the tutorial. For higher n we have a fibration

$$\operatorname{SO}(n) \to \operatorname{SO}(n+1) \to S^n$$

whose long exact sequence of homotopy groups contains the following portion

$$0 = \pi_2(S^n) \to \pi_1(\operatorname{SO}(n)) \xrightarrow{\cong} \pi_1(\operatorname{SO}(n+1)) \to \pi_1 S^n = 0$$

10. Problems

PROBLEM 10.1. An algebra is a vector space A together with a bilinear map $\cdot : A \times A \to A$. Let A be now an associative algebra and define $[\ ,\ ] : A \times A \to A$ by $[a, b] = a \cdot b - b \cdot a$. Show that with this operation A forms a Lie algebra.

A special case of the previous is the algebra $\mathsf{End}(V)$ of endomorphisms of a vector space V together with their compositions. The induced Lie algebra is denoted by $\mathfrak{gl}(V)$. The bracket of two endomorphisms φ, ψ is

$$[\varphi, \psi] = \varphi \circ \psi - \psi \circ \varphi$$

PROBLEM 10.2. Let A be an algebra. A linear map $D : A \to A$ is called a *derivative* if for all $a, b \in A$

$$D(a \cdot b) = D(a) \cdot b + a \cdot D(b)$$

Show that derivatives form a Lie subalgebra $\mathsf{Der}(A) \subseteq \mathfrak{gl}(A)$.

PROBLEM 10.3. Let $C^\infty M = C^\infty(M, \mathbb{R})$ denote the algebra of all smooth functions on M. Then every vector field X on M determines a mapping

$$\begin{aligned} C^\infty M &\longrightarrow C^\infty(M) \\ f &\longmapsto Xf = \mathrm{d}f(X) \end{aligned}$$

Show that this mapping is a derivative (in the algebraic sense). Also show that all derivatives of $C^\infty M$ are of this form.

Let us now describe the Lie bracket of vector fields from this point of view: $[X, Y]$ is simply the vector field corresponding to the bracket of the two derivatives X and Y of $C^\infty M$. This means that $[X, Y]f = XYf - YXf$ and this formula determines a unique vector field $[X, Y]$.

It also holds that algebra homomorphisms $C^\infty N \to C^\infty M$ are in bijection with smooth maps $M \to N$. One may then rewrite the f-relatedness of vector fields X and Y as

$$\begin{array}{ccc} C^\infty N & \xrightarrow{f^*} & C^\infty M \\ {\scriptstyle Y}\downarrow & & \downarrow{\scriptstyle X} \\ C^\infty N & \xrightarrow[f^*]{} & C^\infty M \end{array}$$

It is then a simple matter to show that $X_i \sim_f Y_i$ implies $[X_1, X_2] \sim_f [Y_1, Y_2]$.

Problem 10.4. Compute the Lie algebra of the additive Lie group $\mathbb{R}^n$.

Problem 10.5. Compute the Lie algebra of the Lie group $\mathrm{GL}(n, \mathbb{R})$ from the definition.

Problem 10.6. Compute the Lie algebra of the Lie group $\mathrm{GL}(n, \mathbb{R})$ from the formula $[A, B] = \frac{\partial^2}{\partial s \partial t}\Big|_{(s,t)=(0,0)} \varphi(t)\psi(s)\varphi(t)^{-1}\psi(s)^{-1}$.

Problem 10.7. Compute the Lie algebra of the Lie group $S^3 = \mathrm{Sp}(1)$ of unit quaternions and show that it is isomorphic to $\mathbb{R}^3$ with the vector product $\times$.

Problem 10.8. Let $B : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a bilinear form and denote by

$$G(B) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T B A = B\} \subseteq \mathrm{GL}(n, \mathbb{R})$$

the closed subgroup of all automorphisms preserving the form B. Compute the Lie algebra of $G(B)$.

Problem 10.9. Compute the Lie algebra of $\mathrm{SO}(n, \mathbb{R})$.

Problem 10.10. Let A be an algebra and denote by $\mathsf{Aut}(A)$ the group of all algebra automorphisms of A. Compute its Lie algebra.

Problem 10.11. Determine all Lie algebras of dimension 2 over $\mathbb{R}$.

Problem 10.12. Prove that the element $\left(\begin{smallmatrix} -2 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ of $\mathrm{GL}(2, \mathbb{R})$ lies in the component of the unit E but not in the image of exp.

Problem 10.13. Let

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(3, \mathbb{R}) \;\middle|\; a, b, c \in \mathbb{R} \right\}$$

denote the Heisenberg group. Show that the bracket on $\mathrm{Lie}(G)$ is non-trivial and exp is a global diffeomorphism.

Problem 10.14. Show that for $G = S^3 = \mathrm{Sp}(1)$ the map exp is not a local diffeomorphim at all points of $\mathfrak{g}$.

Problem 10.15. Find all connected one-dimensional Lie groups.

Problem 10.16. Show that a discrete normal subgroup of a connected Lie group must Lie in the centre.

PROBLEM 10.17. Let $f : M \to G$ be a smooth map from a manifold M to a Lie group G. Denote by $\delta_l f$ the $\mathfrak{g}$-valued 1-form called the *left logarithmic derivative of f* given by

$$\delta_l f(x, X) = (\lambda_{f(x)^{-1}})_* f_* X$$

(with (x, X) denoting a tangent vector $X \in T_x M$). For example

$$\delta_l \mathrm{id}(a, A) = (\lambda_{a^{-1}})_* A = \omega(A)$$

the Maurer-Cartan form. Compute $\delta_l \lambda_b$, $\delta_l \rho_b$, $\delta_l \mu_b$, $\delta_l \nu_b$ and $\delta_l (f \cdot g^{-1})$.

As a corollary, for a connected manifold M two maps $f, g : M \to G$ satisfy $\delta_l f = \delta_f g$ if and only if $f = c \cdot g$ for some $c \in G$. There exists also a criterion for determining whether a $\mathfrak{g}$-valued one-form is a left logarithmic derivative of a map into G. This generalizes the integral calculus of functions.

PROBLEM 10.18. Let $\tilde{G}$ be the universal covering of G. Show that $\pi_1 G \subseteq \tilde{G}$ is a discrete and normal subgroup thus lying in the centre of $\tilde{G}$.

PROBLEM 10.19. Show that the image of the adjoint representation $\mathrm{Ad} : \mathrm{Sp}(1) \to \mathrm{GL}(3, \mathbb{R})$ is $\mathrm{SO}(3, \mathbb{R})$ and that its kernel is the subgroup $\{\pm 1\}$. Thus $\mathrm{Sp}(1)$ is the 2-fold (universal) covering of $\mathrm{SO}(3, \mathbb{R})$.

PROBLEM 10.20. Let $\varphi : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SO}(4, \mathbb{R})$ be the map sending (a, b) to the orthogonal transformation of the quaternions $x \mapsto axb^{-}1$. Show that this map is a 2-fold (universal) covering.

PROBLEM 10.21. Compute the centre of $\mathrm{SO}(n, \mathbb{R})$ or even better the centralizer $C_{\mathrm{SO}(n,\mathbb{R})} \mathrm{GL}(n, \mathbb{R})_+$. Try to determine all connected Lie groups with Lie algebra $\mathfrak{so}(n, \mathbb{R})$.

PROBLEM 10.22. Try to determine the first few terms in the Baker-Campbell-Hausdorff formula for

$$\log(\exp X \cdot \exp Y) : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

where log is the (locally defined) inverse to exp.

A semidirect product of groups is a split short exact sequence

$$1 \longrightarrow K \longrightarrow G \underset{i}{\overset{p}{\rightleftarrows}} H \longrightarrow 1$$

The subgroup $K \subseteq G$ is normal being a kernel of p. The map $f : H \xrightarrow{i} G \xrightarrow{\mathrm{int}} \mathsf{Aut}(K)$ given by $f(x)(y) = xyx^{-1}$ is a group homomorphism. For $a \in G$ there are uniquely determined $k \in K$ and $h \in H$ such that $a = k \cdot i(h)$. Namely $h = p(a)$ and $k = a \cdot i(h)^{-1}$. Therefore as sets $G \cong K \times H$ and the multiplication is given by

$$(k_1, h_1) \cdot (k_2, h_2) = k_1 \cdot i(h_1) \cdot k_2 \cdot i(h_2) = k_1 \cdot f(h_1)(k_2) \cdot i(h_1 h_2) = (k_1 \cdot f(h_1)(k_2), h_1 \cdot h_2)$$

The resulting group is denoted by $K \rtimes H = K \rtimes_f H$.

PROBLEM 10.23. Show that $\mathrm{GA}(n, \mathbb{R})$ is a semidirect product $\mathrm{GA}(n, \mathbb{R}) \cong \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$ where the action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ is the standard one.

PROBLEM 10.24. Let G be a Lie group. Show that $\mu_* : TG \times TG \to TG$ endows TG with a structure of a Lie group.

PROBLEM 10.25. Show that TG is a semidirect product $TG \cong \mathfrak{g} \rtimes G$ and identify the involved action of G on $\mathfrak{g}$.

PROBLEM 10.26. Compute the Lie algebra of a semidirect product $K \rtimes_f H$.

PROBLEM 10.27. Determine the Lie algebra of TG.