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- Recall that $\mathbf{N} = [\![x^\omega = 0]\!] = \bigcup_{n\geq 1}[\![x_1 \cdots x_n = 0]\!]$.
  The proof depends on the following key result.

Lemma 7.1

Let S be a finite semigroup with n elements. Then, for every choice of elements $s_1, \ldots, s_n \in S$, there exist indices i, j such that $0 \leq i < j \leq n$ and the following equality holds for all $k \geq 1$:

$$s_1 \cdots s_n = s_1 \cdots s_i (s_{i+1} \cdots s_j)^k s_{j+1} \cdots s_n.$$

Proof.

Consider the n products $p_r = s_1 \cdots s_r$ $(r = 1, \ldots, n)$. If they are all distinct, then at least one of them, say p_r, is idempotent and we may take $i = 0$, $j = r$. Otherwise, there are indices i, j such that $1 \leq i < j \leq n$ and $p_i = p_j$, in which case $p_i = p_j = p_i s_{i+1} \cdots s_j = p_i (s_{i+1} \cdots s_j)^k$. □

- Let $\varphi : A^+ \to S$ be a homomorphism into a semigroup $S \in \mathbf{N}$, say satisfying $x_1 \cdots x_n = 0$. Then, all words of length at least n belong to $\varphi^{-1}(0)$ and for $s \in S \setminus \{0\}$, the words in the language $L = \varphi^{-1}(s)$ have length less than n, and so L is a finite set.

  Thus, every **N**-recognizable language is either finite or cofinite.

- To show that these are precisely the **N**-recognizable languages, it suffices to show that every singleton language $\{w\} \subseteq A^+$ is **N**-recognizable.

  Let $n = |w|$ be the length of the word w. Consider the semigroup S consisting of the words of A^+ of length at most n together with a zero element 0. The product of two words is the word resulting from their concatenation if that word has length at most n and is 0 otherwise.[1] Then S satisfies the identity $x_1 \cdots x_n = 0$, for the natural homomorphism $\varphi : A^+ \to S$, that sends each letter to itself, we have $\varphi^{-1}(w) = \{w\}$.

[1] This amounts to "killing" the ideal of the semigroup A^+ consisting of the words of length greater than n, identifying all the elements in the ideal to a zero. In semigroup theory, such a construction is called a Rees quotient.

Proposition 7.2

A language over a finite alphabet A is $\mathbf{N}$-recognizable if and only if it is finite or its complement in A^+ is finite.

- In view of Theorem 5.9, we deduce the following result:

Proposition 7.3

Let $\mathbf{V}$ be a pseudovariety of semigroups containing $\mathbf{N}$. Then the completion homomorphism $\iota : A^+ \to \overline{\Omega}_A \mathbf{V}$ is injective and A^+ is a discrete subspace of $\overline{\Omega}_A \mathbf{V}$. In particular, a language $L \subseteq A^+$ is $\mathbf{V}$-recognizable if and only if its closure $\overline{L}$ in $\overline{\Omega}_A \mathbf{V}$ is a clopen subset.

Proof.

The injectivity of ι amounts to $\mathbf{V}$ satisfying no identity $u = v$ with $u, v \in A^+$ distinct words. Indeed, $\mathrm{Synt}(\{u\})$ is nilpotent, whence it belongs to $\mathbf{V}$. Since $1u1 \in \{u\}$ while $1v1 \notin \{u\}$, we deduce that u and v are not $\sigma_{\{u\}}$-equivalent and so $\mathrm{Synt}(\{u\}) \not\models u = v$.

We may therefore identify each $w \in A^+$ with $\iota(w) \in \overline{\Omega}_A\mathbf{V}$.

For $w \in A^+$, we have $\overline{\{w\}} = \{w\}$, because $\overline{\Omega}_A\mathbf{V}$ is a metric space. Since $\{w\}$ is $\mathbf{V}$-recognizable, its closure $\overline{\{w\}}$ is an open subset of $\overline{\Omega}_A\mathbf{V}$ by Theorem 5.9. Hence A^+ is a discrete subset of $\overline{\Omega}_A\mathbf{V}$. □

Proposition 7.4

The semigroup $\overline{\Omega}_A\mathbf{N}$ is obtained by adding to A^+ a zero element. The open sets containing zero consist of zero together with a cofinite subset of A^+.[2]

Proof.

It suffices to observe that a non-eventually constant sequence $(w_n)_n$ of words of A^+ is a Cauchy sequence with respect to the metric $d_\mathbf{N}$ if and only if $\lim |w_n| = \infty$. In the affirmative case, for every homomorphism $\varphi : A^+ \to S$ into $S \in \mathbf{N}$, we have $\lim \varphi(w_n) = 0$. Thus, all non-eventually constant Cauchy sequences converge to the same point ot $\overline{\Omega}_A\mathbf{N}$, which is a zero.

The open subsets of $\overline{\Omega}_A\mathbf{N}$ containing 0 have complement which is a closed, whence compact, subset of A^+. Since A^+ is a discrete subset of $\overline{\Omega}_A\mathbf{N}$, that complement must be finite. The converse is clear. □

[2] This is known as the Alexandroff or one-point compactification, which in general is obtained by adding one point and declaring the open sets containing it to consist also of the complement of a compact subset of the original space.

$\mathbf{K}$: FINITE SEMIGROUPS SATISFYING $es = e$

- Recall that $\mathbf{K} = [\![x^\omega y = x^\omega]\!]$. Note that

$$\mathbf{K} = \bigcup_{n \geq 1} \mathbf{K}_n \text{ where } \mathbf{K}_n = [\![x_1 \cdots x_n y = x_1 \cdots x_n]\!].$$

- Let $A^{\mathbb{N}}$ denote the set of all right infinite words over A, i.e., sequences of letters.
- Endow the set $S = A^+ \cup A^{\mathbb{N}}$ with the operation

$$u \cdot v = \begin{cases} uv & \text{if } u \in A^+ \\ u & \text{otherwise} \end{cases}$$

and the function $d : S \times S \to \mathbb{R}_{\geq 0}$ defined by $d(u, v) = 2^{-r(u,v)}$, where $r(u, v)$ is the length of the longest common prefix of u and v.

Proposition 7.5

The set S is a pro-$\mathbf{K}$ semigroup for the above operation and distance function d. The unique continuous homomorphism $\overline{\Omega}_A\mathbf{K} \to S$ that sends each letter $a \in A$ to itself is an isomorphism.

Proof.

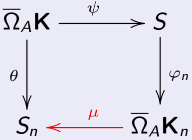
It is easy to check that the multiplication defined on S is associative and that d is an totally bounded complete ultrametric.
Consider the set S_n consisting of all words of A^+ of length at most n, endowed with the operation

$$u \cdot v = \begin{cases} uv & \text{if } |uv| \leq n \\ \mathrm{i}_n(u) & \text{otherwise} \end{cases}$$

where $\mathrm{i}_n(w)$ denotes the longest prefix of length at most n of the word w. This operation is associative and $S_n \in \mathbf{K}_n$. Moreover, every n-generated semigroup from $\mathbf{K}_n$ is a homomorphic image of S_n. Hence $S_n \simeq \overline{\Omega}_A\mathbf{K}_n$.
Note also that the mapping $\varphi_n : S \to S_n$ which sends each $w \in S$ to $\mathrm{i}_n(w)$ is a continuous homomorphism.
Hence, given two distinct points u and v from S, for $n = r(u, v) + 1$, the mapping φ_n is a continuous homomorphism into a semigroup from $\mathbf{K}$ which distinguishes u from v. Thus, S is a pro-$\mathbf{K}$ semigroup.

(. . .)

Consider next the unique continuous homomorphism $\psi : \overline{\Omega}_A\mathbf{K} \to S$ which maps each letter $a \in A$ to itself. Since $\mathbf{K} = \bigcup_{n\geq 1} \mathbf{K}_n$, given distinct $u, v \in \overline{\Omega}_A\mathbf{K}$, there exists a continuous homomorphism $\theta : \overline{\Omega}_A\mathbf{K} \to S_n$ such that $\theta(u) \neq \theta(v)$.



The fact that the above diagram can always be completed by a homomorphism μ shows that $\psi(u) \neq \psi(v)$. Hence ψ is injective. □

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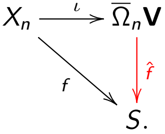
Implicit operations

- Let n be a positive integer.
- An *n-ary implicit operation* on pro-**V** semigroups is a correspondence π associating to each pro-**V** semigroup S an n-ary operation $\pi_S : S^n \to S$ such that , for every continuous homomorphism $\varphi : S \to T$ between pro-**V** semigroups, the following diagam commutes:

$$\begin{array}{ccc} S^n & \xrightarrow{\pi_S} & S \\ \downarrow{\scriptstyle \varphi^n} & & \downarrow{\scriptstyle \varphi} \\ T^n & \xrightarrow{\pi_T} & T, \end{array}$$

  i.e., $\varphi\big(\pi_S(s_1, \ldots, s_n)\big) = \pi_T\big(\varphi(s_1), \ldots, \varphi(s_n)\big)$ for all $s_1, \ldots, s_2 \in S$.
- Examples: the binary multiplication $(s_1, s_2) \mapsto s_1 s_2$ and the component projections $(s_1, \ldots, s_n) \mapsto s_i$ are implicit operations. Composing implicit operations we also obtain implicit operations.

- If A and B are finite sets with the same cardinality n, then $\overline{\Omega}_A\mathbf{V} \simeq \overline{\Omega}_B\mathbf{V}$. We denote by $\overline{\Omega}_n\mathbf{V}$ any of them. Usually, we identify $\overline{\Omega}_n\mathbf{V}$ with $\overline{\Omega}_{X_n}\mathbf{V}$, where $X_n = \{x_1, \ldots, x_n\}$ has cardinality n.
- To each $w \in \overline{\Omega}_n\mathbf{V}$, we may associate an n-ary implicit operation π_w on pro-$\mathbf{V}$ semigroups as follows:
  - for a pro-$\mathbf{V}$ semigroup S, given $s_1, \ldots, s_n \in S$, let $f : X_n \to S$ be the function defined by $f(x_i) = s_i$ $(i = 1, \ldots, n)$;
  - let $(\pi_w)_S(s_1, \ldots, s_n) = \hat{f}(w)$ where $\hat{f}$ completes the following diagram:

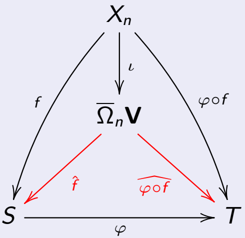


Proposition 8.1

1. *For each $w \in \overline{\Omega}_n\mathbf{V}$, π_w is indeed an n-ary implicit operation on pro-**V** semigroups.*
2. *The correspondence $w \in \overline{\Omega}_n\mathbf{V} \mapsto \pi_w$ is injective and in fact π_w is completely characterized by the operations $(\pi_w)_S$ with $S \in \mathbf{V}$.*

Proof.

1. Let $\varphi : S \to T$ be a continuous homomorphism between two pro-$\mathbf{V}$ semigroups and let $s_1, \ldots, s_n$ be elements of S. Let $f : X_n \to S$ be defined by $f(x_i) = s_i$ $(i = 1, \ldots, n)$. Then we have the following commutative diagram:



which shows that

$$\begin{aligned}\varphi\big((\pi_w)_S(f(s_1), \ldots, f(s_n))\big) &= \varphi\big(\hat{f}(w)\big) = \widehat{\varphi \circ f}(w) \\ &= (\pi_w)_T\big(\varphi(f(s_1)), \ldots, \varphi(f(s_n))\big).\end{aligned}$$

(...)

2. Let $u, v \in \overline{\Omega}_n\mathbf{V}$ be two distinct elements. Then there exists a continuous homomorphism $\varphi : \overline{\Omega}_n\mathbf{V} \to S$ into a semigroup $S \in \mathbf{V}$ such that $\varphi(u) \neq \varphi(v)$. Let $s_i = \varphi(x_i)$ $(i = 1, \dots, n)$. For $w \in \overline{\Omega}_n\mathbf{V}$, by definition of π_w we have

$$(\pi_w)_S(s_1, \dots, s_n) = \varphi(w).$$

Since $\varphi(u) \neq \varphi(v)$, we deduce that

$$(\pi_u)_S(s_1, \dots, s_n) \neq (\pi_v)_S(s_1, \dots, s_n)$$

and so, certainly $\pi_u \neq \pi_v$. □

- We identify w with π_w.
- Note that $S \in \mathbf{V}$ satisfies the **V**-pseudoidentity $u = v$ if and only if $u_S = v_S$.
- We say that a pro-**V** semigroup S satisfies the **V**-pseudoidentity $u = v$ if $u_S = v_S$.

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Implicit signatures

- By an implicit signature we mean a set σ of implicit operations (on **S**) which includes the binary operation of multiplication.
- Example: $\kappa = \{\_ \cdot \_,\ \_^{\omega-1}\}$.
- Given an implicit signature σ, each profinite semigroup S becomes a natural σ-algebra in which each operation $w \in \sigma$ is interpreted as w_S.
- In particular, each $\overline{\Omega}_A\mathbf{V}$ becomes a σ-algebra. The σ-subalgebra generated by $\iota(A)$ is denoted $\Omega_A^\sigma\mathbf{V}$.
- For the minimum implicit signature σ, consisting only of multiplication, we denote $\Omega_A^\sigma\mathbf{V}$ simply by $\Omega_A\mathbf{V}$.[3]
- A formal term constructed from the letters $a \in A$ using the operations from the implicit signature σ is called a σ-term over A. Such σ-term w determines an element $w_\mathbf{V}$ of $\Omega_A^\sigma\mathbf{V}$ by evaluating the operations within $\Omega_A^\sigma\mathbf{V}$.

[3] The bar in the notation $\overline{\Omega}_A\mathbf{V}$ comes from the fact $\Omega_A\mathbf{V} = \iota(A^+)$ is dense in $\overline{\Omega}_A\mathbf{V}$. This notation (without reference to **V**) was introduced by Reiterman [Rei82].

- The following result is an immediate consequence of Theorem 5.3.

Proposition 9.1

The σ-algebra $\Omega_A^\sigma \mathbf{V}$ is a $\mathbf{V}$-free σ-algebra freely generated by A in the sense of the following universal property: for every mapping $\varphi : A \to S$ into a semigroup $S \in \mathbf{V}$, there is a unique homomorphism $\hat{\varphi}$ of σ-algebras such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\iota} & \Omega_A^\sigma \mathbf{V} \\ & {\scriptstyle\varphi}\searrow & \downarrow {\scriptstyle\hat{\varphi}} \\ & & S. \end{array}$$

□

Examples:

- $\Omega_A^\kappa \mathbf{N} = \overline{\Omega}_A \mathbf{N}$;
- for $|A| \geq 2$, since $\overline{\Omega}_A \mathbf{K}$ is uncountable, we have $\Omega_A^\sigma \mathbf{K} \subsetneq \overline{\Omega}_A \mathbf{K}$ for every countable implicit signature σ;
- $\Omega_A^\kappa \mathbf{J} = \overline{\Omega}_A \mathbf{J}$ [Alm95, Section 8.1];
- $\Omega_A^\kappa \mathbf{G}$ is the free group freely generated by $\iota(A) = A$;
- $\Omega_A^\kappa \mathbf{CR}$ is the free completely regular (union of groups) semigroup freely generated by $\iota(A) = A$.

- A key problem for the applications is to be able to solve the word problem in the free σ-algebra $\Omega_A^\sigma \mathbf{V}$: to find an algorithm, if one exists, that given two σ-terms over A, determines whether $u_{\mathbf{V}} = v_{\mathbf{V}}$.
  If such algorithm exists, then we say that the word problem is decidable; otherwise, we say that it is undecidable.

Examples:

- The word problem for $\Omega_A^\kappa \mathbf{N}$: two κ-terms coincide in $\Omega_A^\sigma \mathbf{N}$ if and only if they are equal or they both involve the operation $\_^{\omega-1}$.
- The word problem for $\Omega_A^\kappa \mathbf{G}$ is well known: the operation $\_^{\omega-1}$ is inversion in profinite groups, so all κ-terms can be effectively reduced (over $\mathbf{G}$) to κ-terms in which that operation is only applied to letters; then use, in any order, the reduction rules $aa^{\omega-1} \to 1$ and $a^{\omega-1}a \to 1$ ($a \in A$) to obtain a canonical form for κ-terms over $\mathbf{G}$; two κ-terms are equal over $\mathbf{G}$ if and only if they have the same canonical form.
- Word problem for $\overline{\Omega}_A \mathbf{K}$: exercise.
- The solution of the word problem for $\Omega_A^\kappa \mathbf{J} = \overline{\Omega}_A \mathbf{J}$ gives the structure of $\overline{\Omega}_A \mathbf{J}$ [Alm95, Section 8.1].
- The word problem for $\Omega_A^\kappa \mathbf{CR}$ has been solved by Kad'ourek and Polák [KP86].
- The word problem for $\Omega_A^\kappa \mathbf{A}$ has been solved by McCammond [McC01].

Part II

Separating words and regular languages

Outline

σ-FULLNESS

PRO-$\mathbf{V}$-METRICS

A separation problem

- Let **V** be a pseudovariety of semigroups.
- Suppose that a regular language $L \subseteq A^+$ and a word $w \in A^+$ are given. How do we find out whether a proof that $w \notin L$ exists using **V**-recognizable languages?
- More precisely, we wish to decide whether, given such L and w, there exists a **V**-recognizable language $K \subseteq A^+$ such that $L \subseteq K$ and $w \notin K$.
- For instance, how do we determine whether there exists a finite permutation automaton such that no word from L ends in the same state as w does?
- Another example of the same type of problem: is there some integer n such that no word from L has the same subwords of length at most n as w does?

- Our problem sounds like a topological separation problem, and indeed it admits such a formulation in the profinite world.

Proposition 10.1

Let $\mathbf{V}$ *be a pseudovariety of semigroups,* $L \subseteq A^+$ *a regular language and* w *a word in* A^+*. Then there is a* $\mathbf{V}$*-recognizable language* $K \subseteq A^+$ *such that* $L \subseteq K$ *and* $w \notin K$ *if and only if* $\iota_{\mathbf{V}}(w)$ *does not belong to the closure of* $\iota_{\mathbf{V}}(L)$ *in* $\overline{\Omega}_A\mathbf{V}$*.*

Proof.

By Proposition 5.4, the condition $\iota_{\mathbf{V}}(w)$ belongs to the closure $\overline{\iota_{\mathbf{V}}(L)}$ in $\overline{\Omega}_A\mathbf{V}$ holds if and only if every clopen subset of $\overline{\Omega}_A\mathbf{V}$ which contains $\iota_{\mathbf{V}}(w)$ has nontrivial intersection with $\iota_{\mathbf{V}}(L)$. By Theorem 5.9, such clopen subsets are precisely the sets of the form $\overline{\iota_{\mathbf{V}}(K)}$ where K is a $\mathbf{V}$-recognizable subset of A^+. It remains to observe that, $\iota_{\mathbf{V}}(w) \in \overline{\iota_{\mathbf{V}}(K)}$ and $\overline{\iota_{\mathbf{V}}(K)} \cap \iota_{\mathbf{V}}(L) = \emptyset$ if and only if $w \in K$ and $K \cap L = \emptyset$, which follows from the facts that $K = \iota_{\mathbf{V}}^{-1}(\overline{\iota_{\mathbf{V}}(K)})$ and $L \subseteq \iota_{\mathbf{V}}^{-1}(\iota_{\mathbf{V}}(L))$. □

- Note that, while $\overline{\Omega}_A\mathbf{V}$ is in general uncountable, by Theorem 5.9 it has only countably many clopen subsets, since there are only that many $\mathbf{V}$-recognizable subsets of A^+ (for instance since they are all recognized by finite automata).
- An idea due to Pin and Reutenauer [PR91] in the case of the pseudovariety $\mathbf{G}$ of all finite groups is to somehow "compute" the closure of $\iota_{\mathbf{V}}(L)$ not in $\overline{\Omega}_A\mathbf{G}$ but in the free group $\Omega_A^\kappa\mathbf{G}$, or even in A^+.
- Under the assumption of a conjectured property for the pseudovariety $\mathbf{G}$, they produced an algorithm for computing the required closure, which solves our problem for $\mathbf{G}$.
- We proceed to introduce the required property in general, returning later to their algorithm.

- For a subset L of A^+, denote by $\mathrm{cl}_{\sigma,\mathbf{V}}(L)$ and $\mathrm{cl}_{\mathbf{V}}(L)$ respectively the closure of $\iota_{\mathbf{V}}(L)$ in $\Omega_A^\sigma \mathbf{V}$ and in $\overline{\Omega}_A \mathbf{V}$.
- Note that $\mathrm{cl}_{\sigma,\mathbf{V}}(L) = \mathrm{cl}_{\mathbf{V}}(L) \cap \Omega_A^\sigma \mathbf{V}$.
- Denote by $p_{\mathbf{V}}$ the natural continuous homomorphism $\overline{\Omega}_A \mathbf{S} \to \overline{\Omega}_A \mathbf{V}$.
- Since $\overline{\Omega}_A \mathbf{S}$ is compact and $p_{\mathbf{V}}$ is a onto continuous mapping, we always have the equality $\mathrm{cl}_{\mathbf{V}}(L) = p_{\mathbf{V}}(\mathrm{cl}_{\mathbf{S}}(L))$.
  - In general, for a continuous function $f : S \to T$, and a subset X of S, we have $f(\overline{X}) \subseteq \overline{f(X)}$. The reverse inclusion also holds if f is onto and S is compact.
- We say that the pseudovariety $\mathbf{V}$ is σ-full if, for every regular language $L \subseteq A^+$, the following equality holds:

$$\mathrm{cl}_{\sigma,\mathbf{V}}(L) = p_{\mathbf{V}}(\mathrm{cl}_{\sigma,\mathbf{S}}(L)).$$

In other words, membership of $w \in \Omega_A^\sigma \mathbf{V}$ in $\mathrm{cl}_{\sigma,\mathbf{V}}(L)$ is witnessed by some $w' \in \mathrm{cl}_{\sigma,\mathbf{S}}(L)$ such that $p_{\mathbf{V}}(w') = w$.

Examples:

- The pseudovariety **N** is κ-full: for a regular language $L \subseteq A^+$ and a κ-term w, $w_{\mathbf{N}} \in \mathrm{cl}_{\kappa,\mathbf{N}}(L)$ if and only if w is a word from L or w involves the operation $\_^{\omega-1}$ and L is infinite; in the latter case, by compactness there is some κ-term v such that $v_{\mathbf{S}} \in \mathrm{cl}_{\kappa,\mathbf{S}}(L) \setminus A^+$ and so $w_{\mathbf{N}} = 0 = p_{\mathbf{N}}(v_{\mathbf{S}})$.
- That the pseudovariety **J** is κ-full follows from the structure theorem for $\overline{\Omega}_A\mathbf{J}$.
- The pseudovariety **G** is κ-full: the essential ingredient is a seminal theorem of Ash [Ash91]; the details follow from [AS00] and [Del01].
- The pseudovariety **Ab** is κ-full [Del01].

- The pseudovariety $\mathbf{G}_p$ is not κ-full: this follows from a weak version of Ash's theorem proved by Steinberg [Ste01] for $\mathbf{G}_p$ together with fact that the conjunction of this weaker property with κ-fullness implies that the pseudovariety is defined by pseudoidentities in which both sides are given by κ-terms [AS00]; however, such a definition does not exist since, by a theorem of Baumslag [Bau65], the free group is residually a finite p-group.
- That the pseudovarieties $\mathbf{A}$ and $\mathbf{R}$ are κ-full has been proved by JA-JCCosta-MZeitoun using the solution of the word problems for $\Omega_A^\kappa\mathbf{A}$ [McC01][4] and $\Omega_A^\kappa\mathbf{R}$ [AZ07].

[4] plus refinements from an alternative proof obtained by the same authors including the fact that $\Omega_A^\kappa\mathbf{A}$ is closed for taking factors in $\overline{\Omega}_A\mathbf{A}$.

Outline

σ-fullness

pro-**V**-metrics

PRO-**V** METRICS

- The same way we defined a pseudo-ultrametric on the free semigroup A^+ associated with a pseudovariety **V**, we may define a pseudo-ultrametric on an arbitrary semigroup S: let

$$d(s_1, s_2) = 2^{-r(s_1,s_2)},$$

  where $r(s_1, s_2)$ is the smallest cardinality of a semigroup $T \in \mathbf{V}$ for which there is a homomorphism $\varphi : S \to T$ such that $\varphi(s_1) \neq \varphi(s_2)$.
- Similar arguments show that d is indeed a pseudo-ultrametric on S, with respect to which the multiplication in S is uniformly continuous. If S is finitely generated, then the completion $\hat{S}$ is again a pro-**V** semigroup, but it may not be a free pro-**V** semigroup.
- The pseudo-ultrametric d is an ultrametric if and only if S is residually in **V**.
- Every homomorphism $S \to T$ into $T \in \mathbf{V}$ is uniformly continuous.

Pro-H metric on groups

- Traditionally, one denotes by $\mathbf{H}$ an arbitrary pseudovariety of groups.
- Because a group is highly symmetrical, the pro-$\mathbf{H}$ metric structure looks similar everywhere.

Lemma 11.1

Let G be a group and consider the pro-$\mathbf{H}$ metric on G. Then, for every $u, v, w \in G$, the equalities $d(uw, vw) = d(u, v) = d(wu, wv)$ hold. In particular, for $\epsilon > 0$, we have $B_\varepsilon(u) = uB_\varepsilon(1) = B_\varepsilon(1)u$ and a subset X is open (respectively closed) if and only if so is Xw. Moreover, for $\epsilon > 0$, the ball $B_\epsilon(1)$ is a clopen normal subgroup of G such that $G/B_\epsilon(1) \in \mathbf{H}$. A subgroup H is open if and only if it contains some open ball $B_\epsilon(1)$.

Proof.

This is a simple exercise. □

- For a subgroup H of a group G, denote by H_G the largest normal subgroup of G which is contained in H. It is given by the formula
$$H_G = \bigcap_{g \in G} g^{-1} H g.$$
- If we let G act on the set of right cosets of H in G by right translation, then we obtain a homomorphism $\varphi : G \to S_{G/H}$ into the full symmetric group $S_{G/H}$ (of all permutations of the set G/H) such that $\varphi^{-1}(\mathrm{id}) = H_G$.
- It follows that, if the index $(G : H)$ of the subgroup H in G is finite, then so is $(G : H_G)$ and $(G : H_G)$ is a divisor of $(G : H)!$.

Lemma 11.2

A subgroup H of G is (cl)open in the pro-$\mathbf{H}$ metric if and only if $G/H_G \in \mathbf{H}$.

Proof.

Suppose first that H is open. By Lemma 11.1, H contains a normal subgroup K of G such that $G/K \in \mathbf{H}$. Then $K \subseteq H_G$ and so $G/H_G \simeq (G/K)/(H_G/K)$ belongs to $\mathbf{H}$. Conversely, if $G/H_G \in \mathbf{H}$ then H_G is an open set, because the natural homomorphism $G \to G/H_G$ is (uniformly) continuous. Since H contains H_G, H is a union of cosets of H_G, and so is its complement. Hence H is clopen. □

- Another natural question is whether, for a subgroup H of G, the intersection with H of an open subset of G in the pro-$\mathbf{H}$ metric of G is also open in the pro-$\mathbf{H}$ metric of H.
- In general, the answer is negative, but there are important situations in which it is affirmative.

Example 11.3

Let G be the free group on two free generators a, b and consider the homomorphism $\varphi : G \to S_3$ defined by $\varphi(a) = (12)$ and $\varphi(b) = (13)$. Let $K = \varphi^{-1}(1)$ and let $H = \varphi^{-1}\langle(123)\rangle$ be the inverse image of the subgroup of index 2. Then H is clopen in the pro-$\mathbf{Ab}$ metric of G and K is clopen in the pro-$\mathbf{Ab}$ metric of H but K is not clopen in the pro-$\mathbf{Ab}$ metric of G.

- Note that, for pseudovarieties of groups $\mathbf{K}$ and $\mathbf{H}$, $\mathbf{K} * \mathbf{H}$ consists of all groups G which have a normal subgroup K such that both $K \in \mathbf{K}$ and $G/K \in \mathbf{H}$.[5]
- If $\mathbf{H} * \mathbf{H} = \mathbf{H}$, then we say that $\mathbf{H}$ is closed under extension.
- A condition for the answer to the above question to be affirmative is drawn from the following result.

Lemma 11.4

Let H be a clopen subgroup of G in the pro-$\mathbf{H}$ metric of G and suppose that U is a normal subgroup of H such that $H/U \in \mathbf{H}$. Then the normal subgroup U_G of G is such that $G/U_G \in \mathbf{H} * \mathbf{H}$.

[5] For those unfamiliar with semidirect products, take this as the definition of $\mathbf{K} * \mathbf{H}$ and show that it is a pseudovariety of groups.

Proof.

Consider also the normal subgroup H_G and let $g \in G$. By Lemma 11.2, G/H_G belongs to $\mathbf{H}$. For each $x \in H_G$, the conjugate gxg^{-1} belongs to H and so the mapping $\varphi_g : H_G \to H/U$ which sends x to $gxg^{-1}U$ is a group homomorphism. Moreover, for $x \in H_G$, we have

$$\begin{aligned} x \in U_G &\Leftrightarrow x \in g^{-1}Ug \text{ for all } g \in G \\ &\Leftrightarrow gxg^{-1} \in U \text{ for all } g \in G \\ &\Leftrightarrow \varphi_g(x) = 1 \text{ for all } g \in G. \end{aligned}$$

It follows that H_G/U_G embeds in a finite power of H/U and so $H_G/U_G \in \mathbf{H}$. The result now follows from the observation that $G/H_G \simeq (G/U_G)/(H_G/U_G)$. □

- A first application of the preceding lemma is the following answer to the above question.

Proposition 11.5

Suppose that $\mathbf{H}$ *is closed under extension. Let* H *be a clopen subgroup of* G *in the pro-*$\mathbf{H}$ *metric of* G*. Then a subset of* H *is open in the pro-*$\mathbf{H}$ *metric of* H *if and only if it is open in the pro-*$\mathbf{H}$ *metric of* G*.*

Proof.

By Lemma 11.1, a subgroup L of H is open in the pro-$\mathbf{H}$ metric of H if and only if it contains a normal subgroup U of H such that $H/U \in \mathbf{H}$. By Lemma 11.4, the normal subgroup U_G of G is such that $U/U_G \in \mathbf{H} * \mathbf{H} = \mathbf{H}$. Hence U is open in the pro-$\mathbf{H}$ metric of G by Lemma 11.2. Since L is a union of cosets of U, L is also open in the pro-$\mathbf{H}$ metric of G. □

- In terms of the pro-**H** metrics, we obtain the following more precise result.

Proposition 11.6

Suppose that **H** *is closed under extension and* G *is a group residually in* **H**. *Let* H *be a clopen subgroup of* G *in the pro-***H** *metric of* G. *Then the pro-***H** *metric* d_H *of* H *and the restriction to* H *of the pro-***H** *metric* d_G *of* G *have the same Cauchy sequences.*

PROOF.

Let d be the restriction of d_G to H and let r be the corresponding partial function $H \times H \to \mathbb{N}$. Denote by d' the pseudo-metric d_H and by r' the corresponding partial function. We start by establishing the following function inequalities:

$$r' \leq r \leq \big((G : H) \cdot r'\big)!. \tag{1}$$

The first inequality in (1) follows from the observation that, if a homomorphism from G into a member of $\mathbf{H}$ distinguishes two elements of H then its restriction to H also distinguishes them. Suppose next that $u, v \in H$ and the homomorphism $\varphi : H \to K$ with $K \in \mathbf{H}$ are such that $\varphi(u) \neq \varphi(v)$. Let $U = \varphi^{-1}(1)$. Then H/U embeds in K and, therefore, it belongs to $\mathbf{H}$. By Lemma 11.4, U_G is a normal subgroup of G of finite index such that $G/U_G \in \mathbf{H} * \mathbf{H} = \mathbf{H}$ and, by an earlier observation, $(G : U_G)$ divides $(G : U)!$. If we choose above K so that $|K|$ is minimum, then $(H : U) = r'(u, v)$ and so, since $uU_G \neq vU_G$,

$$r(u, v) \leq (G : U_G) \leq (G : U)! = \big((G : H) \cdot (H : U)\big)! = \big((G : H) \cdot r'(u, v)\big)!$$

which proves (1).

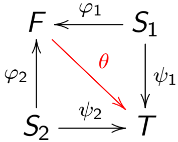
. . .

From the first inequality in (1) we deduce that every Cauchy sequence with respect to d' is also a Cauchy sequence with respect to d. For the converse, let $f(n) = \big((G : H) \cdot n\big)!$. Then f is an increasing sequence and a simple calculation shows that, for every $\varepsilon > 0$,

$$d \leq 2^{-f(\lceil -\log_2 \varepsilon \rceil)} \implies d' \leq \varepsilon.$$

This implies that Cauchy sequences for d are also Cauchy sequences for d'. □

- A free product in a variety $\mathcal{V}$ of semigroups is given by two homomorphisms $\varphi_i : S_i \to F$ $(i = 1, 2)$, with $S_1, S_2, F \in \mathcal{V}$ such that, given any other pair of homomorphisms $\psi_i : S_i \to T$, with $T \in \mathcal{V}$, there exists a unique homomorphism $\theta : F \to T$ such that the following diagram commutes:



- By the usual argument, if the free product exists, then it is unique up to isomorphism.

Exercise 11.7

Show that, for every variety $\mathcal{V}$ and semigroups $S_1, S_2 \in \mathcal{V}$, the free product of S_1 and S_2 in $\mathcal{V}$ exists.

- For semigroups S and T in a variety $\mathcal{V}$, we say that S is a free factor of T if there exists $U \in \mathcal{V}$ such that T is a free product of S and U in $\mathcal{V}$. Note that every semigroup is a free factor of itself.

Exercise 11.8

Suppose that S is a free factor of T in the variety $\mathcal{V}$ generated by a pseudovariety $\mathbf{V}$. Show that:

1. the pseudo-metric $d_{\mathbf{V}}^{S}$ and the restriction of the pseudo-metric $d_{\mathbf{V}}^{T}$ to S coincide;
2. the open sets in pro-$\mathbf{V}$ metric of S are the intersection with S of the open sets of T in the pro-$\mathbf{V}$ metric of T.

Section 12

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